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KELDYSH FORMALISM FOR MULTIPLE PARALLEL WORLDS

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We present a compact and self-contained review of the recently developed Keldysh formalism for multiple parallel worlds. The formalism has been applied to consistent quantum evaluation of the flows of informational quantities, in particular, to the evaluation of Renyi and Shannon entropy flows. We start with the formulation of the standard and extended Keldysh techniques in a single world in a form convenient for our presentation. We explain the use of Keldysh contours encompassing multiple parallel worlds. In the end, we briefly summarize the concrete results obtained with the method.

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1. INTRODUCTION

The seminal work of Leonid Keldysh [1] has paved the way to the modern understanding of quantum systems out of equilibrium. One can do much work in the area armed just with the Fermi Golden Rule and some defiance, yet a consistently scientific approach would almost necessary involve the Keldysh formalism. The formalism has been successfully applied for derivations of dynamical equations of complex systems where intuition ceases to work, like superconductors [2], strongly correlated systems [3], and nonlinear sigma models [4].

For many years, the formalism was considered too complicated for a practical researcher and hardly applied beyond several specific fields. The "Keldysh approach" sounded as a synonym of unnecessary theorization and an antonym to clear physical reasoning. One of us (Y. N.) remembers a talk given by a high-class theorist with a taste for abstract models, young experimentalists being his primary audience. Somewhere in the middle of the talk he said: "Now let us move to physical quantities, namely, Keldysh Green's functions". A burst of laugh lasted for quite a while.

The situation began to change in the 1990s, and is quite different nowadays. The formalism receives much more practical attention, more and more theorists and numerical researchers become qualified, nice modern reviews [5, 6] have appeared in addition to the classical ones [7]. A unique property of the Keldysh formalism that distinguishes it from all other diagram techniques [8] is that the zeroth-order approximation is generally unstable with respect to perturbative corrections. This property is widely appreciated now and makes the technique an indispensable tool for complex quantum dynamics.

Recent extensions of the Keldysh technique to nonunitary evolution of the density matrix [9, 10] allow accessing nontrivial problems of quantum statistics and analyzing large deviations from equilibrium [11]. A finite-element approach to the Keldysh Green's functions for electrons, so-called quantum circuit theory [9, 12], proved to be useful to build adequate models of quantum nanostructures.

The Keldysh technique permits a natural formulation in terms of path integrals [6], providing a very instructive picture of the "doubling" of a classical stochastic variable when it is put on a Keldysh contour. This provides a fundamental link between the Keldysh and Feynman–Vernon formalism. The Keldysh action arising in this context can be evaluated by blocks, each block being obtained from a nonunitary evolution [13]. The Landauer–Buttiker [14, 15] scattering approach is given a compact and general formulation in terms of the Keldysh action [13, 16]

All these extensions are still based on time evolution along a single "doubled" Keldysh contour. In this paper, we discuss a recent extension in a different direction. Technically, the extension involves time evolu-

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tion along many "doubled" contours. We refer to these pairs of contours as parallel worlds (this terminology has nothing to do with the attempt to interprete quantum mechanics with the use of parallel worlds). The closing of the contours is typically different for different subparts of the quantum system under consideration: for some, the contours are closed separately within each world, while for others they can go back and forth through all the worlds.

As we show below, this formalism is natural and indispensable for evaluating the quantities that are nonlinear in the density matrix. The physical meaning of such quantities is not obvious since they do not conform to the standard definition of a physical observable, although they are commonly used in quantum information theory [17], for instance, for entanglement characterization. Most work and applications have been done for evaluation of Renyi entropies [18–20]. We follow these papers here.

The paper is organized as follows. In Sec. 2, we formulate the standard Keldysh formalism in a way convenient for the further presentation, putting emphasis on the link between the Keldysh technique and the master or Bloch equations. In Sec. 3, we explain the extension of the formalism to nonunitary evolution, mostly concentrating on the example of full counting statistics of energy flows [21], which is useful in the context of Renyi entropy flows. Then we explain the use of the parallel world concept for evaluating the conserved quantities related to the products of density matrices of subparts (Sec. 4) and formulate a diagram technique for this situation in Sec. 5. The relations between different Keldysh correlators for a (sub)system in thermal equilibrium, the so-called Kubo–Martin–Schwinger (KMS) [22] relations, are important for single-world techniques. We discuss their generalization to multiple worlds in Sec. 6.

The rest of the paper is devoted to specific examples in which the general theory can be simplified and elaborated. We concentrate on second-order diagrams in Sec. 7 and explain the specifics of higher-order diagrams in Sec. 8. We briefly review our recent results on quantum heat engines in Sec. 9. In Sec. 10, we discuss a rather general correspondence between the statistics of the energy flows and Renyi entropy flows. We conclude in Sec. 11.

2. STANDARD KELDYSH FORMALISM

We first formulate the standard Keldysh formalism in a way that illustrates its potential and at the same time makes direct connections with the problems to be considered in what follows. The starting point of the formalism is the formal expression for the unitary time evolution of the density matrix $\hat{R}$ of a quantum system governed by a (generally time-dependent) Hamiltonian $\hat{H}(t)$,

$$\hat{R}(t) = \mathrm{Texp}\left(i \int_{t'}^{t} d\tau \hat{H}(\tau) \right) \hat{R}(t') \times \times \tilde{\mathrm{T}}\mathrm{exp}\left(-i \int_{t'}^{t} d\tau \hat{H}(\tau) \right), \quad (1)$$

where Texp ($\tilde{\mathrm{T}}$exp) denotes time(anti)ordering in the evolution exponents.

If we were up to exact quantum evolution of the whole system, we would not need any Keldysh technique: the Schrödinger equation would suffice. At the same time, the resulting density matrix would keep the memory of the initial density matrix for infinite time. This is rather unphysical. To address physical situations, we need to separate quantum variables into relevant and less relevant ones. Quite generally, this can be achieved by bipartition of the Hilbert space: we represent it as a direct product $A \otimes B$ of two subparts A and B. The Hamiltonian decomposes as

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_{AB}, \quad (2)$$

where $\hat{H}_{AB}$ is an operator that involves degrees of freedom in both subspaces, while H_A, H_B act in their respective partitions only.

This opens up the opportunities to treat a great variety of physical situations. For instance, system A can be a small system with a finite number of states while B can be an environment with an infinite number of degrees of freedom. In this case, the density matrix of B can be regarded as unchangeable in the course of evolution, to play the role of a (thermal) reservoir for A: the density matrix R_A would then try to adjust to the reservoirs. Alternatively, B can be a collection of independent reservoirs kept at different conditions (like temperatures and chemical potentials): the system A would then try to adjust to these competing reservoirs providing the flows of physical quantities, e. g., charge or heat, between the reservoirs. Yet another possibility is that A and B are both reservoirs and H_{AB} represents a junction between the two. In this case, both R_A and R_B are unchangeable, while the junction provides the flows to both reservoirs.

We assume that the completely separated systems, whose dynamics are governed by $\hat{H}_A + \hat{H}_B$, form a reasonable zeroth-order approximation and implement

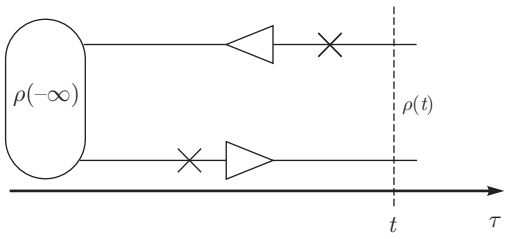

Fig. 1. Perturbation theory for a single density matrix on the Keldysh contour

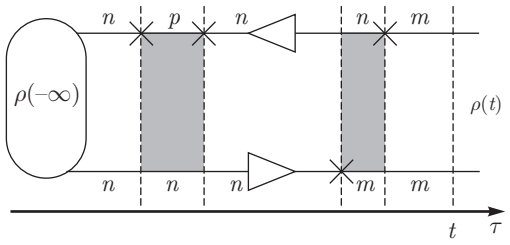

Fig. 2. Master equation in the Keldysh formalism is obtained from a resummation of the perturbation series whereby the time line is separated into diagonal and nondiagonal (grey-shaded) blocks. The state index here encompasses the indices in both subspaces

a perturbation technique in $\hat{H}_{AB}$ keeping the calculations as general as possible. We assume the "adiabatic switching" of the perturbation [23]: in the remote past, the coupling is absent, and the density matrix is a direct product over subspaces A and B,

$$\hat{R}(-\infty) = \hat{R}_A(-\infty) \otimes \hat{R}_B(-\infty),$$

$$\hat{R}_A(-\infty) = \sum_a p_a |a\rangle\langle a|; \quad \hat{R}_B(-\infty) = \sum_\alpha p_\alpha |\alpha\rangle\langle\alpha|.$$

Here, we label the states in subspaces A (B) with Latin (Greek) indexes. The coupling slowly increases achieving actual values at a time long before t. The time evolution of the density matrix is given by

$$\hat{R}(t) = \mathrm{Texp}\left(i \int\limits_{-\infty}^{t} d\tau \hat{H}_{AB}(\tau) \right) \hat{R}(-\infty) \times \\ \times \tilde{\mathrm{T}}\mathrm{exp}\left(-i \int\limits_{-\infty}^{t} d\tau \hat{H}_{AB}(\tau) \right), \quad (3)$$

with $\hat{H}_{AB}(\tau)$ taken in the interaction representation. Expanding this in $H_{AB}(\tau)$ gives a perturbation series most conveniently presented as diagrams involving the Keldysh contour (Fig. 1). The operators in the perturbation series are ordered along the contour. Two parts of the contour correspond to time evolution of bras and kets in the density matrix. The crosses represent the (time-dependent) perturbation $H_{AB}(t)$ at a certain time moment. The integration over time moments of all perturbations is implied. There is a state index associated with each piece of the contour. Since $\hat{R}(-\infty)$ is diagonal, this index does not change when passing this element. The index changes if a nondiagonal matrix element of the perturbation is involved. Summation over the indices is implied.

In contrast to most perturbation theories, the zeroth-order approximation in the Keldysh formalism is not stable with respect to small perturbations. For instance, if A is small and B is a reservoir, $\hat{R}_A$ is determined by the reservoir at an arbitrarily small coupling strength and can have nothing to do with the initial $\hat{R}_A(-\infty)$. This implies that we need to re-sum the perturbation series. In a single world, there is a simple way to re-sum the perturbation series and arrive at a master equation that contains only diagonal elements of density matrix (Fig. 2). For a diagram, we split the time-line by perturbations into the blocks as shown in Fig. 2. The blocks are of two sorts: diagonal ones, that have the same state index on both contours, and nondiagonal ones. To compute a diagram, we need to integrate over time duration of each block. For nondiagonal blocks, the integrand is an oscillatory function of time and the integral has a chance to converge. For nondiagonal blocks, the integrand is a constant, and the integral diverges. This indicates that the diagrams have to be resumed. If we look at the time derivative of the density matrix, it is contributed by the first nondiagonal block. Summation over the subsequent diagonal blocks replaces $\hat{R}(-\infty)$ with the density matrix at the time moment immediately after the first nondiagonal block. With this, the evolution equation for diagonal matrix elements $p_{a\alpha}(t) \equiv R_{a\alpha,a\alpha}$ can be written as (assuming summation over repeated indices)

$$\frac{d}{dt} p_{a\alpha}(t) = \int\limits_0^\infty d\tau W_{a\alpha,b\beta}(\tau) p_{b\beta}(t-\tau) \quad (4)$$

with $W_{a\alpha,b\beta}(\tau)$ being the sum of the perturbation expansion comprising a nondiagonal block that starts from the second order in $\hat{H}_{AB}$. It is natural to require that the matrix elements of H_{AB} are only nondiagonal, that is, $H^{(AB)}_{a\alpha,b\beta} = 0$ if either $a = b$ or $\alpha = \beta$.

If $p_{a\alpha}(t)$ changes slowly in comparison with the typical time scale of the blocks, we can neglect this time dependence in the integrand. The integration over

the time duration of the blocks with different indices gives the transition rates $\Gamma_{a\alpha,b\beta} = \int_0^\infty d\tau\, W_{a\alpha,b\beta}(\tau)$. The unitarity guarantees that the integration over the duration of the blocks with the same indices gives $\int_0^\infty d\tau W_{a\alpha,a\alpha}(\tau) = -\Gamma_{a\alpha,b\beta}$, the total transition rate from the state $|a\alpha\rangle$, which is the sum of the partial transition rates, $\Gamma_{a\alpha} = \sum_{b\beta}\Gamma_{a\alpha,b\beta}$. In this way, we arrive at the master equation in the traditional form

$$\frac{d}{dt}p_{a\alpha} = -\Gamma_{a\alpha}p_{a\alpha} + \Gamma_{a\alpha,b\beta}p_{b\beta}. \tag{5}$$

There are situations where the nondiagonal elements of the density matrix are also relevant for the dynamics. For instance, a relevant subset of quantum states in A can be approximately degenerated such that their energy separations are of the order of the rates Γ, or such degeneracy is provided by a coherent drive with a frequency that cancels the energy separations. A generic example is the quantum heat engine described in Sec. 9. We treat system B as a reservoir for system A and sum over its states assuming the unchanged $\hat{R}_B$. Instead of diagonal blocks, we define the blocks where the states at two parts of the contour belong to the relevant subset for A and the same state for B. After the re-summation over these blocks, the evolution of the density matrix in the relevant subset is given by

$$\frac{d}{dt}\rho_{ab} = i\left(H^r_{ac}\rho_{cb} - \rho_{ac}H^r_{cb}\right) + \\ + \int_0^\infty W_{ab,cd}(\tau)\rho_{cd}(t-\tau), \tag{6}$$

where $\hat{H}^r$ is an operator accounting for a weak degeneracy lifting in the relevant set and typically includes the coherent drive. Under the assumptions of a slow change of this matrix, $\rho_{cd}(t-\tau) \approx \rho_{cd}(t)$, we can integrate over the time duration τ of the blocks to arrive at the Bloch equation in its traditional local-in-time form,

$$\frac{d}{dt}\rho_{ab} = i\left(H^r_{ac}\rho_{cb} - \rho_{ac}H^r_{cb}\right) + \Gamma_{ab,cd}\rho_{cd}. \tag{7}$$

The common feature of Eqs. (4)–(7) is the existence of a stationary solution. Mathematically, the linear operator acting on the density matrix in the right-hand side of the equations has a zero eigenvalue. The existence of a stationary solution is obvious from physical reasons and is a consequence of unitary dynamics. The system approaches the stationary solution irrespective of the initial condition at the start of its evolution: it forgets the initial conditions. As we see in what follows, extensions of the Keldysh formalism typically do not have a stationary solution.

3. EXTENDED KELDYSH TECHNIQUE

The extended Keldysh technique is formally defined through an evolution with the Hamiltonians $\hat{H}^{+,-}$ that are different at the forward and backward parts of the Keldysh contour [9, 10]:

$$\hat{R}(t) = \mathrm{Texp}\left(i\int_{-\infty}^{t} d\tau \hat{H}^+(\tau)\right)\hat{R}(-\infty)\times \\ \times\,\tilde{\mathrm{T}}\mathrm{exp}\left(-i\int_{-\infty}^{t} d\tau \hat{H}^-(\tau)\right). \tag{8}$$

While this equation is very similar to Eq. (1), the evolution for different Hamiltonians is not unitary. Consequently, $\hat{R}(t)$ is not a density matrix, in particular, its trace is not 1. It is natural to call it a pseudo-density matrix. Apparently, it is not physical: what is the physical use of it?

We set the Hamiltonians to $\hat{H}^\pm(\tau) = \hat{H}_0 + \chi^\pm(\tau)\hat{I}$ and compute the trace $\mathrm{Tr}[\hat{R}(t)]$. This depends on the values of $\chi^\pm(\tau)$ for all time moments preceding t, $\mathrm{Tr}[\hat{R}(t)] \equiv \exp(\mathcal{S}\{\chi^\pm(\tau)\})$. By expanding Eq. (8) in $\chi^\pm(\tau)$, we see that $\mathcal{S}\{\chi^\pm(\tau)\}$ is nothing but the generating function of all possible Keldysh cumulants of the operator $\hat{I}$ taken at different moments of time. Therefore, it completely characterizes time-dependent quantum fluctuations.

The functional $\mathcal{S}\{\chi^\pm(\tau)\}$ is called the Keldysh action and is routinely applied in the context of a path-integral formulation of the formalism [6]. In this case, H_0 describes a subsystem subject to a quantum field $\chi^\pm$ that typically arises in the path-integral representation of this variable, and $\mathcal{S}\{\chi^\pm(\tau)\}$ describes the response and back-action of the subsystem to this field. It can be used as a block in the Feynman–Vernon action that describes the fluctuation dynamics of the field [13].

Another application of the extended Keldysh formalism is the full counting statistics (FCS) [10]. We set $\chi^\pm = -\chi/2$, with χ being a constant in the time interval $(0, \mathcal{T})$, called the counting field. The expansion of the Keldysh action in χ produces the Keldysh-time-ordered cumulants of the quantum variable

$$Q = \int_0^{\mathcal{T}} d\tau\, I(\tau).$$

Under certain conditions [10], the inverse of this generating function gives the propability of a change Q of this variable during the time interval,

$$P(Q) = \int d\chi e^{i\chi Q} e^{\mathcal{S}(\chi)}. \tag{9}$$

This technique has been implemented for the FCS of the charge transferred between the reservoirs [9, 10, 24].

An accurate definition of the FCS for conserved quantities implements a gauge transformation in a bipartition. We consider an operator of a conserved quantity $\hat{O}$ that is separable in the bipartition, $\hat{O} = \hat{O}_A + \hat{O}_B$. We define a unitary transformation $\hat{U_A}(\chi) = \exp(i\chi\hat{O}_A)$ and the Hamiltonians on two parts of the contour as

$$\hat{H}^{\pm} = \hat{U_A}\left(\pm\frac{\chi}{2}\right)\ \hat{H}\ \hat{U_A}\left(\mp\frac{\chi}{2}\right). \tag{10}$$

Since $H_{A,B}$ commute with $\hat{O}$, the coupling H_{AB} is the only part modified by this transformation,

$$\hat{H}^{\pm} = \hat{H}_A + \hat{H}_B + \hat{H}^{\pm}_{AB}. \tag{11}$$

The evolution of the pseudo-density matrix is given by an extension of Eq. (3),

$$\hat{R}(t) = \mathrm{Texp}\left(i\int\limits_{-\infty}^{t} d\tau\, \hat{H}^{+}_{AB}(\tau)\right)\hat{R}(-\infty)\times \\ \times\,\tilde{\mathrm{T}}\mathrm{exp}\left(-i\int\limits_{-\infty}^{t} d\tau\, \hat{H}^{-}_{AB}(\tau)\right). \tag{12}$$

The trace of $\hat{R}(t)$ defines a Keldysh action $\mathcal{S}(\chi)$ that gives the statistics of transfers of the quantity $\hat{O}$ to/from the subsystem A.

This can describe the statistics of conserved quantities such as current and energy flows, the latter being of special interest for us. In this case, the conserved quantity is the energy $H_A + H_B$ [11, 21]. The unitary transformation is equivalent to a time shift of the operators in the interaction representation. The coupling $\hat{H}_{AB}$ can be quite generally represented as a sum of the products of the operators $\hat{A}_i$, $\hat{B}_i$ acting in the corresponding subspaces, $\hat{H}_{AB} = \hat{A}_i\hat{B}_i$. The modified $\hat{H}_{AB}$ is then given by

$$\hat{H}^{\pm}_{AB}(t) = \hat{A}_i\left(t \mp \frac{\chi}{2}\right)\hat{B}_i(t). \tag{13}$$

The re-summation of the perturbation series made in the preceding section is also relevant and is to be done for the extended Keldysh formalism. The analogous equations can be derived. Importantly, since the dynamics is nonunitary, the blocks and rates do not satisfy the sum rules imposed by unitarity and are generally dependent on counting fields. For instance, in the extended master equation (cf. Eq. (5)),

$$\frac{d}{dt}p_{a\alpha} = -\tilde{\Gamma}_{a\alpha}p_{a\alpha} + \Gamma_{a\alpha,b\beta}p_{b\beta}, \tag{14}$$

where $\tilde{\Gamma}_{a\alpha} \neq \sum_{b,\beta}\Gamma_{b\beta,a\alpha}$.

Hence, there is no stationary solution of these equations even for stationary counting fields. Diagonalizing the linear evolution operator gives a set of solutions of the form

$$\hat{R}(t) \propto \exp(-D_i t), \tag{15}$$

where D_i are the eigenvalues of the operator. In the long-time limit, the general solution is given by the eigenvalue with the smallest real part, D_0. This gives a remarkably simple and constructive expression for the Keldysh action in the limit of long time intervals $\mathcal{T}$:

$$\mathcal{S}(\chi) = -\mathcal{T}D_0. \tag{16}$$

4. WHY MULTIPLE WORLDS?

Although this fact is rarely discussed, in addition to physical conserved quantities that are represented by operators, there are conserved quantities that are characteristics of the density matrix. They are formally unphysical since they are not associated with any physical operator observable. An example is provided by the Rényi entropies, which are defined as traces of integer powers of the density matrix $\hat{R}$ of a closed system

$$S_M = \mathrm{Tr}\left\{\hat{R}^M\right\}. \tag{17}$$

Since the quantum evolution of the system is governed by a Hamiltonian $\hat{H}$ and

$$-i\hbar\frac{d\hat{R}}{dt} = [\hat{H}, \hat{R}],$$

the density matrices at different moments of time are related by a unitary transformation and the trace of any power of $\hat{R}$ does not depend on time, $dS_M/dt = 0$. The definition can be obviously extended to noninteger M. The more common Shannon entropy is obtained by taking the limit

$$S = -\mathrm{Tr}\{\hat{R}\ln\hat{R}\} = -\lim_{M\to 1}\frac{\partial S_M}{\partial M} = \\ = -\lim_{M\to 1}\left(\frac{\ln S_M}{M-1}\right). \tag{18}$$

We note that $\ln S_M$ is an extensive quantity proportional to the system volume.

We now return to the context of bipartition. For two systems A and B, we can define two sets of Rényi entropies,

$$S^{(A)}_M = \mathrm{Tr}_A\left\{\left(\hat{R}^{(A)}\right)^M\right\}, \\ S^{(B)}_M = \mathrm{Tr}_B\left\{\left(\hat{R}^{(B)}\right)^M\right\}, \tag{19}$$

where the reduced density matrices in the two subspaces are defined via the respective partial traces

$$\hat{R}^{(A)} = \mathrm{Tr}_B\{\hat{R}\}, \quad \hat{R}^{(B)} = \mathrm{Tr}_A\{\hat{R}\}. \tag{20}$$

If the quantum evolutions of the systems are completely independent,

$$\hat{H} = \hat{H}_A + \hat{H}_B,$$

where $H_{A,B}$ are operators involving the corresponding subspaces only, the entropies of both sets provide conserved measures,

$$\frac{d}{dt}S_M^{(A)} = \frac{d}{dt}S_M^{(B)} = 0.$$

If we take the coupling $\hat{H}_A B$ into account, the Rényi entropies are no longer conserved. We temporarily assume that the systems A and B are infitely large and are characterized by a continuous excitation spectrum while H_{AB} couples a relatively small number of degrees of freedom in both systems. This situation is similar to that of two metallic leads kept at different chemical potentials and containing a practically infinite number of electrons. If the leads are connected by a small junction, finite electric current flows through the junction, while the distribution of electrons in the infinite leads remains unchanged. From this analogy, it is natural to conjecture that a finite Rényi entropy flow, Re-flow, flows between subsystems A and B. We define the flows as time derivatives of extensive quantities,

$$\mathcal{F}_M^{(A),(B)} =\equiv \frac{d}{dt}\ln S_M^{(A),(B)}. \tag{21}$$

Owing to the conservation of the Rényi entropy in each system, the Re-flows are independent of the exact bipartition of the system and are determined by the properties of the coupling, which is in principle described by $\hat{H}_{AB}$ rather than by the properties of systems A and B, in full analogy with electrical current. But there is an important difference. For physical quantities, the conservation holds in the whole system as well as in each subsystem. For instance, elecrtical currents to each lead must satisfy the relation $I_A + I_B = 0$. As far as Rényi entropies are concerned, there is no exact conservation law for the sum $\ln S_M^{(A)} + \ln S_M^{(B)}$ at finite $\hat{H}_{AB}$, although these quatities are extensive. There is a conservation law for the total Rényi entropy $\ln S^{(A+B)}$. However, the latter at a finite $\hat{H}_{AB}$ is the sum $\ln S_M^{(A)} + \ln S_M^{(B)}$ only approximately, up to terms proportional to the volume of the system. Therefore, in general,

$$\mathcal{F}_M^{(A)} + \mathcal{F}_M^{(B)} \neq 0.$$

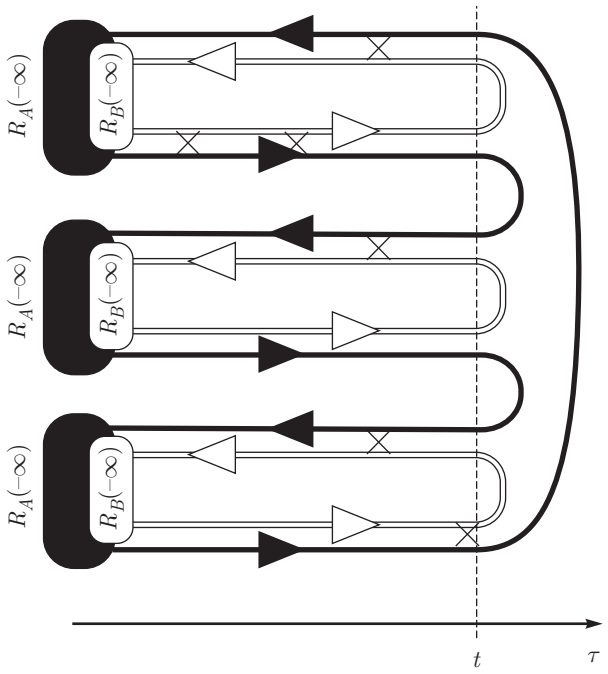

Fig. 3. A perturbation theory diagram for $S_M^{(A)}$ with $M = 3$. It involves three parallel worlds. Reconnection of Keldysh contours for subspaces A (black) and B (white) accounts for the partial trace over B and matrix multiplication in A

How to compute the flows? The crucial observation is that the standard Keldysh formalism as expressed by Eq. (3) can be straightforwardly generalized to any integer number M of density matrices. These matrices undergo independent unitary evolution in the time interval $(-\infty, t)$. It is constructive to think of a set of M "parallel worlds" and draw the diagrams for the perturbation series using M parallel bra and ket contours. To compute $S_M^{(A)}(t)$ with this set, we first need to "split" the contours to account for a possibly different ordering of operators in subspaces A and B (black and white curves in Fig. 3, where $M = 3$). Then we need to reconnect the contours at $\tau = t$. All white contours are closed within each world, which corresponds to the partial trace over B for each density matrix involved. In contrast to this, the black contours are connected to form a single loop going through all the worlds; this corresponds to matrix multiplication in definition (17) of the Rényi entropy. This conveniently represents the rules for operator ordering for any diagram of a particular order in H_{AB}.

It is interesting to note that the sets of Rényi entropies are not the only conserved measures characteristic for a bipartition. Any polynomial in the density matrix that is invariant under the group $U_A \otimes U_B$ of

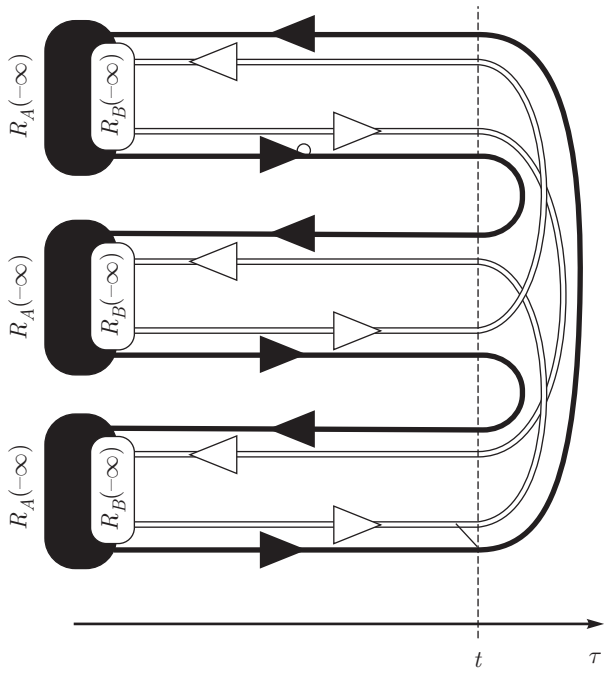

Fig. 4. Reconnection of Keldysh contours for conserved measure K defined by Eq. (22)

unitary transformations in two subspaces would provide such a measure. To give a minimal example, we label the states in A (B) with Latin (Greek) indices. The quantity

$$K \equiv R_{a\alpha,b\gamma}R_{b\beta,c\alpha}R_{c\gamma,a\beta} \tag{22}$$

is a conserved measure that can be reduced neither to the Rényi entropies of the systems nor to the Rényi entropy of the whole system. It is interesting to note that reconnecting the contours in a different fashion gives rise to the perturbation theory for other conserved measures. For instance, for K, the contours are reconnected as shown in Fig. 4. Characterizing all such measures is an interesting research task beyond the scope of this article.

5. DIAGRAM TECHNIQUES FOR MULTIPLE WORLDS

We illustrate the diagram techniques arising in this new context. We concentrate on evaluating $S_M^{(A)}$ and omit the index A for brevity. It is natural to require that the matrix elements of H_{AB} are only nondiagonal, that is, $H^{(AB)}_{a\alpha,b\beta} = 0$ if either $a = b$ or $\alpha = \beta$. In this case, the first nonvanishing diagram giving a correction to S_M is of the second order in H_{AB}. Expressing it in terms of the corrections to $\hat{R}_A$, we find

$$\delta S_M^{(2)} = M \sum_{0\le N\le M} \mathrm{Tr}_A\left[\delta\hat{R}_A^{(1)}\hat{R}_A^N\hat{R}_A^{(1)}\hat{R}_A^{M-N-2}\right] + \\ + M\mathrm{Tr}_A\left[\delta\hat{R}_A^{(2)}\hat{R}_A^N\right]. \tag{23}$$

Here, we use the symmetry of the parallel worlds to cyclically permute $\hat{R}_A$ and its corrections under the trace, which gives the factors M in front of the terms. While there is a first-order correction to $\hat{R}$, it is nondiagonal in the B space. Hence, $\delta\hat{R}_A^{(1)} = 0$: we see below that this is not the case for the quantum heat engine (Sec. 9), where the nondiagonal elements are important. We only need to deal with $\delta\hat{R}_A^{(2)}$ that is concentrated in a single world. The expansion in $\hat{H}_A B$ gives four terms that correspond to four ways to place two $\hat{H}_{AB}$ on two parts of the contour in a single world,

$$\delta\hat{R}_A^{(2)} = \int\limits_{-\infty}^{t} dt_1 \int\limits_{-\infty}^{t_1} dt_2 \Big(-\hat{H}_{AB}(t_1)\hat{H}_{AB}(t_2)\hat{R}_A\hat{R}_B - \\ - \hat{R}_A\hat{R}_B\hat{H}_{AB}(t_2)\hat{H}_{AB}(t_1) + \\ + \hat{H}_{AB}(t_1)\hat{R}_A\hat{R}_B\hat{H}_{AB}(t_2) + \\ + \hat{H}_{AB}(t_2)\hat{R}_A\hat{R}_B\hat{H}_{AB}(t_1)\Big). \tag{24}$$

We need to substitute this in Eq. (23). We now assume that $\hat{H}_{AB} = \hat{A}_i\hat{B}_i$ with $\hat{A}_i$ and $\hat{B}_i$ acting on the corresponding subspaces. We introduce the correlators of these operators. Since the contours for the space B are closed within each world, the correlator takes the usual form

$$C_{ij}(t_1,t_2) \equiv \mathrm{Tr}_B\left[\hat{B}_i(t_1)\hat{B}_j(t_2)\hat{R}_B\right]. \tag{25}$$

A general two-operator correlator in the space A is defined as

$$K_{ij}^{N,M} \equiv \mathrm{Tr}_A\left[\hat{A}_i(t_1)\hat{R}_A^N\hat{A}_j(t_2)\hat{R}_A^{M-N}\right]S_M^{-1}, \tag{26}$$

with the indices $N, M, 0 \le N \le M$ corresponding to different arrangements on the contour traversing M parallel worlds. We divide by S_M^{-1} to keep this correlator an extensive quantity. With this,

$$\delta S_M^{(2)}/S_M = \int\limits_{-\infty}^{t} dt_1 \int\limits_{-\infty}^{t_1} dt_2 W(t_1-t_2) = \\ = \int\limits_{-\infty}^{t} dt_1 \int\limits_{0}^{\infty} d\tau\, W(\tau),$$

where the block $W(t_1,t_2)$ is expressed as

$$W(t_1,t_2) = -C_{ij}(t_1,t_2)K_{ij}^{0,M}(t_1,t_2) - \\ - C_{ji}(t_2,t_1)K_{ji}^{0,M}(t_2,t_1) + \\ + C_{ji}(t_2,t_1)K_{ij}^{1,M}(t_1,t_2) + \\ + C_{ij}(t_1,t_2)K_{ij}^{0,M}(t_1,t_2), \quad (27)$$

with the four terms in this equation corresponding to the four terms in Eq. (24). Therefore, the Re-flow is expressed in terms of the block W as

$$\mathcal{F}_M = \int_0^\infty d\tau\, W(\tau). \quad (28)$$

More complex diagrams are expressed in terms of diagrams and higher-order correlators that have a similar structure.

Expectedly, the correction to S_M diverges as t increases, and hence the Keldysh formalism for multiple parallel worlds also requires re-summation. We can introduce one big density matrix $R_{\mathbf{a}\boldsymbol{\alpha},\mathbf{b}\boldsymbol{\beta}}$, where the M-dimensional "vector" index $\mathbf{a}$ comprises the state indices in the space A for all bra contours, and all other indices are defined similarly. The reduction of this density matrix and the resummation of diagonal blocks leads to analogues of Eqs. (4)–(7). For instance, the analogue of Eq. (4) in parallel worlds is given by

$$\frac{d}{dt}p_{\mathbf{a}\boldsymbol{\alpha}}(t) = \int_0^\infty d\tau\, W_{\mathbf{a}\boldsymbol{\alpha},\mathbf{b}\boldsymbol{\beta}}(\tau)p_{\mathbf{b}\boldsymbol{\beta}}(t-\tau), \quad (29)$$

where W are the blocks computed similarly to those in Eq. (27). Similarly, for the extended Keldysh technique (see Eq. (15)), this equation has a set of nonstationary solutions $\hat{R}(t) \propto \exp(-D_i t)$. The eigenvalues D_i and the form of the solution are affected by the way the contours are reconnected at t. For the connection way that gives the Réneyi entropies, the Re-flows are expressed in terms of the eigenvalue with the smallest real part, D_0, which depends on the number of the worlds involved,

$$\mathcal{F}_M = D_0(M). \quad (30)$$

6. KMS RELATIONS FOR MULTIPLE WORLDS

The correlators in a general nonequilibrium system are independent. The state of thermal equilibrium gives rise to extra relations between the correlators, which are important since they reduce the number of independent parameters in the models of quantum systems. These relations are traditionally called the Kubo–Martin–Schwinger relations [22]. For instance, the correlators C_{ij} (Eq. (25)) in the frequency representation are expressed in a KMS state at a temperature T in terms of the real part of the dynamical susceptibility $\tilde{\chi}_{ij}(\omega)$

$$C_{ij}(\omega) = n_B(\omega)\tilde{\chi}_{ij}(\omega), \quad (31)$$

where $n_B(\omega) \equiv 1/(e^{\beta\omega}-1)$, $\beta = \hbar/k_BT$.

We show that similar relations hold for the multiworld correlators $K_{ij}(\omega)$ defined by Eq. (26). In the frequency representation,

$$K_{ij}^{N,M}(\omega) = \int \frac{d\tau\, e^{i\nu\tau}\mathrm{Tr}\left\{\hat{A}_i(0)\hat{R}_A^N\hat{A}_j(\tau)\hat{R}_A^{M-N}\right\}}{\mathrm{Tr}\,\hat{R}_A^M}.$$

This correlator can be rewritten in the energy basis,

$$K_{i,j}^{N,M} = \int d\tau\, e^{i\omega\tau}\sum_{n,m}\left(A_{i,nm}\frac{e^{-\beta NE_m}}{Z(\beta)^N}\times\right. \\ \left.\times\, A_{j,mn}e^{i(E_m-E_n)\tau}\frac{e^{-\beta E_n(M-N)}}{Z(\beta)^{M-N}}\right)\frac{Z(\beta)^M}{Z(\beta M)} = \\ = 2\pi\delta\left(E_m - E_n + \omega\right)\frac{A_{i,nm}A_{j,mn}e^{-\beta E_n M}}{Z(\beta M)}e^{\beta N\omega}, \quad (32)$$

where $Z(\beta)$ is the partition function defined as

$$Z(\beta) = \sum_i e^{-\beta E_i}.$$

The standard one-world correlator is

$$K_{ij}^{0,1}(\omega) = \int d\tau \exp(i\omega\tau)\mathrm{Tr}\left\{\hat{A}_i(0)\hat{A}_j(\tau)\hat{R}_A\right\}/\mathrm{Tr}\,\hat{R}_A,$$

which after simplifying becomes $2\pi\delta\left(E_m - E_n + \nu\right)\times$ $\times\, A_{i,nm}B_{j,mn}e^{-\beta E_n}/Z(\beta)$. The KMS relation links this to the dynamical susceptibility: $K_{ij}^{0,1}(\nu) = \tilde{\chi}_{ij}(\nu)n_B(\nu/T)$. Substituting this in Eq. (32) gives the generalized KMS relation

$$K_{ij}^{N,M}(\omega) = n_B\left(M\omega\right)e^{\beta\omega N}\tilde{\chi}_{ij}\left(\omega,\beta^*\right). \quad (33)$$

While the correlators are for the system at the inverse temperature β, the dynamical susceptibility is taken at a different inverse temperature $\beta^* \equiv M\beta$. Such temperature rescaling looks surprising in the context of KMS relations. However, this is natural in the context of Rényi entropies. In the state of thermal equilibrium, the Rényi entropy is expressed in terms of free energy at the native and rescaled temperatures,

$$\ln S_M(\beta) = M\beta\left(F(\beta^*) - F(\beta)\right). \quad (34)$$

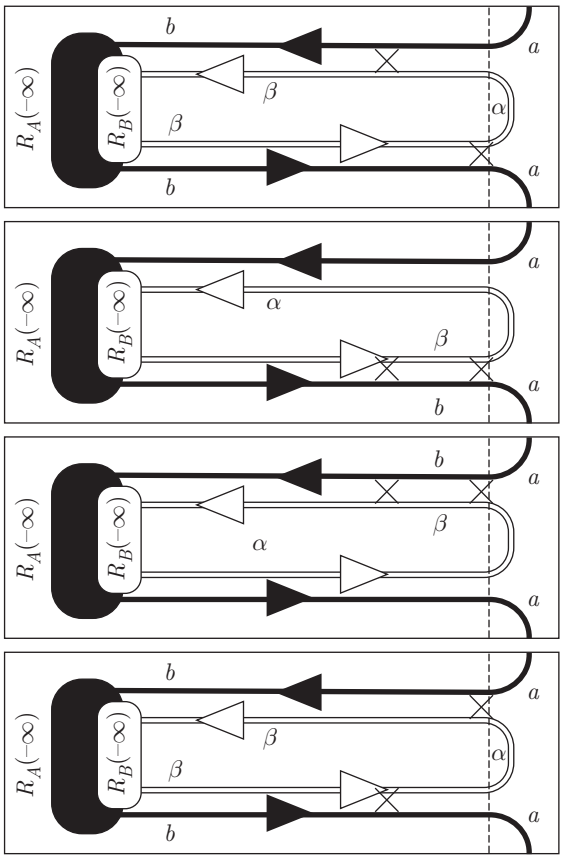

Fig. 5. Second order diagrams for the time derivative of a Rényi entropy. The contributions come only from perturbations $\hat{H}^{(AB)}$ in the same world, only this world is shown in each diagram. For all diagrams, the perturbations are taken at time moments t and $t' < t$. The letters at the contours label the states involved

7. EXAMPLE: SIMPLICITY WITH SECOND-ORDER DIAGRAMS

We start with examples of the multi-world Keldysh approach. In this section, we elaborate on second-order diagrams and obtain a rather general picture of Re-flows in this approximation. In a single world, higher-order diagrams change the values of the rates but do not change the dynamics qualitatively. As we see in the next section, this is not the case in multiple worlds: there, the higher-order diagrams do bring a qualitative change.

We compute the Re-flows in the second order in H_{AB} in a way sligtly different from that used in the preceding section. It is proficient to directy compute the time derivative of S_M. For diagrams, this corresponds to placing one of the perturbations at $\tau = t$. The only way to satistfy the continuity of the state index along the white contours is to place the second perturbation in the same world. Four contributing diagramms are given in Fig. 5. We note that the same four diagrams arise in the derivation of the Golden Rule transition rate in the standard Keldysh formalism. The specifics of Rényi entropies is reflected in extra factors p_a^{M-1} that the diagrams acquire in comparison with the case of a single density matrix. We do not separate $\hat{H}_{AB}$ into subspaces and do not use the correlators, but rather express the answer in terms of the matrix elements of this operator,

$$\frac{\partial}{\partial t}S_M = \left(-M \sum_{a,\alpha;b,\beta} |H^{(AB)}_{a\alpha,b\beta}|^2 p_a^M p_\alpha + \right.$$
$$\left. + M \sum_{a,\alpha;b,\beta} |H^{(AB)}_{a\alpha,b\beta}|^2 p_b p_\beta p_a^{M-1} \right) \times$$
$$\times \int_{-\infty}^{t} dt' 2\mathrm{Re}\left(e^{i(t-t')(E_i+E_\alpha-E_j-E_\beta)} \right). \quad (35)$$

The integral over time t' reduces to

$$2\pi\delta(E_a + E_\alpha - E_b - E_\beta),$$

manifesting energy conservation between the initial state $|a\alpha\rangle$ and the final state $|b\beta\rangle$.

This suggests that we can rewrite the whole expression in terms of Golden Rule rates $\Gamma_{a\alpha,b\beta}$ of the transitions between the states $|a\alpha\rangle$ and $|b\beta\rangle$,

$$\Gamma_{a\alpha,b\beta} = 2\pi|H^{(AB)}_{a\alpha,b\beta}|^2\delta(E_a + E_\alpha - E_b - E_\beta). \quad (36)$$

With this, the flow is given by

$$(S_M)\,\mathcal{F}_M = M \sum_{a,\alpha;b,\beta} \Gamma_{a\alpha;b\beta}(p_b p_\beta - p_a p_\alpha) p_a^{M-1}. \quad (37)$$

We see that the flow vanishes if the systems are in thermodynamic equilibrium at the same temperature. Indeed, in this case $p_b p_\beta / p_a p_\alpha = \exp((E_b + E_\beta - E_a - E_\alpha)/k_B T) = 1$.

Since the transition rates $\Gamma_{a\alpha,b\beta}$ in the Golden rule approximation are symmetric under the permutation $a\alpha \leftrightarrow b\beta$, we can regroup the terms to arrive at

$$(S_M)\,\mathcal{F}_M = M \sum_{a,b} \Gamma_{a\to b} p_a (p_b^{M-1} - p_a^{M-1}), \quad (38)$$

where

$$\Gamma_{a\to b} = \sum_{\alpha,\beta} \Gamma_{a\alpha;b\beta} p_\alpha$$

gives the total transition rate from the state $|a\rangle$ to the state $|b\rangle$ averaged over all possible configurations of system B.

We use Eq. (38) to derive a simplified expression valid in the zero-temperature limit. In this limit, system A is initially in the ground state $|0\rangle$, with $p_0 = 1$ and $p_a = 0$ for $a \neq 0$, $S_M = 1$. We obtain

$$\mathcal{F}_M = -M\Gamma_0, \tag{39}$$

where Γ_0 is the total transition rate from the ground state to any other state. Remarkably, this involves no assumption concerning system B: it can be very far from equilibrium.

Equation (38) is also a convenient starting point to derive the expression for the flow of the Shannon entropy S. Taking the limit $M \to 1$, we obtain

$$-\frac{\partial S}{\partial t} = \sum_{a,b} \ln\left(\frac{p_b}{p_a}\right) \Gamma_{a\to b} p_a. \tag{40}$$

We assume thermal equilibrium for A. In this case, $\ln(p_b/p_a) = (E_a - E_b)/k_BT$. Adding the energy changes $E_b - E_a$ in the course of individual transitions from a to b, we prove that the energy flow to system A equals

$$\frac{dE}{dt} = \sum_{a,b} \Gamma_{a\to b}(E_b - E_a)p_a.$$

Comparing this with Eq. (40), we recover the textbook relation between the heat and entropy flows

$$\frac{dS}{dt} = \frac{1}{k_BT}\frac{dE}{dt}, \tag{41}$$

which appears to be universally valid within the second-order perturbation theory. Remarkably, this invloves no assumption about system B.

8. EXAMPLE: HIGHER-ORDER DIAGRAMS

We analyse the fourth-order diagrams for the time derivative of S_M. As in the foregoing, we assume that H_{AB} does not contain diagonal elements. Since white contours are closed within each world, the four perturbations can come either all in the same world or in two pairs in two different worlds. If all four come in the same world, they describe a correction to one of the Golden Rule transition rates. This correction does not bring anything new and we disregard these diagrams in what follows.

A diagram involving two different worlds is given in Fig. 6. We see that in general the black contour entering a world with perturbations exits it with a different state index. In the particular case where these indices are the same, $a = b$, the diagram diverges upon integration over time. This is not surprising since we expand

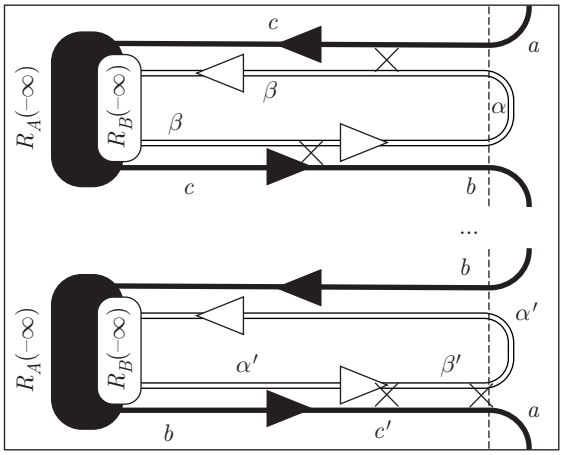

Fig. 6. A fourth-order "quantum" diagram for Rényi entropy flows. The contibutions come from perturbations $\hat{H}^{(AB)}$ in two different worlds, only these two worlds are shown. The letters on the contours label the states involved

$S_M(t) \propto \exp(\mathcal{F}_M t)$. The fourth-order expansion thus contains terms proportional to $(\mathcal{F}_M^{(2)})^2 t/2$, with $\mathcal{F}^{(2)}$ being the second-order contribution to the rate that we have already calculated. Indeed, the diagram with $a = b$ is proportional to $(\mathcal{F}^{(2)})^2$ and therefore does not contribute to fourth-order correction to the flow. We therefore concentrate on the case $a \neq b$. We call this diagram quantum because we see below that it does not permit an interpretation in terms of "classical" transition events. All expressions for $\mathcal{F}$ and dS/dt in this section give fourth-order corrections to these quantities.

There are 16 diagramms of this sort corresponding to the number of ways the pairs of $\hat{H}^{(AB)}$ in each world can be placed on the bra and ket contours. Adding all of them, we can represent the fourth-order correction in the form

$$\frac{d}{dt}S_M^{(A)} = \pi \sum_{a,b} |A_{ab}|^2 \delta(E_a - E_b)\frac{p_a^{M-1} - p_b^{M-1}}{p_a - p_b}, \tag{42}$$

$$A_{ab} = \sum_{c,\alpha,\beta} H^{(AB)}_{a\alpha,c\beta} H^{(AB)}_{c\beta,b\alpha} \left(\pi \left((p_a + p_b)p_\alpha - 2p_c p_\beta \right) \times \right.$$

$$\left. \times\, \delta\left(E_a + E_\alpha - E_c - E_\beta\right) - i\frac{p_a - p_b}{E_a + E_\alpha - E_c - E_\beta} \right).$$

The structure of the matrix elements in the "amplitude" A_{ab} is the same as for the amplitude of the transition from the state $|a\alpha\rangle$ to the state $|b\alpha\rangle$, that is, without a change of the state of subsystem B. Such a transition would seem to involve a virtual state $|c, \beta\rangle$. However,

the rest of the expression for A_{ab} does not support this interpretation: rather, probabilities enter in a form suggesting that the transition takes place between one of the states $|a\alpha\rangle$ and $|b\alpha\rangle$ and the state $|c\beta\rangle$. Therefore, the expression can be associated with no "classical" transition and corresponds to no actual transition rate.

We assume that the probabilities in system A depend only on energies of the corresponding states. It then follows from $E_a = E_b$ that $p_a = p_b$. The term in A_{ab} with the energy difference in the denominator vanishes and the flow reduces to

$$S_M \mathcal{F}_M = (M-1)\pi \sum_{a,b} |A_{ab}|^2 \delta(E_a - E_b) p_a^{M-2}, \quad (43)$$

$$A_{ab} = 2\pi \sum_{c,\alpha,\beta} H^{(AB)}_{a\alpha,c\beta} H^{(AB)}_{c\beta,b\alpha} \left(p_a p_\alpha - p_c p_\beta\right) \times \\ \times \delta\left(E_a + E_\alpha - E_c - E_\beta\right).$$

We note that if both systems are in thermal equilibrium, it follows from $E_a + E_\alpha = E_c + E_\beta$ that $p_a p_\alpha = p_c p_\beta$ and the "amplitudes" A_{ab} vanish.

The "quantum" contibution derived manifests serious problems with the term-by-term perturbation theory in the zero-temperature limit, indicating a nonanalytic dependence of the flows on the coupling strength in the limit of weak couplings and zero temperatures. The contibution seems to have an evident zero-temperature limit, namely zero, at least if the ground state of system A is not degenerate. Indeed, the delta-funcion in Eq. (42) cannot be satisfied for any state $b \neq a$. However, the analytic continuation to noninteger M gives rise to problems.

To see this, we can attempt to derive the Shannon entropy flow by taking the limit $M \to 1$ in Eq. (43). We obtain

$$\frac{dS}{dt} = \sum_{a,b} |A_{ab}|^2 \delta(E_a - E_b) \frac{1}{p_a}, \quad (44)$$

that is, the states with lesser probabilities p_a contibute most to the entropy flow! Since the probabilities of the excited states quickly decrease with descreasing the temperature, we expect a divergence of the Shanon entropy flow as $T \to 0$, in contrast to vanishing Re-flows.

In Ref. [18], the general expression has been elaborated for a typical quantum transport setup where systems A and B are metallic leads kept at the same temperature but at different chemical potentials shifted by eV, and $\hat{H}_{AB}$ describes electron tunneling between the leads. The fourth-order Shannon entropy flow was found to diverge exponentially as $T \to 0$. This indicates an intriguing nonanaliticity of the entropy flows in the coupling strength.

9. EXAMPLE: FLOWS IN A QUANTUM HEAT ENGINE

We give an example of the computation of Re-flows in an interesting system.

A quantum heat engine (QHE) is a system of several discrete quantum states connected to the environments that are kept at different temperatures. The motivation for research in QHEs comes from studying models of photocells and photosynthesis. The thermodynamics of QHEs and their fluctuations in the quantum regime is not a continuation of classical results in discrete energies; instead, features such as quantum coherence, which have no classical analogue, contribute to the heat exchange [19].

We consider a quantum system with discrete states $|n\rangle$ separated into two sets $\{u\}$ and $\{d\}$. All states within a set have approximately the same energy E_u (E_d), the splitting ϵ_n within a set being much smaller than $E_u - E_d > 0$. The system is subject to the external field with the frequency $\omega \approx E_u - E_d$ (we set $\hbar, k_B = 1$ where appropriate), described by the Hamiltonian $H_{dr} = \sum_{m,n} \Omega_{mn} |m\rangle\langle n| e^{-i\omega t} + \text{H.c.}$, and the relevant matrix elements are between the states of two sets. To distinguish the sets, we introduce a matrix η_{nm}, $\eta_{nm} = 1$ if $n \in \{u\}$ and $m \in \{d\}$, $\eta_{nm} = -1$ if $n \in \{d\}$ and $m \in \{u\}$, and $\eta_{nm} = 0$ otherwise.

The quantum system is coupled to a number of environments labeled by a, which are kept at different temperatures T_a. We thus have a multipartition: the whole space is separated into the space of QHE states and the spaces of the environments. The interaction with an environment is described by $H_{int} = \sum_{mn} |m\rangle\langle n| \hat{X}^{(a)}_{mn}$, with $\hat{X}^{(a)}_{mn}$ being the operators in the space of environment a. We assume a linear response of each environment to the state of the quantum system. In this case, each environment is completely characterized by a set of frequency-dependent generalized susceptibilities $\chi^{(a)}_{mn,pq}(\nu)$ that are related to the correlators of $\hat{X}^a$ defined as

$$S^{(a)}_{mn,pq}(t) \equiv \mathrm{Tr}_a\{\hat{X}^a_{mn}(0)\hat{X}^a_{pq}(t)\rho_a\}.$$

The fluctuation–dissipation theorem yields relations in the frequency domain:

$$S_{mn,pq}(\nu) = n_B(\nu/T)\tilde{\chi}_{mn,pq}(\nu),$$

where

$$\tilde{\chi}_{mn,pq}(\nu) \equiv (\chi_{mn,pq}(\nu) - \chi_{pq,mn}(-\nu))/i,$$

and the Bose distribution

$$n_B(\nu/T) \equiv 1/(\exp(\beta\nu) - 1).$$

We concentrate on the Re-flows in one of the environments, which we call a probe environment. The rates induced by the probe environment are assumed to be smaller than all other rates. In this case, we can concentrate on the second-order diagrams. We implement the M-world Keldysh formalism where the contours of the QHE and all environments except the probe one are closed within each world, while the contour of the probe environment traverses all the worlds. There are two sorts of second-order diagrams. The diagrams of the first sort, which we call incoherent, are whitin a single world and are similar to those considered in Sec. 7. The presence of nondiagonal elements of the density matrix in the QHE gives rise to a new type of diagrams, which we call coherent one. In this case, two perturbations are located in different worlds.

Collecting all diagrams (see Appendix B in [19]), we express $\mathcal{F}_M$ as

$$\mathcal{F}_M = \frac{Mn_B(M\omega/T)}{n_B((M-1)\omega/T)n_B(\omega/T)\omega}(Q_i - Q_c). \quad (45)$$

Thus, the R-flow is naturally separated into two parts, which come from incoherent and coherent diagrams. The corresponding quantities $Q_{i,c}$ are expressed in terms of the density matrix of the engine ρ and the dynamical susceptibilities of the probe environment,

$$Q_i = \omega \sum_{mnp;\eta_{np}=1} \rho_{mn}\tilde{\chi}_{pm,np}(\omega)(1 + n_B(\omega/T)) - \\ - \rho_{mn}\tilde{\chi}_{np,pm}(\omega)n_B(\omega/T), \quad (46)$$

$$Q_c = \omega \sum_{mnpq;\eta_{pq}=1} \rho_{nm}\rho_{qp}\tilde{\chi}_{mn,pq}(\omega). \quad (47)$$

The same-world diagrams contribute to the incoherent part that is proportional to Q_i, Q_i is linear in ρ and is therefore an observable. The different-world diagrams form the coherent part proportional Q_c that is quadratic in ρ and in principle would not be observable. The M dependence is identical for both parts.

We interpret the parts and the quantities $Q_{i,c}$: Q_i is an observable, being the total energy flow to the probe environment. The terms proportional $1 + n_B$ describe absorption of energy quanta $\hbar\omega$ by the environment, while those proportional n_B correspond to the emission to the system. Upon taking the limit $M \to 1$, the incoherent part reproduces the textbook equation for the entropy flow, $\mathcal{F}_S = Q_i/T_b$.

The interpretation of the coherent part is more involved and interesting. We replace $|m\rangle\langle n|$ in H_{int} with classical external forces f_{mn} with the time dependence $f_{mn} \propto \exp(-i\omega\eta_{mn})$. These classical forces would cause energy dissipation to the probe environment, which is determined from the forces and the dissipative part of the susceptibility $\tilde{\chi}$. This energy dissipation is Q_c.

Both parts of R-flows can be extracted from the measurement results, although in a different way. The entropy flow is not directly related to the energy flow. Rather,

$$\mathcal{F}_S = (Q_i - Q_c)/T_b, \quad (48)$$

the difference being due to quantum coherent effects in our heat engine. A similar relation holds for the Renyi entropy flow in the low-temperature limit

$$\mathcal{F}_M = M(Q_i - Q_c)/\omega \quad (49)$$

(this limit does not commute with the limit $M \to 1$ since $\mathcal{F}_S$ diverges at low temperatures). In the absence of coherent effects, the low-temperature R-flow is readily interpreted semiclassically [18], as the number of events (in our case, $\hbar\omega$ quantum absorptions) per second in M parallel worlds. With coherences, such simple interpretation does not work since $\mathcal{F}_M$ can be negative [20].

10. EXAMPLE: EXACT CORRESPONDENCE

Another example of the Keldysh multi-world formalism is a relation that we derive for coherent and incoherent second-order diagrams in the general time-dependent situation. This relation gives an exact correspondence between formally unphysical Re-flows and physical observables, namely, the full counting statistics of energy transfers considered in Sec. 3.

As discussed in Sec. 4, the Renyi entropies in quantum physics are considered unphysical, or nonobservable, due to their nonlinear dependence on the density matrix. Such quantities cannot be determined from immediate measurements; instead, their quantification seems to be equivalent to determining the density matrix. This requires reinitialization of the density matrix between many successive measurements. Therefore, the Renyi entropy flows between the systems are conserved measures of nonphysical quantities.

An interesting and nontrivial question is as follows: Is there any relation between the Renyi entropy flows and the physical flows? An idea of such a relation was first put forward by Levitov and Klich in [25], where they proposed that the Shannon entropy flow can be

quantified from the measurement of the full counting statistics (FCS) of charge transfers. The validity of this relation is restricted to zero temperature and obviously to the systems where interaction occurs by means of charge transfer. In this section, we present a relation that is similar in spirit (see [20] for details).

We consider two quantum systems A and B. We assume that system A is infinitely large and is kept in thermal equilibrium at a temperature T. System B is arbitrary: it can encompass several degrees of freedom as well as infinitely many of those. It does not have to be in thermal equilibrium and in general is subject to time-dependent forces. It is convenient to assume that these forces are periodic with a period τ. However, this period does not explicitly enter the formulation of our result, which is also valid for aperiodic forces. The only requirement is that there be a stationary limit of the flows of physical quantities to system A. The stationary limit is defined by averaging the instant flow over the period τ. For aperiodic forces, it is determined by averaging over a sufficiently long time interval.

The energy transfer is statistical. In Sec. 3, we discussed the full counting statistics of energy transfers. The FCS of energy transfer in system A during the time interval $[0, \mathcal{T}]$ can be determined from Eq. (12). For quantification of the Renyi entropy flow, we need to define an auxiliary FCS of energy transfer. The most general interaction Hamiltonian is $\hat{H}_{AB} = \sum_n \hat{A}_n \hat{B}_n$ with $\hat{A}_n$ being operators in the space of the system in thermal equilibrium, and $\hat{B}_n$ being those in the space of an arbitrary system. We replace $\hat{B}_n$ with their average values $\hat{B}_n \to \langle \hat{B}_n \rangle$. The resulting Hamiltonian is that of the equilibrium system subject to time-dependent external forces. These forces induce energy transfers to the system to be characterized by an FCS. We discuss possible physical realization of the scheme. So we have two FCSs. In the limit of long $\mathcal{T}$, their cumulant-generating functions (Keldysh actions) are proportional to the time interval, $\mathcal{S}_i(\xi) = -\bar{f}_i(\xi)$ (incoherent) and $\mathcal{S}_c(\xi) = -\bar{f}_c(\xi)$ (coherent), with ξ being the counting field of energy transfer to/from system A.

Our main result is the exact correspondence

$$\begin{gathered} \bar{\mathcal{F}}_M^{(\beta)}/M = \bar{f}_i^{(M\beta)}(\xi^*) - \bar{f}_c^{(M\beta)}(\xi^*), \\ \xi^* = i\beta(M-1), \end{gathered} \tag{50}$$

which indicates that the Renyi entropy flow of the order M to the system kept at the temperature $T = 1/k_B\beta$ is exactly equal to the difference of the FCS of incoherent and coherent energy transfers to the system kept at the temperature T/M at a fixed characteristic parameter ξ^*. This relation is valid in the weak-coupling limit, where the interaction between the systems can be treated perturbatively.

There is an obvious classical limit in the case where quantum system B is considered classical. All operators $\hat{B}_n$ are numbers corresponding to classical forces acting on the system in thermal equilibrium. In this case, the dynamics of the system is governed by the Hamiltonian in the degrees of freedom of the system and is therefore unitary. In this case, there is no entropy flow. This can be separately understood only from looking into the FCS in correspondence (50): in this case, $\bar{f}_i = \bar{f}_c$.

The entropy/FCS correspondence (50) allows us to quantify the time flow of the Renyi as well as Shannon entropy. These quantities are not accessible in direct measurement because they are nonlinear functions of the density matrix. Direct measurements of the density matrix for a probe environment requires characterization of the reduced density matrix of an infinite system, which is a rather nontrivial procedure and requires the complete and precise reinitialization of the initial density matrix. However, measuring the entropy flow from the correspondence requires that some generating functions be extracted from determining the statistical cumulants of transferred energy in experimental data. This can be done equally well for imaginary and real values of the characteristic parameter. The measurement procedures may be complex, yet feasible and physical.

The correspondence can have many other advantages; for instance, a complete understanding of entropy flows may help to identify the sources of fidelity loss in quantum communications and methods to prevent or control them.

11. CONCLUSIONS

We have formulated and illustrated a fascinating extension of the Keldysh formalism to multiple parallel worlds. Keldysh contours in this scheme are different for different sub-parts of a quantum system, which provides dependences between the worlds. We explain that the formalism naturally arises in the context of characterizing the flows of conserved measures (Rényi entropies) and illustrate its similarities with single-world extensions of the Keldysh formalism.

It is a great honor for us to present these results in a special issue celebrating numerous scientific merits of Leonid Veniaminovich Keldysh. We gladly appreciate his pioneering research that provided a powerful and indispensable tool for many generations of quantum

physicists, us including, and wish him many happy returns of the day.

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