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DYNAMICS OF ANISOTROPIC POWER-LAW $f(R)$ COSMOLOGY

M. F. Shamir [*]

Department of Sciences and Humanities,
National University of Computer and Emerging Sciences Lahore Campus, Pakistan

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Modified theories of gravity have attracted much attention of the researchers in the recent years. In particular, the $f(R)$ theory has been investigated extensively due to important $f(R)$ gravity models in cosmological contexts. This paper is devoted to exploring an anisotropic universe in metric $f(R)$ gravity. A locally rotationally symmetric Bianchi type I cosmological model is considered for this purpose. Exact solutions of modified field equations are obtained for a well-known $f(R)$ gravity model. The energy conditions are also discussed for the model under consideration. The viability of the model is investigated via graphical analysis using the present-day values of cosmological parameters. The model satisfies null energy, weak energy, and dominant energy conditions for a particular range of the anisotropy parameter while the strong energy condition is violated, which shows that the anisotropic universe in $f(R)$ gravity supports the crucial issue of accelerated expansion of the universe.

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1. INTRODUCTION

A new picture of the universe has evolved after the experimental evidence of accelerating expansion of the universe [1]. It is thought that the mysterious dark energy is responsible for this. However, another explanation comes from the modified theories of gravity, and it is suggested that higher-order theories of gravity are helpful in addressing the dark energy and late time acceleration issues [2]. Various generalizations or modifications of Einstein's theory of general relativity (GR) have been proposed in the recent decades. Among the various modifications, $f(R)$ and $f(R,T)$ theories of gravity (R is the Ricci scalar and T is the trace of the energy-momentum tensor) have been treated most seriously [3–6]. These theories are considered most suitable due to cosmologically important $f(R)$ models. Viable $f(R)$ gravity models [7] have been proposed that show the unification of early-time inflation and late-time acceleration. Starobinsky [8] proposed a class of models producing viable cosmology that satisfies solar-system and laboratory tests. Hu and Sawicki [9] studied some viable $f(R)$ models that justified accelerated expansion without involving a cosmological constant. The $f(T)$ theory of gravity [10] is an alternate theory which is a generalization of teleparallel gravity in which the Weitzenböck connection is used instead of the Levi–Civita connection. Modified Gauss–Bonnet gravity also known as $f(G)$ gravity is another theory that has gained popularity in the last few years [11].

The energy conditions have gained much importance in the recent cosmological studies. The energy conditions can be divided into four different categories: the null energy condition (NEC), the weak energy condition (WEC), the strong energy condition (SEC), and the dominant energy condition (DEC). Many important phenomena including the Hawking–Penrose singularity theorems and the validity of the second law of black-hole thermodynamics can be discussed using energy conditions [12]. Moreover, violation of the SEC in the context of modified gravity indicates the cosmic expansion [13]. In [14], energy conditions were investigated using metric $f(R)$ gravity. In [15], energy conditions were studied from the standpoint of the stability and viability of cosmological models in $f(R)$ gravity. A charged black-hole metric satisfying the WEC was discussed in [16]. The viability of some $f(G)$ gravity models were investigated in [17] by exploring these energy conditions. In [18], the work in $f(R,G)$ gravity was extended and the viability of some $f(R,G)$ gravity models was discussed using the WEC.

In this paper, we investigate the dynamics of $f(R)$ gravity with an anisotropic background. It is well known that isotropic models are among the best choices to study large-scale structure of the universe. More-

[*] E-mail: farasat.shamir@nu.edu.pk

over, according to the cosmological observations including the cosmic microwave background (CMB) radiation, the current universe is isotropic. However, it is believed that the early universe may not have been exactly uniform. Also, the local anisotropies that we observe today in galaxies and superclusters also motivate us to model the universe with an anisotropic background. Bianchi-type models are among the simplest models with an anisotropic background. In particular, investigating the Bianchi-type universe in the context of modified theories is interesting. There exist some previous works [19–21] on cosmology in the framework of $f(R)$ modified gravity where the exact solutions of field equations are explored for different Bianchi-type spacetimes using the variation law of the Hubble parameter. Here, we consider locally rotationally symmetric (LRS) Bianchi type I spacetime to find exact solutions of the field equations in $f(R)$ gravity. It is worthwhile to mention here that we do not use any conventional assumption like a constant deceleration parameter or the variation law of the Hubble parameter to investigate the solutions in this paper. In addition, the energy conditions are also discussed and the viability of $f(R)$ models is investigated via graphical analysis using the present-day values of cosmological parameters. The SEC is violated, which shows that the anisotropic universe in $f(R)$ gravity supports the phenomenon of expansion of the universe. The paper is planned as follows. Field equations in $f(R)$ gravity are briefly introduced in Sec. 2. In Sec. 3, power-law solutions are discussed in the light of energy conditions. The last section summarizes and concludes the results.

2. MODIFIED FIELD EQUATIONS

The field equations for the $f(R)$ theory of gravity are

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = \kappa T^m_{\mu\nu}, \quad (1)$$

where $F(R)$ denotes the derivative of $f(R)$ with respect to R, κ is the coupling constant in gravitational units, $T^m_{\mu\nu}$ is the standard matter energy–momentum tensor, and

$$\Box \equiv \nabla^\mu\nabla_\mu \quad (2)$$

with ∇_μ defined as the covariant derivative. The field equations can be expressed in an alternative form familiar from the general relativity (GR) field equations as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{eff}_{\mu\nu}, \quad (3)$$

where

$$T^{eff}_{\mu\nu} = \frac{1}{F(R)}\left[\frac{1}{2}g_{\mu\nu}\left(f(R) - RF(R)\right) + F(R)^{;\alpha\beta}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\mu\nu}g_{\alpha\beta}\right) + T^m_{\mu\nu}\right]. \quad (4)$$

This form of gravitational field equations is interesting and we can obtain the corresponding energy conditions using this form. We consider the spatially homogeneous, anisotropic, and LRS Bianchi type I spacetime

$$ds^2 = dt^2 - L^2(t)dx^2 - M^2(t)[dy^2 + dz^2], \quad (5)$$

where L and M are cosmic scale factors. The average scale factor a and the average Hubble parameter H are

$$a = \sqrt[3]{LM^2}, \quad H = \frac{1}{3}\left(\frac{\dot{L}}{L} + \frac{2\dot{M}}{M}\right). \quad (6)$$

Throughout the paper, the dot denotes the derivative with respect to t. Here, we assume that the universe is filled with perfect fluid. Due to the complicated and highly nonlinear nature of $f(R)$ field equations, we consider the physical condition that the shear scalar σ is proportional to the expansion scalar θ, which leads to

$$L = M^n, \quad (7)$$

where n is an arbitrary real number. Then the field equations take the form

$$(2n+1)\frac{\dot{M}^2}{M^2} = \frac{1}{F}\left[\kappa\rho + \frac{f}{2} + \left((n+2)\frac{\ddot{M}}{M} + (n^2+n+1)\frac{\dot{M}^2}{M^2}\right)F - (n+2)\frac{\dot{M}}{M}\dot{F}\right], \quad (8)$$

$$\frac{2\ddot{M}}{M} + \frac{\dot{M}^2}{M^2} = \frac{1}{F}\left[-\kappa p + \frac{f}{2} + \left((n+2)\frac{\ddot{M}}{M} + (n^2+n+1)\frac{\dot{M}^2}{M^2}\right)F - \frac{2\dot{M}}{M}\dot{F} - \ddot{F}\right], \quad (9)$$

$$(n+1)\frac{\ddot{M}}{M} + n^2\frac{\dot{M}^2}{M^2} = \frac{1}{F}\left[-\kappa p + \frac{f}{2} + \left((n+2)\frac{\ddot{M}}{M} + (n^2+n+1)\frac{\dot{M}^2}{M^2}\right)F - (n+1)\frac{\dot{M}}{M}\dot{F} - \ddot{F}\right]. \quad (10)$$

3. POWER-LAW $f(R)$ MODEL

It has been established that dark matter and dark energy phases can be achieved by finding the exact solutions using a power-law $f(R)$ model [22]. In [23], $f(R)$ gravity was investigated using reconstruction methods, and it is shown that perfect fluid may lead to the inflationary universe consistent with the Planck data. Also, it was concluded that power-law $f(R)$ models gave the best-fit values compatible with BICEP2 and Planck data. Thus, it is cosmologically viable and realistic to assume $f(R)$ in the power-law form to solve the field equations.

3.1. Exact solutions

We follow the approach of Nojiri and Odintsov [24] and assume that $F(R) \propto f_0 R^m$, where f_0 is an arbitrary constant. Using this assumption and manipulating Eqs. (8)–(10), we obtain

$$(n^3+2n^2+2n+2)\dot{M}^4+(2n^2+4n+3)M\dot{M}^2\ddot{M}+ \\ +(n+2)M^2\ddot{M}^2-2m(n^2+n+1)\dot{M}^4+ \\ +m(2n^2+n)M\dot{M}^2\ddot{M}+m(n+2)M^2\dot{M}\dddot{M}=0. \quad (11)$$

We here assume that the solution is in a power-law form, $M(t)=(c_1t+c_2)^k$, where c_1, c_2, and k are arbitrary real constants with $k\neq 0$. Using Eq. (11), we obtain the constraint equation

$$(n^3+4n^2+7n+6)k^2-[2m(n^2+2n+3)+2n^2+6n+7]k+ \\ +(1+2m)(n+2)=0. \quad (12)$$

This equation is important because it can be used to reconstruct different forms of $f(R)$ models with suitable solutions of field equations. A detailed discussion on possibility of exact solutions has already been given in [25]. For example, when $n=-1$, Eq. (12) gives

$$k=2m+1, \quad m\neq 0,\ 1. \quad (13)$$

Hence, the solution metric takes the form

$$ds^2=dt^2-(c_1t+c_2)^{-2(2m+1)}dx^2- \\ -(c_1t+c_2)^{2(2m+1)}(dy^2+dz^2). \quad (14)$$

3.2. Energy conditions

In modern-day cosmology, the energy conditions are considered useful to establish some important theorems about black holes. The viability of some important cosmological models is linked with the energy conditions. The energy conditions are reflected by the term $R_{\mu\nu}v^\mu v^\nu\geq 0$ in the Raychaudhuri equation for the expansion of universe

$$\frac{d\theta}{d\tau}=-\frac{\theta^2}{2}-\sigma_{\mu\nu}\sigma^{\mu\nu}+\omega_{\mu\nu}\omega^{\mu\nu}-R_{\mu\nu}v^\mu v^\nu, \quad (15)$$

where θ, $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ respectively denote the expansion, shear, and rotation, while v^μ is a null vector. The Raychaudhuri equation is valid for any geometrical theory of gravity, and can therefore be also used to investigate the energy conditions for modified theories, the $f(R)$ gravity in particular. The NEC, WEC, SEC, and DEC are as follows:

$$\begin{aligned} &\text{NEC:}\ \ \rho^{eff}+p^{eff}\geq 0; \\ &\text{WEC:}\ \ \rho^{eff}\geq 0, \quad \rho^{eff}+p^{eff}\geq 0; \\ &\text{SEC:}\ \ \rho^{eff}+3p^{eff}\geq 0, \quad \rho^{eff}+p^{eff}\geq 0; \\ &\text{DEC:}\ \ \rho^{eff}\geq 0, \quad \rho^{eff}\pm p^{eff}\geq 0. \end{aligned} \quad (16)$$

Using field equations (3), we obtain

$$\rho^{eff}=\frac{1}{F}\left[\kappa\rho+\frac{f}{2}+\left((n+2)\frac{\ddot{M}}{M}+ \right.\right. \\ \left.\left. +(n^2+n+1)\frac{\dot{M}^2}{M^2}\right)F-(n+2)\frac{\dot{M}}{M}\dot{F}\right], \quad (17)$$

$$p^{eff}=\frac{1}{F}\left[\kappa p-\frac{f}{2}-\left((n+2)\frac{\ddot{M}}{M}+ \right.\right. \\ \left.\left. +(n^2+n+1)\frac{\dot{M}^2}{M^2}\right)F+\left(\frac{n+3}{2}\right)\frac{\dot{M}}{M}\dot{F}+\ddot{F}\right]. \quad (18)$$

In cosmology, apart from the Hubble parameter, we can define some important parameters like the deceleration and jerk parameters as

$$q=-\frac{1}{H^2}\frac{\ddot{a}}{a}, \quad j=\frac{1}{H^3}\frac{\dddot{a}}{a}. \quad (19)$$

We simplify these parameters as follows:

$$\dot{H}=-H^2(q+1), \quad \ddot{H}=H^3(j+3q+2). \quad (20)$$

The Ricci scalar in terms of the Hubble parameter and the deceleration parameter takes the form

$$R=-6\left[\frac{3(n^2+2n+3)}{(n+2)^2}-(q+1)\right]H^2, \quad (21)$$

Thus, using Eqs. (16)–(21), the energy conditions are given as follows:

NEC:

$$\left[\frac{3m(n+1)(q+1)}{n+2} + 4m(m-1)(q+1)^2 + \right.$$
$$\left. + 2m[(q+1)^2 + j + 3q + 2]\right] H^2 +$$
$$+ \left[\left(\frac{-18(n^2+2n+3)}{(n+2)^2} + 6(q+1)\right) H^2\right]^{-m} \times$$
$$\times \kappa(\rho + p) \geq 0; \quad (22)$$

WEC:

$$3\left[\frac{m}{m+1}\left(\frac{3(n^2+2n+3)}{(n+2)^2} - q - 1\right) + 2m(q+1)\right] H^2 +$$
$$+ \left[\left(6(q+1) - \frac{18(n^2+2n+3)}{(n+2)^2}\right) H^2\right]^{-m} \times$$
$$\times \kappa\rho \geq 0, \quad \rho^{eff} + p^{eff} \geq 0; \quad (23)$$

SEC:

$$\left[\frac{-6m}{m+1}\left(\frac{3(n^2+2n+3)}{(n+2)^2} - (q+1)\right) - \frac{3m(n+5)(q+1)}{n+2} + \right.$$
$$\left. + 12m(m-1)(q+1)^2 + 6m[(q+1)^2 + j + 3q + 2]\right] H^2 +$$
$$+ \left[\left(\frac{-18(n^2+2n+3)}{(n+2)^2} + 6(q+1)\right) H^2\right]^{-m} \times$$
$$\times \kappa(\rho + 3p) \geq 0, \quad \rho^{eff} + p^{eff} \geq 0; \quad (24)$$

DEC:

$$\left[\frac{6m}{m+1}\left(\frac{3(n^2+2n+3)}{(n+2)^2} - (q+1)\right) + \frac{3m(3n+7)(q+1)}{n+2} - \right.$$
$$\left. - 4m(m-1)(q+1)^2 - 2m[(q+1)^2 + j + 3q + 2]\right] H^2 +$$
$$+ \left[\left(\frac{-18(n^2+2n+3)}{(n+2)^2} + 6(q+1)\right) H^2\right]^{-m} \times$$
$$\times \kappa(\rho - p) \geq 0, \quad \rho^{eff} + p^{eff} \geq 0, \quad \rho^{eff} \geq 0. \quad (25)$$

3.3. Qualitative analysis

Here, we first investigate the viability of $f(R)$ gravity and then give a qualitative analysis of the exact solutions using the energy conditions. For this purpose, we consider present-day values for the parameters

$$q_0 = -0.81 \pm 0.14, \quad j_0 = 2.16^{+0.81}_{-0.75},$$
$$s_0 = -0.22^{+0.21}_{-0.19}, \quad H_0 = 0.718, \quad (26)$$

and the present-day value for the Ricci scalar can be taken as

$$R = -6\left[\frac{3(n^2+2n+3)}{(n+2)^2} - (q_0+1)\right] {H_0}^2. \quad (27)$$

We also consider the vacuum case $\rho = 0 = p$ in the analysis. With these values and Eqs. (22)–(25), the graphical analysis is shown in Fig. 1 and Fig. 2. Exact values of the parameters $H = 0.718$, $q = -0.81$, and $j = 2.16$ are used in drawing Fig. 1 and Fig. 2 to have good qualitative behaviors of the results.

It is clear from Fig. 1 that the WEC is satisfied. This implies that the NEC is also satisfied for the $f(R)$ model under consideration. However, the SEC is violated, as shown in Fig. 2*a*. This is an indication of acceleration of the universe for this cosmological model. The DEC is also satisfied in Fig. 2*b*. Thus, the viability of the model with the present-day parameters is obvious. It is also evident from the figures that when $n = -1$, the NEC, WEC, and DEC are satisfied, while the SEC is violated. Therefore, exact solution (14) is valid for the given range of m. We can reconstruct many other solutions using Eq. (11) and do a similar analysis. Thus, the exact solutions are valid and satisfy the energy conditions for $f(R) = R^{\delta}$, $1 < \delta < 2$. Hence, an anisotropic universe in $f(R)$ gravity supports the current expansion of the universe with the anisotropy parameter range $-2 < n < 0$.

4. CONCLUDING REMARKS

The paper is devoted to investigating the dynamics of an anisotropic universe in $f(R)$ modified gravity. For this purpose, an LRS Bianchi type I spacetime is chosen. The highly nonlinear and complicated nature of the field equations forces us to assume that the shear scalar σ is proportional to the expansion scalar θ. This implies that $L = M^n$, where L and M are the metric coefficients and n is an arbitrary constant. A brief summary and conclusion of the work is as follows.

- We have considered a power-law $f(R)$ gravity model already available in the literature [22]. The interesting feature of the model is that dark matter and dark energy phases can be achieved by finding exact solutions using a power-law $f(R)$ model. We have developed a general differential equation (11), which can be used to explore many solutions. However, only one solution is chosen for the present analysis.

- The energy conditions are developed for the power-law $f(R)$ gravity model. The present-day values of cosmological parameters are assumed to check the viability of the model. It is mentioned here that long mathematical expressions for the energy conditions are not easy to analyze directly. Thus, the graphical analysis is given that shows that the NEC, WEC and DEC

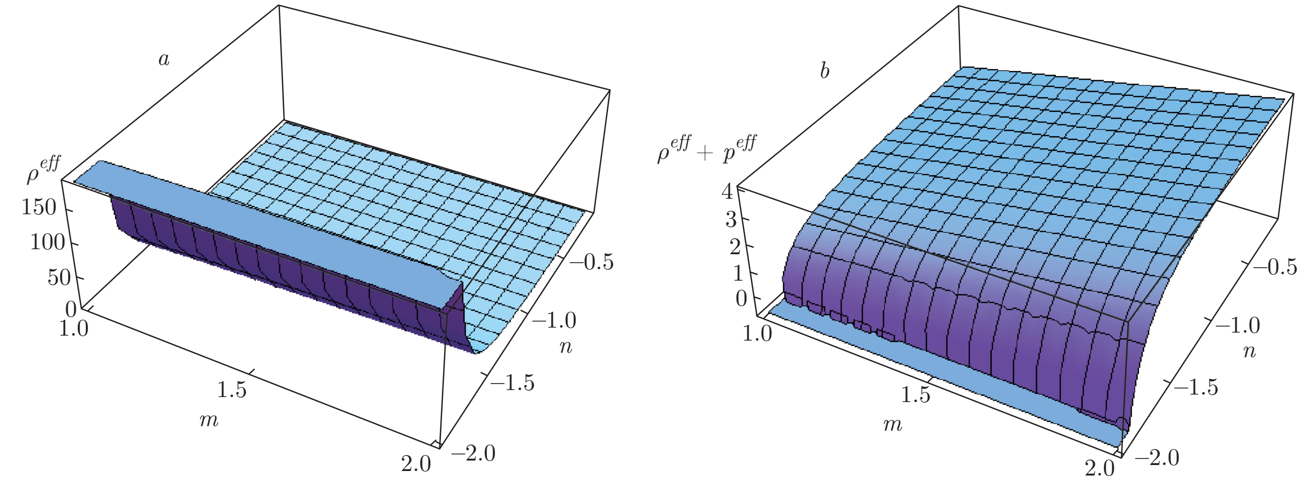

Fig. 1. Plots of the WEC with $H = 0.718$, $q = -0.81$, and $j = 2.16$

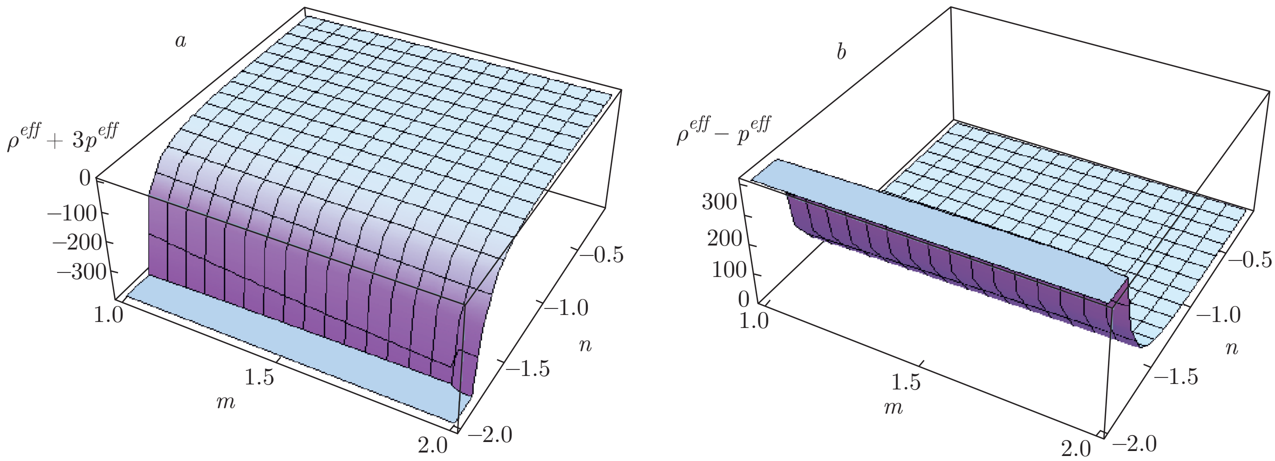

Fig. 2. Plots of the SEC and DEC with $H = 0.718$, $q = -0.81$, and $j = 2.16$

are satisfied while the SEC is violated. This violation indicates expansion of the universe.

• The parameters for the graphical analysis are assumed such that the validity of the exact solutions may be checked. It is concluded that the solutions satisfy the energy conditions and support the current expansion of the universe when $f(R) = R^{\delta}$, $1 < \delta < 2$ and the anisotropy parameter is in the range $-2 < n < 0$.

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