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FERMI-LIQUID THEORY AND POMERANCHUK INSTABILITIES: FUNDAMENTALS AND NEW DEVELOPMENTS

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Despite its apparent simplicity, the Landau Fermi liquid (FL) theory is one of the most non-trivial theories of interacting fermions [1–6]. In general terms, it states that a generic system of interacting fermions in dimension $D > 1$ displays behavior which differs from that for free fermions only qualitatively, but not quantitatively.

A conventional wisdom is that fundamental properties of a FL are determined by fermions near the Fermi surface (FS), while contributions from fermions away from the FS can be safely integrated out in, e. g., renormalization group formalism [7], and only provide some finite renormalization of FL parameters like quasiparticle residue Z and the vertex function $\Gamma^\omega(p_F, q_F)$ for the interaction between fermions on the FS. The potential instabilities of a FL — superconductivity or a spontaneous deformation of a FS (a Pomeranchuk order), are also attributed to the effects of the interaction between fermions at the FS. The condition for a Pomeranchuk instability in a charge or spin channel with orbital momentum l is set to be $F_l^{c(s)} = -1$, where $F_l^{c(s)}$ are partial components of $\Gamma^\omega(p_F, q_F)$.

In this communication, we review earlier and recent work on microscopic theory of a FL, with special attention to the interplay between contributions from fermions at the FS and away from it. Our central message is that conservation laws set up delicate balances between these contributions, with sometimes surprising effects, and some care should be taken when considering the contributions away from the FS.

We first discuss the expressions for quasiparticle residue Z and mass renormalization m^*/m in a generic rotationally-invariant FL, known as Pitaevskii–Landau and Kondratenko relations. These relations can be rewritten as Ward identities [8] associated with conservation of total charge (fermionic mumber), total spin, and total momentum. This leads to five relations between Z, m^*/m, and three-leg vertices associated with conserved "charges" and their currents:

$$\Lambda^c Z = 1, \quad \Lambda^s Z = 1, \quad \Lambda^{mom} Z = 1, \tag{1}$$

$$\frac{m^*}{m}\Lambda^c_J Z = 1 + F_1^c, \quad \frac{m^*}{m}\Lambda^s_J Z = 1 + F_1^s. \tag{2}$$

Here, Λ^c, Λ^s, and Λ^{mom} are three-leg vertices for conserved "charges" (fermionic number/charge, spin, and momentum) and Λ^c_J and Λ^s_J are three-leg vertices for conserved fermionic number/charge and spin "currents". Each three-leg vertex is expressed via charge and spin components of the vertex function $\Gamma^\omega(p_F, q)$ in which one fermion is at the FS, but the other is generally away from it:

$$\begin{aligned}
\Lambda^c &= 1 - 2i\int \Gamma^c(p_F, q)(G_q^2)_\omega \frac{d^{D+1}q}{(2\pi)^{D+1}},\\
\Lambda^s &= 1 - 2i\int \Gamma^c(p_F, q)(G_q^2)_\omega \frac{d^{D+1}q}{(2\pi)^{D+1}},\\
\Lambda^{mom} &= 1 - 2i\int \Gamma^c(p_F, q)(G_q^2)_\omega \frac{\mathbf{p}_F\cdot\mathbf{q}}{p_F^2}\frac{d^{D+1}q}{(2\pi)^{D+1}},
\end{aligned} \tag{3}$$

and

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$$\begin{aligned} \Lambda_J^c = 1 - 2i \int \Gamma^c(p_F, q)(G_q^2)_\omega \frac{\mathbf{p}_F \cdot \mathbf{q}}{p_F^2} \times \\ \times \frac{\partial \epsilon_{\mathbf{q}}}{\partial \epsilon_{\mathbf{q}}^{par}} \frac{d^{D+1}q}{(2\pi)^{D+1}}, \\ \Lambda_J^s = 1 - 2i \int \Gamma^s(p_F, q)(G_q^2)_\omega \frac{\mathbf{p}_F \cdot \mathbf{q}}{p_F^2} \times \\ \times \frac{\partial \epsilon_{\mathbf{q}}}{\partial \epsilon_{\mathbf{q}}^{par}} \frac{d^{D+1}q}{(2\pi)^{D+1}}. \end{aligned} \tag{4}$$

The functions $\Gamma^c(p_F, q)$ and $\Gamma^s(p_F, q)$ are charge and spin component of $\Gamma^\omega_{\alpha\beta,\gamma\delta}(p_F, q) = \delta_{\alpha\gamma}\delta_{\beta\delta}\Gamma^c(p_F, q) + \boldsymbol{\sigma}_{\alpha\gamma}\boldsymbol{\sigma}_{\beta\delta}\Gamma^s(p_F, q)$, $p = (\omega', \mathbf{p})$, the object $\left(G_q^2\right)_\omega$ is the product of two Green's functions with the same momenta and infinitesimally close frequencies, $\epsilon_{\mathbf{q}}$ is the dispersion, which near the FS reduces to $\epsilon_{\mathbf{q}} = \mu + p_F(q - p_F)/m$, and $\epsilon_{\mathbf{q}}^{par} = \mathbf{q}^2/(2m)$ is a parabolic dispersion with the same m as in $\epsilon_{\mathbf{q}}$. The current of momentum (the energy-momentum tensor) is not expressed as a bilinear combination of fermions, and does not give an additional relation for m^*/m in terms of $\Gamma^c(p_F, q)$.

Equations (3) and (4) are valid for both Galilean-invariant and non-Galilean-invariant systems, as long as the total momentum is a conserved. We emphasize that the integrals in (3) and (4) are not confined to the FS and come from fermions with q anywhere in the Brillouin zone.

Next we explore the fact that the set of equations (1) and (2) is overcomplete in the sense that the relations with Λ^c and Λ^c_J alone already express Z and m^*/m in terms of the vertex function:

$$\begin{aligned} &\frac{1}{Z} = 1 - 2i \int \Gamma^c(p_F, q)(G_q^2)_\omega \frac{d^{D+1}q}{(2\pi)^{D+1}}, \\ &\frac{m^*}{m} = (1 + F_1^c)Q, \\ &Q = \\ &= \frac{1-2i \int \Gamma^c(p_F, q)(G_q^2)_\omega \frac{\mathbf{p}_F \cdot \mathbf{q}}{p_F^2} \frac{d^{D+1}q}{(2\pi)^{D+1}}}{1-2i \int \Gamma^c(p_F, q)(G_q^2)_\omega \frac{\mathbf{p}_F \cdot \mathbf{q}}{p_F^2} \frac{\partial \epsilon_{\mathbf{q}}}{\partial \epsilon_{\mathbf{q}}^{par}} \frac{d^{D+1}q}{(2\pi)^{D+1}}} \end{aligned} \tag{5}$$

(in a Galilean-invariant system, where $\epsilon_{\mathbf{q}} = \epsilon^0_{\mathbf{q}}$, $Q = 1$, and $m^*/m = 1 + F_1^c$, Refs. [1–3]). The other equations place constraints on the vertex function: the two additional relations in (3) set conditions on the integrals over $d^{D+1}q$ of $\Gamma^c(p_F, q)$ and $\Gamma^s(p_F, q)$, and the additional relation in (4) can be expressed as

$$\frac{\Lambda^c_J}{\Lambda^s_J} = \frac{1 + F_1^c}{1 + F_1^s}. \tag{6}$$

Equation (6) expresses two essential insights. First, that the ratio of certain integrals, which come from fermions away from the FS, is exactly expressed in terms of Landau parameters, which are determined by the interactions between fermions at the FS. In other words, conservation laws relate the low-energy sector of a FL, where quasiparticles are well defined, and the high-energy sector, where in a generic FL there are no long-lived fermionic states. Second, we see from (6) and (2) that, as long as m^*/m remains finite, the current vertices $\Lambda_{J_{c(s)}}$ scale as $1 + F_1^{c(s)}$ and vanish at $F_1^{c(s)} = -1$.

The second part of our paper is devoted to a study of Pomeranchuk instabilities of the FS to deformations in a given angular momentum channel l, either in the spin or the charge sector. We analyze how conservation laws constrain these instabilities. To this end we introduce a generic order parameter with angular momentum l

$$\hat{\rho}^c_l(\mathbf{q}) = \sum_{\mathbf{p},\alpha} \lambda^c_l(\mathbf{p}) c^\dagger_{\mathbf{p}-\mathbf{q}/2,\alpha} c_{\mathbf{p}+\mathbf{q}/2,\alpha}, \tag{7}$$

$$\hat{\rho}^s_l(\mathbf{q}) = \sum_{\mathbf{p},\alpha\beta} \lambda^s_l(\mathbf{p}) c^\dagger_{\mathbf{p}-\mathbf{q}/2,\alpha} \sigma^{\alpha\beta} c_{\mathbf{p}+\mathbf{q}/2,\beta}, \tag{8}$$

where $\lambda_l^{c(s)}(\mathbf{p})$ is a form-factor. A generic $\lambda_l^{c(s)}(\mathbf{p})$ has momentum dependence dictated by its angular momentum channel (1 for $l = 0$, $\mathbf{p}$ for $l = 1$, etc.) times a function of $|\mathbf{p}|$. For example in 2D

$$\lambda_l^{c(s)}(k) = \cos(l\phi_k)|\mathbf{k}|^l f_l^{c(s)}(|\mathbf{k}|), \tag{9}$$

or equivalently with sin instead of cos.

The starting point for our analysis is the expression for the static susceptibility $\chi_l^{c(s)}$ in a generic FL, originally derived by Leggett [9]:

$$\begin{aligned} \chi_l^{c(s)} &= \left(\Lambda_l^{c(s)} Z\right)^2 \chi_{l,qp}^{c(s)} + \chi_{l,inc}^{c(s)}, \\ \chi_{l,qp}^{c(s)} &= \chi_{l,0} \frac{m^*/m}{1 + F_l^{c(s)}}. \end{aligned} \tag{10}$$

Here, $\Lambda_l^{c(s)}$ is the same triple vertex as before but now for arbitrary order parameter with angular momentum l (still normalized such that $\Lambda_l^{c(s)} = 1$ for free fermions), and $\chi_{l,0}$ is the susceptibility of free fermions (a particle–hole bubble with $\lambda_l^{c(s)}$ in the vertices. $\chi_{l,inc}^{c(s)}$ is the contribution to the susceptibility from fermions strictly away from the FS. We present the diagrammatic derivation of (10).

The $\chi_{l,qp}^{c(s)}$ term is often called the quasiparticle contribution because it is finite in the static limit $\Omega = 0$, $\mathbf{q} \to 0$, but vanishes at $\mathbf{q} = 0$, $\Omega \to 0$, i. e., it comes from the immediate vicinity of the FS, like the susceptibility of free fermions. However, $\Lambda_l^{c(s)}$, Z, and $\chi_{l,inc}^{c(s)}$ (and m^*/m in a non-Galilean system) are inputs coming from fermions away from the FS. The last term,

$\chi^{c(s)}_{l,inc}$, is in general not described at all within FL theory, and its value does not depend on the order of limits Ω goes to zero first or $\mathbf{q}$ goes to zero first.

We analyze the implication of the conservation laws for (10). We first consider the susceptibilities for three conserved order parameters — total charge (particle number), total spin, and total momentum. For the first two $l = 0$ and $\lambda^{c(s)}_{l=0}(\mathbf{p}) = 1$, for the third one $l = 1$ and $\lambda^{c(s)}_{l=1}(\mathbf{p}) = \mathbf{p}$. In all three cases, $\Lambda Z = 1$ and $\chi_{inc} = 0$. Consequently, the full susceptibilities coincide with the corresponding χ_{qp}:

$$\chi^c = \frac{m^*}{\pi}\frac{1}{1+F^c_0}, \quad \chi^s = \frac{m^*}{\pi}\frac{1}{1+F^s_0}, \qquad \chi^{mom} = \frac{m^* k_F^2}{2\pi}\frac{1}{1+F^c_1} \tag{11}$$

(by χ^{mom} we mean the longitudinal component of the susceptibility tensor). We see that in each channel the susceptibility diverges when the proper $F^{c(s)}_l = -1$. This divergence is indeed the signature of a Pomeranchuk instability. The $l = 0$ instability in the charge channel corresponds to phase separation, the one in the spin channel corresponds to ferromagnetism, and the one at $l = 1$ signals the emergence of a charge nematic (dipolar) order. In a Galilean-invariant system, $m^*/m = 1 + F^c_1$, and $l = 1$ Pomeranchuk instability does not occur by elementary reasons.

We next consider the susceptibilities for charge current and spin current orders, i. e., for order parameters with $l = 1$ and form factor $\lambda^{c(s)}_{l=1}(\mathbf{p}) = \partial\epsilon_{\mathbf{p}}/\partial\mathbf{p} = (\mathbf{p}/m)\partial\epsilon_{\mathbf{p}}/\partial\epsilon^0_{\mathbf{p}}$. We label the corresponding susceptibilities as $\chi^{c(s)}_J$. Using (2), we can re-express Eq. (10) for $\chi^{c(s)}_J$ as

$$\chi^{c(s)}_J = \frac{mp_F^2}{2\pi}\frac{m}{m^*}\left(1 + F^{c(s)}_1\right) + \chi^{c(s)}_{J,inc}. \tag{12}$$

We see that the quasiparticle part of the susceptibility of either charge-current or spin-current order parameter actually vanishes when the corresponding Landau parameter reaches -1, i. e., a Pomeranchuk instability does not develop at $F^{c(s)}_1 = -1$, if one probes it by analyzing $\chi_{J_{c(s)}}$ (Refs. [9, 10]). This is indeed the consequence of $\Lambda^{c(s)}_{l=1} \propto 1+F^{c(s)}_{l=1}$. Putting it differently, the vanishing of $\chi^{c(s)}_{J,qp}$ shows that supposedly innocent contributions from fermions away from the FS may qualitatively change the behavior attributed to the interaction between FL quasiparticles in the immediate vicinity of the FS.

We further argue that the cancellation between the vertex $\Lambda^{c(s)}_l$ and $1 + F^{c(s)}_l$ is entirely due to conservation laws and holds only for the order parameters which correspond to spin and charge currents ($\lambda^{c(s)}_{l=1}(k) = \partial\epsilon_k/\partial\mathbf{k}$). For other $l = 1$ order parameters with different $\lambda^{c(s)}_{l=1}(k)$ and for order parameters with other l, we see no reasons why $\Lambda^{c(s)}_l$ should be proportional to $1+F^{c(s)}_l$, or, alternatively speaking, why the integral over $\Gamma^{c(s)}(p_F, q)$ over q away from the FS has to be expressed in terms of $\Gamma^{c(s)}(p_F, q_F)$ between fermions at the FS. Hence, we may expect a Pomeranchuk instability to occur at $F^{c(s)}_l = -1$ (Ref. [11]).

In the last part of the paper, we verify the relations between contributions to the susceptibility of the spin current from fermions away and near the FS in explicit computations to second order in the Hubbard interaction U. We identify a specific relation at order U^2 between the contributions from fermions away from the FS and at the FS. For a Galilean-invariant system, this relation is

$$U^2 \int d_{kl}\left(G_l G_{k-p_F+l} + G_l G_{k+p_F-l}\right) \times \times G_k^2 \frac{\mathbf{p}_F\cdot\mathbf{k}}{p_F^2} = F^c_{l=1} - F^s_{l=1}. \tag{13}$$

The l.h.s. of this relation is the integral which comes from fermions not confined to the FS, while the r.h.s. comes from fermions right at the FS. Using this relation one can explicitly re-express $\Lambda^{c(s)}_J$ in terms of $1 + F^{c(s)}_{l=1}$ and verify Eq. (6).

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