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USEFULNESS OF MULTIQUBIT W-TYPE STATES IN QUANTUM INFORMATION PROCESSING

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We analyze the efficiency of multiqubit W-type states as resources for quantum information. For this, we identify and generalize four-qubit W-type states. Our results show that these states can be used as resources for deterministic quantum information processing. The utility of results, however, is limited by the availability of experimental setups to perform and distinguish multiqubit measurements. We therefore emphasize protocols where two users want to establish an optimal bipartite entanglement using the partially entangled W-type states. We find that for such practical purposes, four-qubit W-type states can be a better resource in comparison to three-qubit W-type states. For a dense coding protocol, our states can be used deterministically to send two bits of classical message by locally manipulating a single qubit. In addition, we also propose a realistic experimental method to prepare the four-qubit W-type states using standard unitary operations and weak measurements.

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1. INTRODUCTION

Quantum entanglement [1] plays a key role in many potential applications in quantum information and computation [2–6]. The optimal success of a quantum communication protocol can be ascertained by the use of maximally entangled states as resources for information transfer. However, in general, the use of nonmaximally entangled resources leads to probabilistic protocols and the fidelity of information transfer is always less than unity. For example, quantum teleportation of a single qubit using a three- or four-qubit W state is always probabilistic and the teleportation fidelity depends on the unknown parameter of the teleported state [7]. On the other hand, Agrawal and Pati [8] proposed a new class of three-qubit W-type states for deterministic teleportation of a single qubit by performing three-qubit joint measurements. The efficiency of these W-type states, however, decreases if one performs standard two-qubit and single qubit measurements only [9] instead of performing a joint three-qubit measurement. We therefore address the question of the usefulness of nonmaximally entangled resources for sending maximum information from a sender to a receiver.

We propose a new class of nonmaximally entangled four-qubit W-type states for quantum information processing and demonstrate the possibility of deterministic teleportation of a single qubit with unit fidelity. For practical purposes, we emphasize a protocol to share optimal bipartite entanglement. For this, we use partially entangled four-qubit W-type states as a starting resource between two users and achieve the optimal bipartite entanglement by performing standard single- and two-qubit measurements only. Our results show that the shared two-qubit entanglement can lead to a maximally entangled resource for certain state parameters. We further demonstrate the need to analyze four-qubit W-type states by comparing the efficiency of three- and four-qubit W-type states as resources in terms of concurrence [10] of the finally shared entangled state between two users. Interestingly, our results show that for certain ranges of parameters, four-qubit W-type states are more efficient resources in comparison to three-qubit W-type states for achieving optimal concurrence.

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For dense coding, we found that in principle a sender can transmit a two-bit classical message to a receiver by locally manipulating his/her single qubit. Moreover, we also generalize teleportation and dense coding protocols using N-qubit W-type states as resources to achieve deterministic information transfer. In order to add another dimension and significance to the results obtained in this article, we finally demonstrate experimental preparation of four-qubit W-type states. The experimental generation of these states is achieved using standard single- and two-qubit unitary operations and weak measurements.

2. TELEPORTATION USING FOUR-QUBIT W-TYPE STATES

Teleportation is a process of transmitting quantum information over arbitrary distances using a shared entangled resource. Although a nonmaximally entangled four-qubit W state can be used as a resource for probabilistic teleportation of a single qubit, one cannot achieve teleportation of a single qubit using the standard four-qubit W state with certainty. Therefore, we propose a new class of four-qubit W states, namely

$$|\Psi_k\rangle_{1234} = \frac{1}{2\sqrt{k+1}}\Big[|1000\rangle + \sqrt{k}e^{i\gamma}|0100\rangle + \sqrt{k+1}e^{i\delta}|0010\rangle + \sqrt{2k+2}e^{i\zeta}|0001\rangle\Big]_{1234}, \quad (1)$$

where k is a real number and γ, δ, and ζ are phases. The states in Eq. (1) can be used as resources to achieve optimal and deterministic quantum teleportation. For example, if Alice wants to teleport an unknown state $|\phi\rangle_a = [\alpha|0\rangle + \beta|1\rangle]_a$, where $|\alpha|^2 + |\beta|^2 = 1$, to Bob, then Alice and Bob need to share the four-qubit state $|\Psi_k\rangle_{1234}$ such that Alice has qubits 1, 2, and 3 and Bob has qubit 4.

Thus, the joint state of five qubits can be represented as

$$|\Phi\rangle_{a1234} = |\phi\rangle_a \otimes |\Psi_k\rangle_{1234}. \quad (2)$$

To teleport the unknown state to Bob, Alice projects her four qubits on the states

$$\begin{aligned} |\eta_k\rangle^{\pm}_{a123} &= \frac{1}{2\sqrt{k+1}}\Big[|0100\rangle + \sqrt{k}e^{i\gamma}|0010\rangle + \sqrt{k+1}e^{i\delta}|0001\rangle \pm \sqrt{2k+2}e^{i\zeta}|1000\rangle\Big]_{a123}, \\ |\xi_k\rangle^{\pm}_{a123} &= \frac{1}{2\sqrt{k+1}}\Big[|1100\rangle + \sqrt{k}e^{i\gamma}|1010\rangle + \sqrt{k+1}e^{i\delta}|1001\rangle \pm \sqrt{2k+2}e^{i\zeta}|0000\rangle\Big]_{a123}. \end{aligned} \quad (3)$$

Although the teleportation protocol works for all k, γ, δ, and ζ, for simplicity we assume $k = 1$ and $\gamma = \delta = \zeta = 0$. Therefore, the joint state of five qubits can be reexpressed using Alice's measurement basis as

$$|\Phi\rangle_{a1234} = \frac{1}{2}\Big[|\eta_1\rangle^{+}_{a123}|\phi\rangle_4 + |\eta_1^{-}\rangle^{+}_{a123}\sigma_z|\phi\rangle_4 + |\xi_1^{+}\rangle^{+}_{a123}\sigma_x|\phi\rangle_4 + |\xi_1^{-}\rangle^{+}_{a123}\iota\sigma_y|\phi\rangle_4\Big], \quad (4)$$

where $|\phi\rangle_4 = [\alpha|0\rangle + \beta|1\rangle]_4$. A four-qubit joint measurement on qubits $a, 1, 2$, and 3 projects the state of Bob's qubit onto one of the four possible states as shown in Eq. (4) with the equal probability of $1/4$.

Hence, teleportation of a single qubit using a nonmaximally entangled four-qubit W-type state is always successful. The use of proposed states as quantum channels also provides flexibility to the experimental setups by relaxing the requirement of a maximally entangled shared resource for a faithful teleportation. Because the teleportation is deterministic, the total probability and fidelity of teleporting a single qubit using a partially entangled four-qubit W-type state is also unity.

3. TELEPORTATION USING N-QUBIT W-TYPE STATES

In the preceding section, we successfully demonstrated the efficient quantum teleportation of a single qubit state using a new class of four-qubit W-type states. We now extend our method to generalize the optimal teleportation protocol using N-qubit W-type states as resources.

For successfully teleporting a single qubit $|\phi\rangle_a$ to Bob, Alice needs to share an N-qubit W-type state

$$\begin{aligned} |\Psi_k\rangle_{12\ldots N} &= \\ &= \frac{1}{\sqrt{(N-2)(2k+N-3)+2}}\Big[|100\ldots N\rangle_{12\ldots N} + \sqrt{k}e^{i\gamma}|010\ldots N\rangle_{12\ldots N} + \sqrt{k+1}e^{i\delta}|001\ldots N\rangle_{12\ldots N} + \\ &+ \ldots + \sqrt{k+(N-3)}e^{i\zeta}|000\ldots 10\rangle_{12\ldots N} + \sqrt{(N-2)k + \frac{(N-2)(N-3)}{2} + 1}e^{i\beta} \times |000\ldots 1\rangle_{12\ldots N}\Big] \end{aligned} \quad (5)$$

with Bob such that qubits 1 to $N-1$ are with Alice and qubit N is with Bob. In this case, the projection basis used by Alice is

$$
\begin{aligned}
&|\eta_k\rangle^{\pm}_{a,1,2\ldots,N-1} = \\
&= \frac{1}{\sqrt{(N-2)(2k+N-3)+2}}\,[|010\ldots N\rangle + \\
&+ \sqrt{k}e^{i\gamma}\,|001\ldots N\rangle + \sqrt{k+1}e^{i\delta}\,|0001\ldots N\rangle + \\
&+ \ldots \sqrt{k+(N-3)}e^{i\zeta}\,|000\ldots 1\rangle \pm \\
&\pm \sqrt{(N-2)k + \frac{(N-2)(N-3)}{2} + 1}e^{i\beta} \times \\
&\times |100\ldots 0\rangle]_{a,1,2\ldots,N-1}, \\
&|\xi_k\rangle^{\pm}_{a,1,2\ldots,N-1} = \\
&= \frac{1}{\sqrt{(N-2)(2k+N-3)+2}}\,[|110\ldots N\rangle + \\
&+ \sqrt{k}e^{i\gamma}\,|101\ldots N\rangle + \sqrt{k+1}e^{i\delta}\,|1001\ldots N\rangle + \\
&+ \ldots \sqrt{k+(N-3)}e^{i\zeta}\,|100\ldots 1\rangle \pm \\
&\pm \sqrt{(N-2)k + \frac{(N-2)(N-3)}{2} + 1}e^{i\beta} \times \\
&\times |000\ldots 0\rangle]_{a,1,2\ldots,N-1}.
\end{aligned}
\tag{6}
$$

Similarly to the teleportation protocol discussed in the preceding section, we can express the joint state of $N+1$ qubits in terms of Alice's projection basis as

$$
\begin{aligned}
|\Phi\rangle_{a12\ldots N} &= |\phi\rangle_a \otimes |\Psi_k\rangle_{123\ldots N} = \\
&= \frac{1}{2}\Big[|\eta_k\rangle^{+}_{a12\ldots N-1}\,|\phi\rangle_N + \\
&+ |\eta_k\rangle^{-}_{a12\ldots N-1}\,\sigma_z\,|\phi\rangle_N + \\
&+ |\xi_k\rangle^{+}_{a12\ldots N-1}\,\sigma_x\,|\phi\rangle_N + \\
&+ |\xi_k\rangle^{-}_{a12..N-1}\,\iota\sigma_y\,|\phi\rangle_N\Big],
\end{aligned}
\tag{7}
$$

where $|\phi\rangle_N = [\alpha\,|0\rangle + \beta\,|1\rangle]_N$. Equation (7) clearly shows that the teleportation protocol is always successful with equal probability of $1/4$ for the four different measurement outcomes of Alice. Therefore, Bob can always recover the original state by performing single-qubit unitary transformations on the state of his qubit, once he receives the two-bit classical message from Alice regarding her measurement outcome.

4. ANALYSIS OF THE EFFICIENCY OF W-TYPE STATES IN TELEPORTATION PROCESS

We have shown that the N-qubit W-type states can be successfully used as an optimal resource for efficient teleportation. The successful completion of a teleportation protocol depends on the availability of an experimental setup to perform and distinguish multiqubit measurements. It is evident that with the present experimental techniques, one can only perform and distinguish different Bell measurements [11]. Therefore, we analyze the efficiency of our states for a protocol where two users want to create an efficient bipartite entangled channel between them using partially entangled four-qubit W-type states $|\Psi_k\rangle_{1234}$. For this, we assume that Alice initially has a two-qubit entangled state $|\phi\rangle_{ab} = [\alpha\,|00\rangle + \beta\,|11\rangle]_{ab}$ in addition to the shared W-type entangled state

$$
\begin{aligned}
|\Psi_k\rangle_{1234} = \frac{1}{2\sqrt{k+1}}\Big[&|1000\rangle + \sqrt{k}\,|0100\rangle + \\
&+ \sqrt{k+1}\,|0010\rangle + \sqrt{2k+2}\,|0001\rangle\Big]_{1234}
\end{aligned}
\tag{8}
$$

with Bob such that qubits 1, 2, and 3 are with Alice and qubit 4 is with Bob. To share a bipartite entanglement with Bob, Alice needs to perform Bell measurements

$$
\begin{aligned}
|\phi\rangle^{\pm} &= \frac{1}{\sqrt{2}}\,[|00\rangle \pm |11\rangle], \\
|\psi\rangle^{\pm} &= \frac{1}{\sqrt{2}}\,[|01\rangle \pm |10\rangle]
\end{aligned}
\tag{9}
$$

on her qubits. There are different combinations in which Alice can perform these Bell measurements to achieve the required two-qubit entanglement. We have examined all possible combinations and measurement outcomes, and we here discuss only four optimal cases where the concurrence of finally shared two-qubit entangled state is optimal and efficient. We now proceed to analyze the efficiency of the protocol in terms of the concurrence of the finally shared entangled state.

Case 1: In the first case, Alice's measurement outcomes are $|\phi^+\rangle_{b1}$ and $|\phi^+\rangle_{23}$. Therefore, the joint state of two qubits shared between Alice and Bob can be represented as

$$
\begin{aligned}
|\psi\rangle_{a4} = \frac{1}{\sqrt{(2k+2)\alpha^2+\beta^2}}\Big[&\sqrt{2k+2}\alpha \times \\
&\times |01\rangle_{a4} + \beta\,|10\rangle_{a4}\Big].
\end{aligned}
\tag{10}
$$

The concurrence of $|\psi\rangle_{a4}$ is

$$
C_4^{(1)} = \frac{2\alpha\sqrt{1-\alpha^2}\sqrt{2k+2}}{(2k+1)\alpha^2+1},
\tag{11}
$$

where the subscript of C represents the number of qubits in the initially shared W-type states and the superscript represents different cases. Equation (11) clearly demonstrates that for any given real positive number k, if $|\alpha|^2$ is varied from 0 to 1, then the concurrence first increases and then decreases to a minimum value. Interestingly, for $\alpha^2 = 1/(2k+3)$, the concurrence of the shared entangled state is equal to unity, i. e., Alice and Bob can share a maximally entangled state. The finally shared optimally entangled state can therefore be used for various information processing protocols. This can be useful in scenarios where the users in a communication protocol only have access to partially entangled multiqubit states. Further, the analysis presented here not only allows the users to create maximum entanglement but also releases the constraints on the experimental setup to perform and distinguish multiqubit measurements.

Case 2: In the second case, Alice's measurement outcomes are $|\phi^+\rangle_{b2}$ and $|\phi^+\rangle_{13}$. Hence, the shared bipartite state and concurrence of this state can be given by

$$|\psi\rangle_{a4} = \frac{1}{\sqrt{(2k+2)\alpha^2 + k\beta^2}} \Big[\sqrt{2k+2}\alpha \times \times |01\rangle_{a4} + \sqrt{k}\beta\,|10\rangle_{a4}\Big] \quad (12)$$

and

$$C_4^{(2)} = \frac{2\alpha\sqrt{1-\alpha^2}\sqrt{2k+2}\sqrt{k}}{(k+2)\alpha^2 + k}. \quad (13)$$

Similarly to the first case discussed above, the concurrence of the shared state first increases, attains a maximum, and then decreases to 0 for any k and $0 < \alpha < 1$. Further, for $\alpha^2 = k/(3k+2)$, the concurrence of the shared state is unity.

Case 3: The third case provides another interesting observation that for Alice's measurement outcomes $|\phi^+\rangle_{b3}$ and $|\phi^+\rangle_{12}$, the concurrence of a shared bipartite state is independent of the parameter k. In this scenario, the shared bipartite state and its concurrence are given by

$$|\psi\rangle_{a4} = \frac{1}{\sqrt{(2k+2)\alpha^2 + (k+1)\beta^2}} \times \times \Big[\sqrt{2k+2}\alpha\,|01\rangle_{a4} + \sqrt{k+1}\beta\,|10\rangle_{a4}\Big] \quad (14)$$

and

$$C_4^{(3)} = \frac{2\sqrt{2}\alpha\sqrt{1-\alpha^2}}{\alpha^2+1}. \quad (15)$$

The concurrence given in Eq. (15) attains its maximum for $\alpha^2 = 1/3$.

Case 4: The fourth case, the one where Alice's measurement outcomes are $|\phi^+\rangle_{a1}$ and $|\phi^+\rangle_{b2}$, is even more interesting because the concurrence of the finally shared bipartite state is independent of both the parameters k and α. In this scenario, the shared bipartite state and its concurrence are

$$|\psi\rangle_{34} = \frac{1}{\sqrt{3k+3}} \Big[\sqrt{2k+2}\,|01\rangle_{34} + + \sqrt{k+1}\,|10\rangle_{34}\Big] \quad (16)$$

and

$$C_4^{(4)} = \frac{2\sqrt{2}}{3}. \quad (17)$$

Clearly, the concurrence given in Eq. (17) does not depend on the parameters of input states.

Figure 1 compares the concurrence of the initial state and the concurrence of the above four cases to analyze the efficiency of the finally shared bipartite state. For $0 < \alpha^2 \le \sqrt{2}-1$ and $2/3 \le \alpha^2 < 1$, the efficiency of the finally shared state is better than the efficiency of the initial bipartite state in terms of concurrence. For $k = 1$, the concurrence in cases 1 and 2 is the same. Similarly, for large k, case 2 and case 3 lead to identical results. Moreover, Fig. 1 also shows a relation between α^2 and a combination of Bell measurements to be performed to achieve the optimal concurrence.

A similar calculation for a shared N-qubit partially entangled state shows that the concurrence of finally shared states dependent on the input parameters can be given as

$$C = \frac{2\alpha\beta\sqrt{k+r}\sqrt{(N-2)k + \frac{(N-2)(N-3)}{2} + 1}}{\left((N-2)k + \frac{(N-2)(N-3)}{2} + 1\right)\alpha^2 + (k+r)\beta^2}, \quad (18)$$

where r is a variable that takes values from 0 to $N-3$ or $1-k$. Equation (18) clearly shows that for $r = 1-k$, entanglement of the finally shared state between Alice and Bob depends on the input state parameters α and k. For $k \to \infty$, the concurrence is given by

$$C = \frac{2\alpha\sqrt{1-\alpha^2}\sqrt{N-2}}{(N-3)\alpha^2 + 1}. \quad (19)$$

Hence, for a given range of α, if k is very large, then the W-type states with a smaller number of qubits is a better resource.

Similarly, the concurrence of final states independent of the input parameters can be expressed as

$$C = \frac{2\sqrt{k+r}\sqrt{(N-2)k + \frac{(N-2)(N-3)}{2} + 1}}{(N-1)k + \frac{(N-2)(N-3)}{2} + 1 + r}. \quad (20)$$

As previously, if k is very large, then the W-type states with a smaller number of qubits is a better resource.

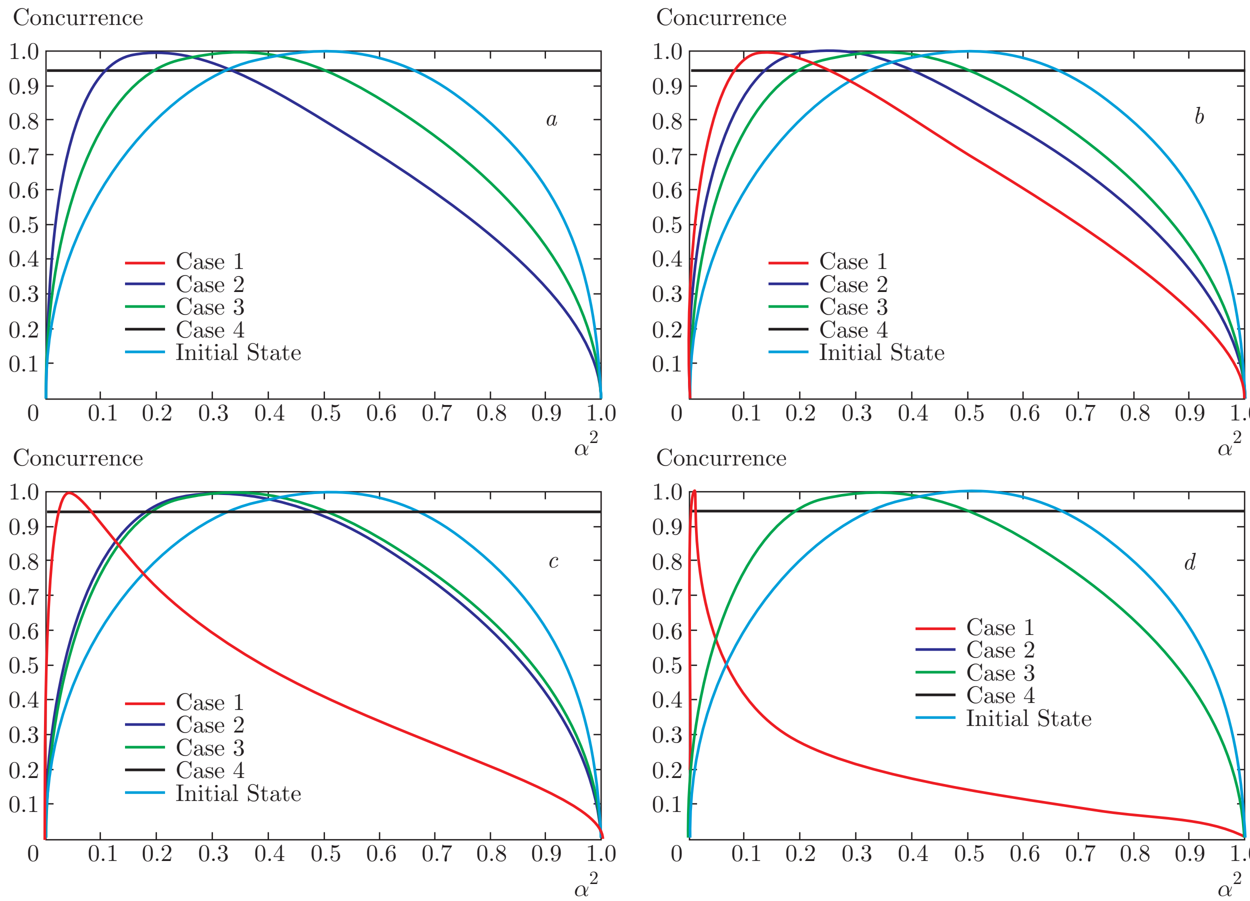

Fig. 1. (Color online) Comparison of the efficiency of shared bipartite states and the initial state for $K = 1$ (a) (Case 1 and Case 2 are superimposed on each other); $K = 2$ (b); $K = 10$ (c); $K = 100$ (d) (Case 2 and Case 3 are approximately superimposed on each other)

In order to analyze the usefulness of four-qubit W-type states for such a protocol, we further compare the efficiency of three- and four-qubit W-type states as resources in terms of concurrence of the finally shared entangled state. We made an interesting observation that for certain ranges of α^2, the four-qubit W-type states are more efficient resources in comparison with three-qubit W-type states for achieving the optimal concurrence shared between two users. For this, we first give the form of three-qubit W-type states as

$$|\Psi_k\rangle_{123} = \frac{1}{\sqrt{2k+2}}\Big[|100\rangle + \sqrt{k}\,|010\rangle + + \sqrt{k+1}\,|001\rangle\Big]_{123}. \quad (21)$$

Similarly to the four-qubit case, there are optimal cases for which the concurrences of finally shared states can be given as

$$C_3^{(1)} = \frac{2\alpha\sqrt{(1-\alpha^2)}\sqrt{k+1}}{(k)\alpha^2+1} \quad (22)$$

and

$$C_3^{(2)} = \frac{2\alpha\sqrt{k(k+1)(1-\alpha^2)}}{\alpha^2+k}. \quad (23)$$

In above two cases, the optimal concurrence of finally shared entangled states is dependent on the input state. But similarly to the four-qubit case, for three-qubit case there also exists an optimal case in which concurrence of the finally shared state is independent of the input state, i. e.

$$C_3^{(3)} = \frac{2\sqrt{k+1}}{(k+2)}. \quad (24)$$

Figure 2 clearly demonstrates the comparison between the efficiencies of three- and four-qubit W-type states in terms of the concurrence of the finally shared bipartite state. Depending on the values of the parameter k, we identify four different cases.

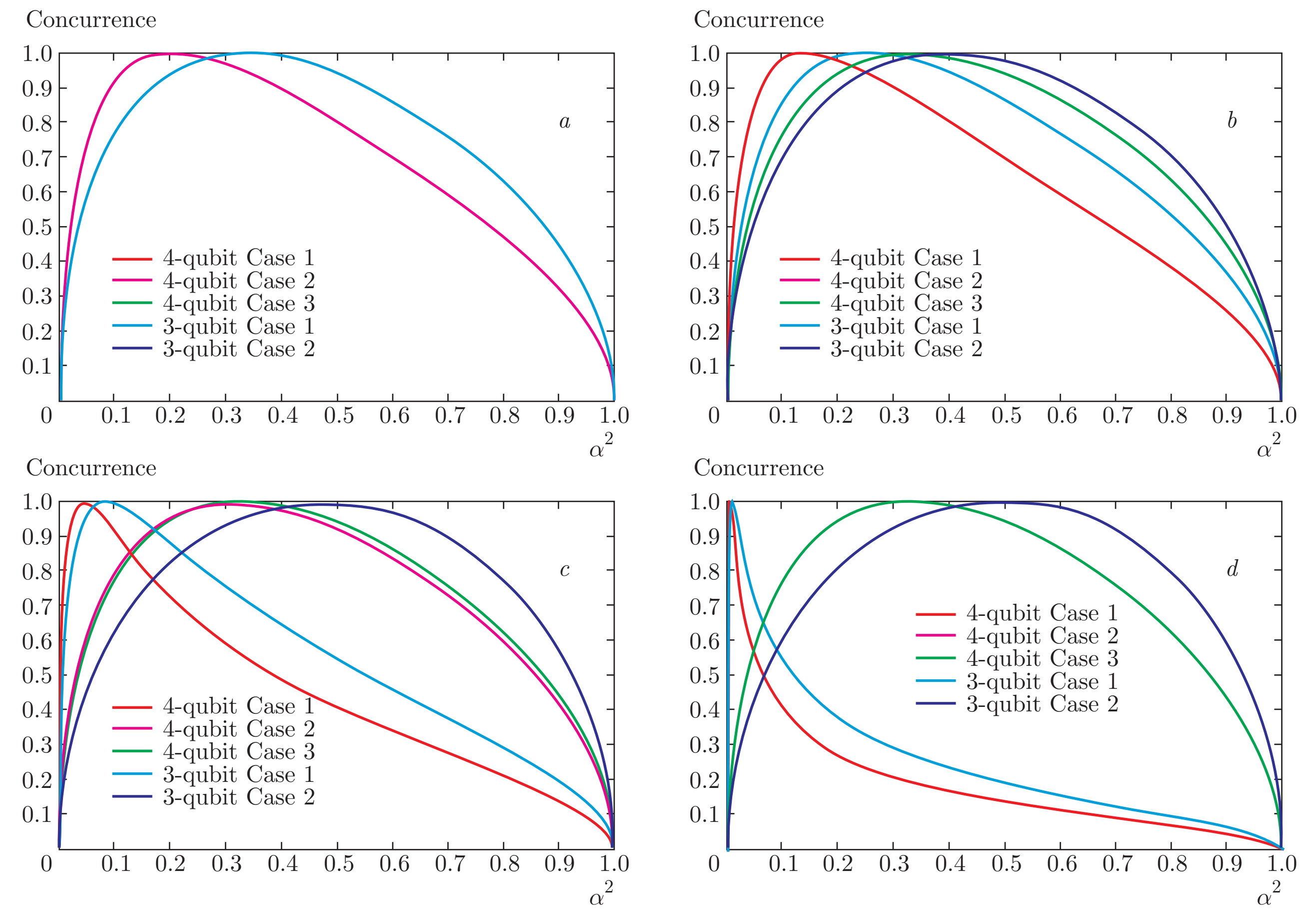

Fig. 2. Comparison of the efficiency of three- and four-qubit W-type states as resources for $K = 1$ (a) (Case 1 and Case 2 for four-qubit, and Case 3 for four-qubit and Case 1 and Case 2 for three-qubit are superimposed on each other); $K = 2$ (b) (Case 1 of three-qubit and Case 2 of four-qubit W-type states are superimposed on each other); $K = 10$ (c); $K = 100$ (d) (Case 2 and Case 3 of the four-qubit W-type state are approximately superimposed on each other)

Case 1: For $k = 1$, if

$$0 < \alpha^2 \leq \frac{k(\sqrt{2}-1)}{((k+2)-\sqrt{2})},$$

then the four-qubit W-type state is a better resource in comparison to the three-qubit W-type state, and otherwise both are equally efficient.

Case 2: For $k = 2$,

- **Range 1**: If

$$0 < \alpha^2 \leq \frac{\sqrt{2}-1}{(2-\sqrt{2})k+1},$$

then the four-qubit W-type state is a better resource than the three-qubit W-type state.

- **Range 2**: If

$$\frac{\sqrt{2}-1}{(2-\sqrt{2})k+1} < \alpha^2 \leq \frac{\sqrt{k+1}-\sqrt{2}}{k\sqrt{2}-\sqrt{k+1}},$$

then the three-qubit W-type state is a better resource than the four-qubit W-type state.

- **Range 3**: If

$$\frac{\sqrt{k+1}-\sqrt{2}}{k\sqrt{2}-\sqrt{k+1}} < \alpha^2 \leq \frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}},$$

then the four-qubit W-type state is a better resource than the three-qubit W-type state.

- **Range 4**: If

$$\frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}} < \alpha^2 < 1,$$

then the three-qubit W-type state is a better resource than the four-qubit W-type state.

Case 3: For $k > 2$,

- **Range 1**: If

$$0 < \alpha^2 \leq \frac{\sqrt{2}-1}{(2-\sqrt{2})k+1},$$

then the four-qubit W-type state is a better resource than the three-qubit W-type state.

• **Range 2**: If

$$\frac{\sqrt{2}-1}{(2-\sqrt{2})k+1} < \alpha^2 \le \frac{k-\sqrt{2k}}{\sqrt{2k}k-(k+2)},$$

then the three-qubit W-type state is a better resource than the four-qubit W-type state.

• **Range 3**: If

$$\frac{k-\sqrt{2k}}{\sqrt{2k}k-(k+2)} < \alpha^2 \le \frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}},$$

then the four-qubit W-type state is a better resource than the three-qubit W-type state.

• **Range 4**: If

$$\frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}} < \alpha^2 < 1,$$

then the three-qubit W-type state is a better resource than the four-qubit W-type state.

Case 4: When k is very large,

• **Range 1**: If

$$0 < \alpha^2 \le \frac{\sqrt{2}-1}{(2-\sqrt{2})k+1},$$

then the foue-qubit W-type state is a better resource than the three-qubit W-type state.

• **Range 2**: If

$$\frac{\sqrt{2}-1}{(2-\sqrt{2})k+1} < \alpha^2 \le \frac{\sqrt{k+1}-\sqrt{2}}{\sqrt{2}k-\sqrt{k+1}},$$

then the three-qubit W-type state is a better resource than the four-qubit W-type state.

• **Range 3**: If

$$\frac{\sqrt{k+1}-\sqrt{2}}{\sqrt{2}k-\sqrt{k+1}} < \alpha^2 \le \frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}},$$

then the four-qubit W-type state is a better resource than the three-qubit W-type state.

• **Range 4**: If

$$\frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}} < \alpha^2 < 1,$$

then the three-qubit W-type state is a better resource than the four-qubit W-type state.

Hence, for practical implementation of an efficient bipartite state sharing protocol, we can choose W-type states as resources according to the range of parameters α^2 and k.

For numerical estimation of the physical efficiency of three- and four-qubit W_k states, we can analyse the ranges given above to understand the efficiency of such states in terms of concurrence of finally shared states. Alternately, Table shows the range of parameters for $k = 1$, $k = 2$, $k = 5$, $k = 10$, and $k = 100$ to compare the efficiency of three- and four-qubit W-type states. For example, for $k = 1$, four-qubit states are better resources that three-qubit states for $\alpha^2 \le 0.261$, and for $\alpha^2 > 0.261$ we can choose either of the states as a starting shared resource. Thus, for $k = 1$, four-qubit W_1 states can always be used as a resource either for a better efficiency or for the same efficiency in comparison to three-qubit W_1 states. Similarly, one can find conclusions for other values of k as well.

5. SUPERDENSE CODING USING N-QUBIT W-TYPE STATES

Superdense coding deals with efficient information transfer between the users in a communication protocol using a shared entangled resource. We use

$$|\eta_1\rangle^+_{1234} = \frac{1}{2\sqrt{2}}\Big[|0100\rangle + |0010\rangle + \\ + \sqrt{2}\,|0001\rangle + 2\,|1000\rangle\Big]_{1234} \quad (25)$$

as a shared resource for a superdense coding protocol between Alice and Bob such that the first qubit is with Alice and the rest of the qubits are with Bob. In order to communicate the classical message to Bob, Alice first encodes her message using one of the four single qubit operations $I, \sigma_x, \sigma_y, \sigma_z$ on her qubit 1. The four operations map the originally shared state between Alice and Bob to four orthogonal states

$$\begin{aligned} (\sigma_x \otimes I \otimes I \otimes I)\,|\eta_1\rangle^+_{1234} &= |\xi_1\rangle^+_{1234}\,, \\ (\sigma_z \otimes I \otimes I \otimes I)\,|\eta_1\rangle^+_{1234} &= |\eta_1\rangle^-_{1234}\,, \\ (i\sigma_y \otimes I \otimes I \otimes I)\,|\eta_1\rangle^+_{1234} &= |\xi_1\rangle^-_{1234}\,, \\ (I \otimes I \otimes I \otimes I)\,|\eta_1\rangle^+_{1234} &= |\eta_1\rangle^+_{1234}\,. \end{aligned} \quad (26)$$

Thus, in principle, Alice can prepare four distinct messages for Bob by locally manipulating her qubit. Once Alice encodes the message, she sends her qubit to Bob. In order to distinguish between the messages sent by Alice, Bob can always perform an appropriate joint measurement on the state of four qubits. Hence, Bob is always able to distinguish between the four messages produced by Alice. The protocol is optimal because by locally manipulating her single qubit, Alice can transmit two bits of classical message to Bob.

Table. Numerical estimation of comparison of three- and four-qubit W_k states

k	α^2	Concurrence of the finally shared two-qubit state using an initially shared three-qubit W_k state as a resource Case			Concurrence of the finally shared two-qubit state using an initially shared four-qubit W_k state as a resource Case			
		(Case 1)	(Case 2)	(Maximum)	(Case 1)	(Case 2)	(Case 3)	(Maximum)
$k=1$	0	0	0	0	0	0	0	0
	0.1	0.771	0.771	0.771	0.923	0.923	0.771	0.923
	0.2	0.943	0.943	0.943	1.000	1.000	0.943	1.000
	0.3	0.997	0.997	0.997	0.965	0.965	0.997	0.997
	0.4	0.990	0.990	0.990	0.891	0.891	0.990	0.990
	0.5	0.943	0.943	0.943	0.800	0.800	0.943	0.943
	0.6	0.866	0.866	0.866	0.700	0.700	0.866	0.866
	0.7	0.762	0.762	0.762	0.591	0.591	0.762	0.762
	0.8	0.629	0.629	0.629	0.471	0.471	0.629	0.629
	0.9	0.447	0.447	0.447	0.324	0.324	0.447	0.447
	1	0	0	0	0	0	0	0
$k=2$	0	0	0	0	0	0	0	0
	0.1	0.866	0.700	0.866	0.980	0.866	0.771	0.980
	0.2	0.990	0.891	0.990	0.980	0.990	0.943	0.990
	0.3	0.992	0.976	0.992	0.898	0.992	0.997	0.997
	0.4	0.943	1.000	1.000	0.800	0.943	0.990	0.990
	0.5	0.866	0.980	0.980	0.700	0.866	0.943	0.943
	0.6	0.771	0.923	0.923	0.600	0.771	0.866	0.866
	0.7	0.661	0.831	0.831	0.499	0.661	0.762	0.762
	0.8	0.533	0.700	0.700	0.392	0.533	0.629	0.629
	0.9	0.371	0.507	0.507	0.267	0.371	0.447	0.447
	1	0	0	0	0	0	0	0
$k=5$	0	0	0	0	0	0	0	0
	0.1	0.980	0.644	0.980	0.990	0.815	0.771	0.990
	0.2	0.980	0.843	0.980	0.866	0.968	0.943	0.968
	0.3	0.898	0.947	0.947	0.738	1.000	0.997	1.000
	0.4	0.800	0.994	0.994	0.629	0.973	0.990	0.990
	0.5	0.700	0.996	0.996	0.533	0.911	0.943	0.943
	0.6	0.600	0.958	0.958	0.447	0.825	0.866	0.866
	0.7	0.499	0.881	0.881	0.365	0.717	0.762	0.762
	0.8	0.392	0.755	0.755	0.283	0.585	0.629	0.629
	0.9	0.267	0.557	0.557	0.191	0.411	0.447	0.447
	1	0	0	0	0	0	0	0

Table. Continuation

k	α^2	Concurrence of the finally shared two-qubit state using an initially shared three-qubit W_k state as a resource Case			Concurrence of the finally shared two-qubit state using an initially shared four-qubit W_k state as a resource Case			
		(Case 1)	(Case 2)	(Maximum)	(Case 1)	(Case 2)	(Case 3)	(Maximum)
$k=10$	0	0	0	0	0	0	0	0
	0.1	0.995	0.623	0.995	0.908	0.795	0.771	0.908
	0.2	0.884	0.823	0.884	0.722	0.957	0.943	0.957
	0.3	0.760	0.933	0.933	0.589	1.000	0.997	1.000
	0.4	0.650	0.988	0.988	0.489	0.982	0.990	0.990
	0.5	0.553	0.999	0.999	0.408	0.927	0.943	0.943
	0.6	0.464	0.969	0.969	0.338	0.845	0.866	0.866
	0.7	0.380	0.898	0.898	0.274	0.739	0.762	0.762
	0.8	0.295	0.777	0.777	0.211	0.605	0.629	0.629
	0.9	0.199	0.577	0.577	0.141	0.428	0.447	0.447
	1	0	0	0	0	0	0	0
$k=100$	0	0	0	0	0	0	0	0
	0.1	0.548	0.602	0.602	0.404	0.774	0.771	0.774
	0.2	0.383	0.802	0.802	0.276	0.944	0.943	0.944
	0.3	0.297	0.918	0.918	0.212	0.997	0.997	0.997
	0.4	0.240	0.981	0.981	0.171	0.989	0.990	0.990
	0.5	0.197	1.000	1.000	0.140	0.941	0.943	0.943
	0.6	0.161	0.979	0.979	0.115	0.864	0.866	0.866
	0.7	0.130	0.915	0.915	0.092	0.760	0.762	0.762
	0.8	0.099	0.798	0.798	0.070	0.626	0.629	0.629
	0.9	0.066	0.598	0.598	0.047	0.445	0.447	0.447
	1	0	0	0	0	0	0	0

We now proceed to demonstrate the optimal dense coding protocol using our N-qubit W-type states

$$|\eta_k\rangle^+_{12\ldots N} = \\ = \frac{1}{\sqrt{(N-2)(2k+N-3)+2}}\,[|010\ldots N\rangle + \\ + \sqrt{k}\,|001\ldots N\rangle + \sqrt{k+1}\,|0001\ldots N\rangle + \\ + \ldots\sqrt{k+(N-3)}\,|000\ldots 1\rangle + \\ + \sqrt{(N-2)k+\frac{(N-2)(N-3)}{2}+1}\,\times \\ \times\,|100\ldots 0\rangle]_{12\ldots N}\,, \quad (27)$$

where qubit 1 is with Alice and rest of the qubits are with Bob. Similarly to the four-qubit case, Alice can produce four distinct messages to Bob using single-qubit unitary transformations I, σ_x, σ_y, σ_z such that

$$\begin{aligned} (I\otimes I\otimes I\otimes I)\,|\eta_k\rangle^+_{12\ldots N} &= |\eta_k\rangle^+_{12\ldots N}\,, \\ (\sigma_x\otimes I\otimes I\otimes I)\,|\eta_k\rangle^+_{12\ldots N} &= |\xi_k\rangle^+_{12\ldots N}\,, \\ (\sigma_z\otimes I\otimes I\otimes I)\,|\eta_k\rangle^+_{12\ldots N} &= |\eta_k\rangle^-_{12\ldots N}\,, \\ (i\sigma_y\otimes I\otimes I\otimes I)\,|\eta_k\rangle^+_{12\ldots N} &= |\xi_k\rangle^-_{12\ldots N}\,. \end{aligned} \quad (28)$$

Therefore, our N-qubit W-type states can also be used for the optimal super dense coding protocol.

6. EXPERIMENTAL GENERATION OF W_k STATES

In the previous sections, we have demonstrated the usefulness of four-qubit W-type states for quantum information processing. Considering the importance of four-qubit W-type states, it is imperative to propose a method for experimental realization of such states. We now proceed to discuss a method for experimental realization of W-type states represented in Eq. (1).

The standard three-qubit W-type states have been experimentally realized using spontaneous parametric down conversion [12]. The fundamental and theoretical framework to analyse the properties of the W class of states have allowed the experimental realization of the W class of states to become an area of extensive research [13–15]. Recently, Wu et al. [16] described the experimental generation of tripartite entangled polarization states using Stokes operators, and Dong et al. [17] proposed the experimental preparation of three-qubit W-type states originally proposed by Pati and Agrawal. In this section, we use the three-qubit W-type states generated in [17] as input to prepare the four-qubit W-type states. Our procedure also involves the use of single- and two-qubit quantum gates [18–22] along with weak measurements [23,24], if required. It is important to mention that the gates and measurements used here can also be realized experimentally. We first define the three-qubit W state prepared in [17]

$$|\Psi_k\rangle_{123} = \frac{1}{\sqrt{2+2k}}\Big[|100\rangle + \sqrt{k}e^{i\gamma}|010\rangle + \sqrt{k+1}e^{i\delta}|001\rangle\Big]_{123}. \quad (29)$$

In the simplest case where all the phases are 0 and $k=1$, we have

$$|\Psi_1\rangle_{123} = \frac{1}{2}\Big[|100\rangle + |010\rangle + \sqrt{2}|001\rangle\Big]_{123}. \quad (30)$$

We use the state $|\Psi_1\rangle_{123}$ for the preparation of the four-qubit W state $|\Psi_1\rangle_{1234}$ given by Eq. (1) for $k=1$. For this, we write

$$|\psi_1\rangle_{1234} = |0\rangle_1 \otimes \frac{1}{2}\Big[|100\rangle + |010\rangle + \sqrt{2}|001\rangle\Big]_{234}. \quad (31)$$

To generate the four-qubit W-type state, we first perform a controlled Hadamard operation on qubits 1 and 2, considering qubit 2 as the controlled qubit such that

$$|\psi_1'\rangle_{1234} = \frac{1}{2\sqrt{2}}\Big[|0100\rangle + |1100\rangle + \sqrt{2}|0010\rangle + 2|0001\rangle\Big]_{1234}. \quad (32)$$

We can now perform a C-NOT operation on qubits 1 and 2 by keeping qubit 1 as the control qubit to obtain the four-qubit W-type state as

$$|\Psi_1\rangle_{1234} = \frac{1}{2\sqrt{2}}\Big[|1000\rangle + |0100\rangle + \sqrt{2}|0010\rangle + 2|0001\rangle\Big]_{1234}. \quad (33)$$

The experimental generation of the generalized four-qubit state requires the input state to be the direct product of a single-qubit state and a three-qubit W-type state expressed in Eq. (29), and hence the joint state of four qubits can be expressed as

$$|\phi_k\rangle_{1234} = |\Psi_k\rangle_{123} \otimes |0\rangle_4 = \frac{1}{\sqrt{2+2k}}\Big[|1000\rangle + \sqrt{k}e^{i\gamma}|0100\rangle + \sqrt{k+1}e^{i\delta}|0010\rangle\Big]_{1234}. \quad (34)$$

On qubits 3 and 4, we perform controlled Hadamard operation keeping qubit 3 as the control to obtain

$$|\phi_k'\rangle_{1234} = \frac{1}{2\sqrt{2+2k}}\Big[2|1000\rangle + 2\sqrt{k}e^{i\gamma}|0100\rangle + \sqrt{2}\sqrt{k+1}e^{i\delta}|0010\rangle + \sqrt{2}\sqrt{k+1}e^{i\delta}|0011\rangle\Big]_{1234}. \quad (35)$$

We now perform a C-NOT operation on qubits 3 and 4 by considering qubit 4 as the control qubit to obtain

$$|\phi_k''\rangle_{1234} = \frac{1}{2\sqrt{2+2k}}\Big[2|1000\rangle + 2\sqrt{k}e^{i\gamma}|0100\rangle + \sqrt{2}\sqrt{k+1}e^{i\delta}|0010\rangle + \sqrt{2}\sqrt{k+1}e^{i\delta}|0001\rangle\Big]_{1234}. \quad (36)$$

We further perform weak measurements

$$M = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1/4} \end{bmatrix}$$

on qubits 1 and 2 and

$$M' = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1/2} \end{bmatrix}$$

on qubit 3 such that $|\phi_k''\rangle_{1234}$ becomes

$$|\Psi_k'\rangle_{1234} = \frac{1}{2\sqrt{k+1}}\Big[|1000\rangle + \sqrt{k}e^{i\gamma}|0100\rangle + \sqrt{k+1}e^{i\delta}|0010\rangle + \sqrt{2k+2}e^{i\delta}|0001\rangle\Big]_{1234}. \quad (37)$$

The state in Eq. (37) is the same as the state in Eq. (1) except for the equal phase factors in the third and the fourth terms. For this, we finally perform a simple unitary operation

$$U = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

on qubit 4. Therefore, we have

$$|\Psi_k\rangle_{1234} = \frac{1}{2\sqrt{k+1}} \Big[|1000\rangle + \sqrt{k} e^{i\gamma} |0100\rangle + \\ + \sqrt{k+1} e^{i\delta} |0010\rangle + \sqrt{2k+2} e^{i\zeta} |0001\rangle \Big]_{1234}.$$

7. CONCLUSION

We have analyzed a class of partially entangled four-qubit W-type states for efficient quantum information processing tasks. Although performing and distinguishing multiqubit measurements is an uphill task, our states can nevertheless be used for deterministic teleportation with unit fidelity. To demonstrate the practical utility of such states, we have discussed and compared the efficiency of three- and four-qubit W-type states for sharing optimal bipartite entanglement between two users. Furthermore, we have also proposed an experimental realization of four-qubit W_k states, which increases the importance of the results obtained in this study. Our results will be of high importance in situations where users only have access to partially entangled states and would like to establish optimal bipartite entanglement for efficient and deterministic information processing. The analytic relations between the range of state parameters and optimal concurrence of the finally shared state are also obtained, allowing one to decide when to use a three- or a four-qubit W-type state for a particular protocol. We have also shown that our states can also be used for optimal dense coding. The protocols have also been generalized for the case of N qubits.

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