ЖЭТФ, 2016, том 149, вып. 6, стр. 1107–1109

VORTEX RINGS IN BOSE GAS

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Received March 31, 2015

We consider excitations that exist, in addition to phonons, in the ideal Bose gas at zero temperature. These excitations are vortex rings whose energy spectrum is similar to the roton one in liquid helium.

DOI: 10.7868/S0044451016060018

1. INTRODUCTION

Superfluid helium-4 is a subject of intensive experimental and theoretical interest since the pionering publications of Landau and Khalatnikov. The low-temperature properties of this quantum liquid can be understood by introduction of phonons and rotons as elementary excitations. The corresponding energy–momentum dependence is presented by the well-known Landau curve that starts with the linear phonon dispersion, reaches the maximum, and then goes down to the roton minimum. Roton–phonon interactions in liquid He-4 are still the subject of ongoing investigations (see, e. g., [1, 2]).

In contrast to He-4, the Bose gas at zero temperature is a simpler subject, open for more detailed descriptions. All particles of an ideal Bose gas at zero temperature are condensed in the lowest energy level, the "condensate". With the switch of a weak repulsive interaction, some particles are pushed above the condensate and are distributed over the higher levels. The structure of this ground state and elementary excitations above it were considered long ago in two articles by the author [3]. Elementary excitations in this system are phonons with the dispersion curve that bends upward (contrary to the Landau curve). Such phonons are not stable and can split into pairs of new phonons. This behavior of phonons was confirmed later in the BEC experiments [4–6].

In the interacting system characterized by the scattering amplitude f_0, the elementary excitations with not very high momenta p are similar to phonons with a correction to the sound velocity u,

$$\epsilon_{\mathbf{p}} = pu\left(1 + \frac{7}{6\pi^2}\sqrt{n_0 f_0^3}\right) \equiv p\tilde{u}, \tag{1}$$

where n_0 is the condensate density.

The question arises if there are other types of elementary excitations, say, vortex rings, which in a gas may be similar to smoke rings. Some results needed for our goal are presented below.

2. VORTEX RINGS

The classical description of such vortex rings is well known and can be found in popular literature (for instance, in Vol. 2 of Feynman's Lectures on Physics). Characteristics of vortex rings are derived in volume IX of the Landau–Lifshitz course of Theoretical Physics (in Problems to Sec. 29). The picture of a roton as the closest quantum mechanical analog of a smoke ring was suggested by Feynman and Cohen [7].

A vortex ring is characterized by the circulation of velocity over the ring, the "vorticity", which in quantum mechanics is quantized as

$$\Gamma = \oint \mathbf{v}_s \cdot d\mathbf{l} = 2\pi\hbar\,\frac{n}{m}, \tag{2}$$

where m is the mass of the gas particles and n is an integer that in our case can be taken as unity.

The magnitude of velocity at small distances r from the ring line (as compared to the ring size R) simply follows from Eq. (2):

$$v_s = \frac{\Gamma}{2\pi r} \equiv \frac{\kappa}{r}. \tag{3}$$

Obviously, the dependence $v_s(r)$ as in Eq. (3) should have some limit for small r. A natural assumption is

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that the central part of the vortex ring has a core of the atomic radius a with the velocity circulation around this core as in Eq. (3) for $r \geq a$ and the stream $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$ inside this core.

The general distribution of velocities due to the vortex ring is given by

$$\mathbf{v} = \frac{\kappa}{2} \int \frac{d\mathbf{l} \times \mathbf{r}}{r^3}, \tag{4}$$

where $\mathbf{r}$ is the radius vector from the integration point on the vortex line to the point of velocity observation. With the use of Eq. (4), we easily find the velocity $\mathbf{v}_0$ at the central point of the ring.

At each integration point, $d\mathbf{l}$ and $\mathbf{r}$ are orthogonal and, for the ring of radius R, the integral gives $2\pi/R$. As a result, the velocity at the central point of the ring is

$$\mathbf{v}_0 = \frac{\pi\kappa}{R} = \frac{\Gamma}{2R}, \tag{5}$$

which indicates that smaller size leads to higher speed.

We should keep in mind that the velocity found in Eq. (5) does not coincide with the velocity of the ring. To find the latter, we should place $\mathbf{r}$ in Eq. (4) to a point on the ring and integrate in Eq. (4) over all other points on the ring. As a result, we obtain the velocity of the ring (which is perpendicular to the ring plane) as

$$v = \frac{\kappa}{2R} \int\limits_0^\pi \frac{d\vartheta}{\sin(\vartheta/2)}. \tag{6}$$

The integral in Eq. (6) diverges at the lower limit and should be cut off at the size of the core radius a of the vortex ring. This determines the velocity of the ring,

$$v \simeq \frac{\kappa}{2R} \ln \frac{R}{a}, \tag{7}$$

where the factor $\ln(R/a)$ should be considered as a large constant quantity. We note that the velocity of the ring in (7) can be both smaller and larger than the velocity at the center of the ring, Eq. (5).

The energy of the vortex ring can be evaluated as the sum of energies of the circulations around the vortex core line,

$$E \simeq \int \frac{\rho v_s^2}{2}\, dV = \frac{\rho}{2} L \int v_s^2 \cdot 2\pi r\, dr, \tag{8}$$

where ρ is the density of the gas. Using Eq. (3) for the velocity v_s and integrating over the volume of the vortex, we obtain

$$E \simeq \frac{\rho}{2} L \int v_s^2 \cdot 2\pi r\, dr = \frac{\rho}{2} L \int \left(\frac{\kappa}{r}\right)^2 \cdot 2\pi r\, dr \approx \approx \pi\rho\kappa^2 L \int \frac{dr}{r}. \tag{9}$$

The last expression in Eq. (9), with $L = 2\pi R$ and the integral estimated as $\ln(R/a)$, gives the energy of the ring in the form

$$E = 2\pi^2 \rho\kappa^2 R \ln \frac{R}{a}. \tag{10}$$

Finally, the parameters of the ring — the velocity v in Eq. (7), the energy E, the momentum P, and the vorticity Γ — can be found, for a large ratio $\ln(R/a)$, as

$$v = \frac{\Gamma}{4\pi R} \ln \frac{R}{a}, \quad E \approx \frac{1}{2} \rho \Gamma^2 R \ln \frac{R}{a}, \quad P = \pi\rho\Gamma R^2, \quad \Gamma = \frac{2\pi\hbar}{m}. \tag{11}$$

Equations (11) contain known results for vortex rings in classical hydrodynamics with the only quantized parameter, the vorticity Γ.

3. CREATION OF THE RINGS BY PHONONS

The vortex rings are moving objects and, in a closed volume, they disappear on the walls. Some creation mechanism is needed for their presence. To create a ring, one needs a powerful push of gas trough a short round channel. Such a push can be provided by some particle (proton, neutron, nucleus) or by the density wave (phonon). We assume that one phonon can provide the needed push and create a vortex ring.

Let the energy and momentum of the phonon be $(\epsilon_i, \mathbf{p}_i)$, and $(\epsilon_f, \mathbf{p}_f)$ before and after the process, respectively. The energy and momentum conservation conditions in the process of creating the ring with energy E and momentum $\mathbf{P}$ can be written as

$$\epsilon_i - \epsilon_f = E, \quad \mathbf{p}_i - \mathbf{p}_f = \mathbf{P}. \tag{12}$$

Since a particle or a phonon creates a ring in the direction of its velocity, the last equality in Eq. (12) can be considered as a scalar one. Using Eq. (11), we then obtain

$$\frac{\epsilon_i - \epsilon_f}{p_i - p_f} = \frac{E}{P} = \frac{\Gamma}{2\pi a} \frac{a}{R} \ln \frac{R}{a} = \tilde{u}, \tag{13}$$

where $\tilde{u}$ is the velocity of the particle or the phonon (Eq. (1)). As follows from the preceding section, considering $\ln(R/a)$ as a constant, we obtain the following energy–momentum dependence for the rings:

$$E \propto \sqrt{P}. \tag{14}$$

To investigate whether Eq. (13) can be satisfied with real parameters of the rings, we write it as an equation for $x \equiv \ln(R/a)$:

$$\frac{a}{R} \ln \frac{R}{a} = x e^{-x} \approx \frac{2\pi a}{\Gamma} \tilde{u}. \tag{15}$$

The left-hand side of Eq. (15), as a function of x, starts growing from zero and goes through the maximum at $x = 1$. Therefore, Eq. (15) can be satisfied with the right-hand side being less than the maximum of the left-hand side, $1/e$, or

$$\frac{2\pi a}{\Gamma}\tilde{u} \le \frac{1}{e}. \tag{16}$$

The process of ring creation has to satisfy an additional restriction: the phonon can create a ring with the velocity at the center, Eq. (5), not exceeding the velocity of the phonon,

$$v_0 = \frac{\Gamma}{2R} \le \tilde{u}. \tag{17}$$

Inequality (16) does not contain R and leads to a consistency requirement for general physical parameters,

$$\frac{am}{\hbar n}\tilde{u} \le \frac{1}{e}, \tag{18}$$

which can be considered a bound for the core parameter a.

Equation (17) defines the lower bound for the size R of created rings,

$$R \ge \frac{\Gamma}{2\tilde{u}} = \frac{\pi\hbar n}{m\tilde{u}}. \tag{19}$$

The lower bound for the radius of the ring leads to the bounds for its energy and momentum. As follows from Eq. (19), these bounds are

$$E_{min} = \frac{1}{4}\rho\frac{\Gamma^3}{\tilde{u}}\ln\frac{\Gamma}{2\tilde{u}a}, \quad P_{min} = \frac{1}{4}\pi\rho\frac{\Gamma^3}{\tilde{u}^2}. \tag{20}$$

From the point (E_{min}, P_{min}), the function $E(R)$ continues, according to Eq. (4), as $E \propto \sqrt{P}$.

The spectrum of rotons near its minimum is usually approximated by a parabolic Landau curve,

$$E(P) = E(P_{min}) + (P - P_{min})^2/2\mu, \tag{21}$$

where μ is the effective mass parameter. Due to the restrictions in the preceding section, we have considered only the rings that correspond to the upper right branch of this spectrum. We can only conclude that the spectrum of vortex rings does not contradict that of rotons.

4. CONCLUSION

Elementary excitations additional to phonons were found in a Bose gas at zero temperature. They are the vortex rings that have the energy–momentum dispersion law similar to what is assumed for rotons.

The equilibrium condition between phonons and rotons (vortex rings) in the Bose gas depends on phonon–roton and roton–roton scatterings and interaction of rotons with the walls. All these questions need specific considerations, but a coarse picture may be seen as follows: rotons are continuously created by phonons and annihilated with their reflections from the walls of the volume.

The author is thankful for V. Zelevinsky for the fruitful discussions.

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