ЖЭТФ, 2015, том 148, вып. 5 (11), стр. 883–898

NONCLASSICAL PROPERTIES AND QUANTUM RESOURCES OF HIERARCHICAL PHOTONIC SUPERPOSITION STATES

T. J. Volkoff *

Department of Chemistry, University of California
94720, Berkeley, California, USA

Received April 24, 2015

We motivate and introduce a class of "hierarchical" quantum superposition states of N coupled quantum oscillators. Unlike other well-known multimode photonic Schrödinger-cat states such as entangled coherent states, the hierarchical superposition states are characterized as two-branch superpositions of tensor products of single-mode Schrödinger-cat states. In addition to analyzing the photon statistics and quasiprobability distributions of prominent examples of these nonclassical states, we consider their usefulness for high-precision quantum metrology of nonlinear optical Hamiltonians and quantify their mode entanglement. We propose two methods for generating hierarchical superpositions in $N = 2$ coupled microwave cavities, exploiting currently existing quantum optical technology for generating entanglement between spatially separated electromagnetic field modes.

DOI: 10.7868/S004445101511005X

1. INTRODUCTION

Maximally entangled quantum states occupy a distinguished position in the theory of quantum information. One has only to consider the central role of Greenberger–Horne–Zeilinger (GHZ) states in $(\mathbb{C}^2)^{\otimes N}$ [1] in many quantum algorithms and quantum teleportation protocols [2] to be convinced of their importance. From a practical perspective, such states comprise the most valuable quantum resource, in terms of both entanglement and usefulness for quantum metrology [3, 4]. Unfortunately, in the case of N two-level quantum systems such as spin-1/2 chains, the maximally entangled states are sensitive to local errors (e. g., phase flips) and can quickly lose all nonclassical resources. However, because of the countably infinite dimension of the Hilbert space of a chain of quantum oscillators (isomorphic to $(\ell^2(\mathbb{C}))^{\otimes N}$), one may hope to engineer maximally entangled states of a subspace isomorphic to $(\mathbb{C}^2)^{\otimes N}$ that are robust under quantum evolutions corresponding to relevant sources of decoherence.

The entangled coherent states (see Ref. [5, 6] and the references therein) are paradigmatic examples of entangled states in $\ell^2(\mathbb{C}) \otimes \ell^2(\mathbb{C})$. However, only a strict subset of these are maximally entangled. Explicit conditions for maximal entanglement of linear combinations of products of coherent states were found in [7] and multimode entangled coherent states have been studied in the context of encoding continuous-variable quantum information [8–10]. To extend the notion of GHZ states to a separable Hilbert space (i. e., a Hilbert space with a countable orthonormal basis) in a general way, we first introduce the two-branch, N mode states

$$\frac{1}{\sqrt{2+2\,\mathrm{Re}(z^N)}}\left(\mathbb{I}_N+U^{\otimes N}\right)|\phi\rangle, \quad z := \langle\phi|U|\phi\rangle, \tag{1}$$

where $|\phi\rangle$ is a single-mode pure quantum state of the Hilbert space $\mathcal{H}$ and U is a partial isometry with $|\phi\rangle$ in its domain. When $z = 0$, the branches $|\phi\rangle^{\otimes N}$ and $U^{\otimes N}|\phi\rangle^{\otimes N}$ are orthogonal in $\mathcal{H}^{\otimes N}$ and the resulting state is a strict analog of a GHZ state. For example, superpositions of oscillator Fock states proportional to

$$|n\rangle^{\otimes N} + e^{i\varphi}|m\rangle^{\otimes N},$$

with $\varphi \in [0, 2\pi]$, fall into this class and are interesting for their quantum optical properties. Such superpositions represent the most obvious generalization of GHZ states to the Hilbert space of N oscillators, $(\ell^2(\mathbb{C}))^{\otimes N}$. The case $z \neq 0$, although deviating from the strict notion of GHZ states due to nonorthogonality of the branches, contains many important macroscopic N-mode superpositions. The entangled coherent states (having $|\phi\rangle = |\alpha\rangle$ and $U = e^{i\theta a^\dagger a}D(\beta)$, where $\theta \in [0, 2\pi]$ and $D(\beta)$ is the oscillator displacement operator for α, $\beta \in \mathbb{C}$) serve as well-studied examples.

* E-mail: adidasty@gmail.com

However, the focus of this paper is on revealing certain new states of the $z = 0$ set. In particular, we consider taking

$$|\phi\rangle = |e_1\rangle := \left(\frac{\mathbb{I} + V}{\sqrt{2 + 2w}}\right)|\phi'\rangle, \quad |\phi'\rangle \in \mathcal{H},$$

where V is a single-mode partial isometry containing $|\phi'\rangle$ in its domain and $w = \langle\phi'|V|\phi'\rangle \in \mathbb{R}$. Then the state

$$|e_2\rangle := \left(\frac{\mathbb{I} - V}{\sqrt{2 - 2w}}\right)|\phi'\rangle$$

is orthogonal to $|e_1\rangle$. Taking

$$U = e^{i\theta/N}(|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|)$$

produces two-branch superpositions of the following form, which we refer to as hierarchical cat states (HCS):

$$|\mathrm{HCS}_N^\theta\rangle := \frac{1}{\sqrt{2}}\left(\left(\frac{(\mathbb{I} + V)|\phi'\rangle}{\sqrt{2+2w}}\right)^{\otimes N} + \right. \\ \left. + e^{i\theta}\left(\frac{(\mathbb{I} - V)|\phi'\rangle}{\sqrt{2-2w}}\right)^{\otimes N}\right). \quad (2)$$

The origin of the name "hierarchical cat state" is self-evident: $|\mathrm{HCS}_N^\theta\rangle$ is an equal-weight superposition of two orthogonal branches in $\mathcal{H}^{\otimes N}$ (i.e., an N-mode "cat" state), while each branch is a tensor product of "kitten" superpositions in the single-mode Hilbert space $\mathcal{H}$. This construction allows considering maximally entangled states in a $(\mathbb{C}^2)^{\otimes N}$ subspace of $(\ell^2(\mathbb{C}))^{\otimes N}$ that retain single-mode quantum coherence in the states $|\phi'\rangle \pm V|\phi'\rangle$ even after intermode decoherence processes reduce the superposition state $|\mathrm{HCS}_N^\theta\rangle$ to an equal-probability statistical mixture of tensor products.

When $\mathcal{H} \cong \ell^2(\mathbb{C})$, $|\mathrm{HCS}_N^\theta\rangle$ is in a (complex) two-dimensional subspace of $(\ell^2(\mathbb{C}))^{\otimes N}$. By appropriate choices of V and $|\phi'\rangle$, the branches of $|\mathrm{HCS}_N^\theta\rangle$ can take the form of tensor products of single-mode photonic Schrödinger-cat states such as the even and odd coherent states $|\psi_\pm\rangle \propto |\alpha\rangle \pm |-\alpha\rangle$ [11] or superpositions of squeezed states. For these photonic HCS states, it is clear that if the single-mode coherence time is sufficiently long (e.g., greater than the intermode coherence time), a statistical mixture of N-mode Schrödinger-cat states remains even after the intermode coherence is lost by some decohering process.

The photonic HCS state obtained by taking $V = e^{i\pi a^\dagger a}$ to be the oscillator π-phase shift, $|\phi'\rangle = |\alpha\rangle$ a coherent state of the quantum oscillator with the amplitude $|\alpha|$, and $\theta = 0$ or $\theta = \pi$ in (2) was introduced in Ref. [12]. We label these states by $|\mathrm{HCS}_N^\pm(\alpha)\rangle$ (the "+" symbolizing $\theta = 0$ and the "−" symbolizing $\theta = \pi$), and they serve as the canonical examples of photonic HCS in this work. Each branch of $|\mathrm{HCS}_N^\pm(\alpha)\rangle$ is an N-fold tensor product of either the even coherent state $|\psi_+\rangle$ or the odd coherent state $|\psi_-\rangle$. If N is an odd natural number, then the even (odd) branch is an eigenvector of the photon parity operator $\exp\left\{i\pi\sum_{j=1}^N a_j^\dagger a_j\right\}$ with the eigenvalue 1 (−1), and hence

$$\exp\left\{i\pi\sum_{j=1}^N a_j^\dagger a_j\right\}|\mathrm{HCS}_N^\pm(\alpha)\rangle = |\mathrm{HCS}_N^\mp(\alpha)\rangle.$$

If N is an even natural number, $|\mathrm{HCS}_N^\pm(\alpha)\rangle$ is invariant under such a local π rotation. Independent of N, a photon parity measurement results in a projection of $|\mathrm{HCS}_N^\pm(\alpha)\rangle$ onto either the even or odd branch. The entangled coherent states

$$|\mathrm{ECS}_N^\pm(\alpha)\rangle \propto |\alpha\rangle^{\otimes N} \pm |-\alpha\rangle^{\otimes N}$$

are invariant under the bosonic algebra freely generated by $a_i a_j$, $i, j \in \{1, \ldots, N\}$, and they are therefore considered to be Barut–Girardello coherent states of $\mathfrak{sp}(N, \mathbb{C})$ [13]. The state $|\mathrm{HCS}_N^\pm(\alpha)\rangle$ is invariant under the algebra freely generated by the identity operator and the two-photon annihilation operators a_j^2, $j = 1, \ldots, N$. This algebraic property allows $|\mathrm{HCS}_N^\pm(\alpha)\rangle$ to be considered as superpositions of $\mathfrak{sp}(N, \mathbb{C})$ Barut–Girardello coherent states. However, while properties such as quasiprobability densities, photon statistics, and dissipative evolutions of the entangled coherent states have been thoroughly documented [14, 15], a detailed analysis of the properties of photonic HCS states is lacking.

When $|\mathrm{HCS}_2^\pm(\alpha)\rangle$ is shared between two spatially separated parties, the state $|\mathrm{HCS}_2^\pm(\alpha)\rangle$ serves as an entanglement resource for teleportation of an arbitrary superposition of coherent states of the form $c_1|\alpha\rangle + c_2|-\alpha\rangle$ in the same way as the GHZ state $(1/\sqrt{2})(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ serves as an entanglement resource for teleportation of an arbitrary qubit pure state. This follows from the fact that $|\mathrm{HCS}_2^\pm(\alpha)\rangle$ are maximally entangled states in the 4-dimensional sub-Hilbert space spanned by $\{|(-1)^j\alpha\rangle \otimes |(-1)^\ell\alpha\rangle\}_{j,\ell\in\{0,1\}}$[1]. The states $|\mathrm{HCS}_N^\pm(\alpha)\rangle$ are also useful probes for high-precision phase estimation of Hamiltonians of $\mathfrak{sl}(2, \mathbb{C})$ [19] (see also Sec. 3.1). These intriguing attributes motivate a more thorough description and analysis of HCS states.

[1] In fact, the states $|\mathrm{HCS}_2^\pm(\alpha)\rangle$ exhibit the same amount (1 ebit) of entanglement entropy as $|\mathrm{ECS}_2^-(\alpha)\rangle$ [16, 17]. A "Bell basis" of maximally entangled states for this 4-dimensional sub-Hilbert space is given by $|\mathrm{HCS}_2^+(\alpha)\rangle$, $|\mathrm{HCS}_2^-(\alpha)\rangle$, $|\mathrm{ECS}_2^-(\alpha)\rangle$, and $(e^{i\pi a^\dagger a} \otimes \mathbb{I})|\mathrm{ECS}_2^-(\alpha)\rangle$ [18].

The remainder of this paper is structured as follows: in Sec. 2, we indicate some nonclassical properties of $|\mathrm{HCS}_N^{\pm}(\alpha)\rangle$, citing the nonclassical properties of the entangled coherent states for comparison; Sec. 3 is devoted to exploration of the quantum resources, in particular the metrological usefulness and entanglement entropy, of $|\mathrm{HCS}_N^{\pm}(\alpha)\rangle$; in Sec. 4, we propose two methods for generating $|\mathrm{HCS}_N^{+}(\alpha)\rangle$ using techniques that are accessible by current quantum optical technology; in Sec. 5, we demonstrate how the idea of hierarchically encoding continuous-variable quantum information can be deepened with many levels of hierarchy. We do not attempt an exhaustive analysis of photonic hierarhical cat states, but rather try to show the most salient properties of these states by considering basic examples.

2. NONCLASSICAL PROPERTIES OF HCS

Here, we note the basic photon statistics of $|\mathrm{HCS}_N^{+}(\alpha)\rangle$ for arbitrary N, derive some of its quasiprobability distributions for $N = 2$, and show a duality between Pauli matrices and photon operations in the subspace of $\ell^2(\mathbb{C})$ spanned by the even and odd coherent states. Throughout this section, we compare the nonclassical properties of $|\mathrm{HCS}_N^{\pm}(\alpha)\rangle$ to those of the entangled coherent states, which are more familiar. We show the inner products of $|\mathrm{ECS}_N^{\pm}(\alpha)\rangle$ and $|\mathrm{HCS}_N^{\pm}(\alpha)\rangle$ immediately:

$$\langle \mathrm{ECS}_N^{+}(\alpha)|\mathrm{HCS}_N^{+}(\alpha)\rangle = \frac{1}{\sqrt{1+e^{-2N\alpha^2}}}\begin{cases} \left(\dfrac{1}{2}+\dfrac{1}{2}e^{-2\alpha^2}\right)^{N/2} + \left(\dfrac{1}{2}-\dfrac{1}{2}e^{-2\alpha^2}\right)^{N/2}, & N \text{ even}, \\ \left(\dfrac{1}{2}+\dfrac{1}{2}e^{-2\alpha^2}\right)^{N/2}, & N \text{ odd}, \end{cases} \tag{3}$$

$$\langle \mathrm{ECS}_N^{+}(\alpha)|\mathrm{HCS}_N^{-}(\alpha)\rangle = \frac{1}{\sqrt{1+e^{-2N\alpha^2}}}\begin{cases} \left(\dfrac{1}{2}+\dfrac{1}{2}e^{-2\alpha^2}\right)^{N/2}, & N \text{ even}, \\ \left(\dfrac{1}{2}+\dfrac{1}{2}e^{-2\alpha^2}\right)^{N/2} - \left(\dfrac{1}{2}-\dfrac{1}{2}e^{-2\alpha^2}\right)^{N/2}, & N \text{ odd}, \end{cases} \tag{4}$$

$$\langle \mathrm{ECS}_N^{-}(\alpha)|\mathrm{HCS}_N^{\pm}(\alpha)\rangle = \begin{cases} 0, & N \text{ even}, \\ \pm\dfrac{(1/2-(1/2)e^{-2\alpha^2})^{N/2}}{\sqrt{1-e^{-2N\alpha^2}}}, & N \text{ odd}, \end{cases} \tag{5}$$

where $\alpha \in \mathbb{R}$. It is intriguing to note the $\alpha \to \infty$ asymptotic form of these inner products (the $N \to \infty$ asymptotic is always zero). For any odd N and for reasonably large α,

$$\langle \mathrm{ECS}_N^{+}(\alpha)|\mathrm{HCS}_N^{-}(\alpha)\rangle \approx 0,$$

while for any even N and any α,

$$\langle \mathrm{ECS}_N^{-}(\alpha)|\mathrm{HCS}_N^{\pm}(\alpha)\rangle = 0$$

identically. The total expected photon number in all of these states is asymptotically $N\alpha^2$, i.e., $|\mathrm{HCS}_N^{\pm}(\alpha)\rangle$ and $|\mathrm{ECS}_N^{\pm}(\alpha)\rangle$ differ mainly in photon statistics and quantum correlations, not in intensity. In particular, if P_e ($P_o = \mathbb{I} - P_e$) is the projection onto the even (odd) photon number subspace[2)] of $(\ell^2(\mathbb{C}))^{\otimes N}$, it is clear that

$$P_o|\mathrm{ECS}_N^{+}(\alpha)\rangle = P_e|\mathrm{ECS}_N^{-}(\alpha)\rangle = 0$$

while

$$P_o|\mathrm{HCS}_N^{+}(\alpha)\rangle = P_e|\mathrm{HCS}_N^{+}(\alpha)\rangle = \frac{1}{\sqrt{2}}.$$

Because of its symmetry, $|\mathrm{HCS}_N^{\pm}(\alpha)\rangle$ has a simple expression as a superposition of tensor products of coherent states. We first consider the direct product $\mathbb{Z}_2\times\ldots\times\mathbb{Z}_2$ (N times) with the group operation given by addition modulo 2. This group is isomorphic to the Abelian group $\mathfrak{U}$ with elements

$$\bigotimes_{j=1}^{N} \exp(ik_j\pi a_j^\dagger a_j), \quad k_j \in \{0,1\},$$

and the group operation being operator multiplication. Let $\mathfrak{U}_1$ ($\mathfrak{U}_2$) be the subgroup of elements corresponding to $\mathbf{k}$ such that the number of nonzero entries of $\mathbf{k}$ is even (odd). Then we can write (again for $\alpha \in \mathbb{R}$)

[2)] Explicitly, $P_e = \sum_{\mathbf{n}:\|\mathbf{n}\|^2=0 \bmod 2} P_{\mathbf{n}}$, where $P_{\mathbf{n}}$ is the rank-one projector onto the ray $|n_1\rangle\otimes\ldots\otimes|n_N\rangle$.

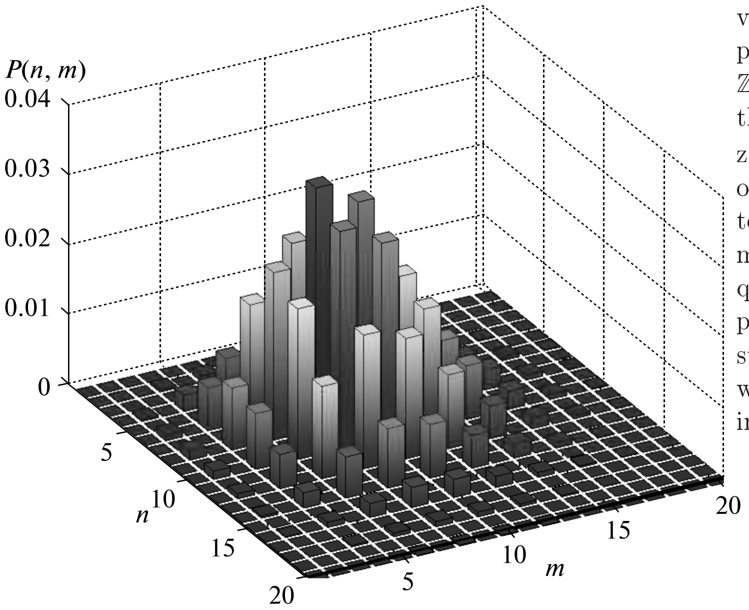

Fig. 1. Photon number distribution $P(n,m)$ for $|\mathrm{HCS}^+_{N=2}(\alpha=3)\rangle$; $n, m \in \{1,\ldots,20\}$

$$|\mathrm{HCS}^{\pm}_N(\alpha)\rangle = \frac{e^{N\alpha^2/2}}{2^{N+1/2}} \times \\ \times \left(\sum_{j=1,2} \left(\mathrm{ch}^{-N/2}\,\alpha^2 + (-1)^{j\mp 1}\,\mathrm{sh}^{-N/2}\,\alpha^2 \right) \sum_{u\in\mathfrak{U}_j} u \right) \times \\ \times |\alpha\rangle^{\otimes N}. \quad (6)$$

From the above expression, the expansion in the Fock state basis can be made explicit by using the fact that

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

The photon number distribution of $|\mathrm{HCS}^+_N(\alpha)\rangle$ for any $\alpha \in \mathbb{C}$ is given by

$$P(\mathbf{n}) = |\langle \mathbf{n}|\mathrm{HCS}^+_N(\alpha)\rangle|^2 = \frac{e^{-N|\alpha|^2}}{2^{N+1}} \times \\ \times \left| \sum_{j=0,1} \frac{1}{(1+(-1)^j e^{-2|\alpha|^2})^{N/2}} \prod_{k=1}^{N} (1+(-1)^{n_k+j}) \times \right. \\ \left. \times \frac{\alpha^{n_k}}{\sqrt{n_k!}} \right|^2, \quad (7)$$

where $|\mathbf{n}\rangle := |n_1\rangle \otimes \ldots \otimes |n_N\rangle$ for $\mathbf{n} \in \mathbb{N}^N$. It is clear from the above expression that if not all entries of $\mathbf{n}$ are even or odd, then the photon number distribution vanishes. For $N=2$, this results in a checkerboard pattern of zero and nonzero probabilities on the lattice $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ (Fig. 1). This feature stands in contrast to the distribution $|\langle \mathbf{n}|\mathrm{ECS}^+_N(\alpha)\rangle|^2$, which is identically zero if and only if $\sum_{k=1}^N n_k$ is odd. Our present focus on the photon statistics of $|\mathrm{HCS}^+_N(\alpha)\rangle$ is merely due to the fact that they are the hierarchical cat states of most immediate practical use for continuous-variable quantum information processing. Indeed, more complex photon statistics are furnished by hierarchical cat states formed, e.g., from the $\mathbb{Z}/4\mathbb{Z}$ coherent states, which generalize the even/odd coherent states by forming a $\mathbb{C}^4$ subspace of $\ell^2(\mathbb{C})$ having an orthonormal basis

$$\begin{aligned} |e_0\rangle &= \frac{|\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle}{2e^{-|\alpha|^2/2}\sqrt{2\,\mathrm{ch}\,|\alpha|^2 + 2\cos|\alpha|^2}}, \\ |e_1\rangle &= \frac{|\alpha\rangle - |-\alpha\rangle - i|i\alpha\rangle + i|-i\alpha\rangle}{2e^{-|\alpha|^2/2}\sqrt{2\,\mathrm{sh}\,|\alpha|^2 + 2\sin|\alpha|^2}}, \\ |e_2\rangle &= \frac{|\alpha\rangle + |-\alpha\rangle - |i\alpha\rangle - |-i\alpha\rangle}{2e^{-|\alpha|^2/2}\sqrt{2\,\mathrm{ch}\,|\alpha|^2 - 2\cos|\alpha|^2}}, \\ |e_3\rangle &= \frac{|\alpha\rangle - |-\alpha\rangle + i|i\alpha\rangle - i|-i\alpha\rangle}{2e^{-|\alpha|^2/2}\sqrt{2\,\mathrm{sh}\,|\alpha|^2 - 2\sin|\alpha|^2}}, \end{aligned} \quad (8)$$

where $\langle n|e_j\rangle \neq 0$ if and only if $n \equiv j \mod 4$.

The variance of a single-mode quadrature

$$x_j^{(\theta)} := \frac{1}{\sqrt{2}}(a_j e^{-i\theta} + a_j^\dagger e^{i\theta})$$

in $|\mathrm{HCS}^{\pm}_N(\alpha)\rangle$ is

$$\frac{1}{2} + |\alpha|^2(\coth 2|\alpha|^2 + \cos(2\,\mathrm{Arg}\,\alpha - 2\theta)) \quad (9)$$

for all θ. Because this variance is greater than 1/2, $|\mathrm{HCS}^{\pm}_N(\alpha)\rangle$ is therefore not squeezed in any quadrature. The $|\mathrm{HCS}^{\pm}_N(\alpha)\rangle$ do not exhibit second-order squeezing [20] (also referred to as amplitude squared squeezing) because they are eigenvectors of a_j^2 for all $j = 1,\ldots,N$. The single-mode Mandel Q_M parameter of $|\mathrm{HCS}^{\pm}_N(\alpha)\rangle$ is always negative, but exhibits a dip for $|\alpha| \approx 3/2$ as shown in Fig. 2.

We now turn to the quasiprobability distributions and functional representations of $|\mathrm{HCS}^{\pm}_2(\alpha)\rangle$ with the focus being on the functional form of the hierarchical coherences. In addition, we use the explicit expressions of the quasiprobability distributions to infer nonclassical features of the states. Each quasiprobability distribution for a quantum state ρ of $(\ell^2(\mathbb{C}))^{\otimes N}$ is obtained by the Fourier transformation on $\mathbb{C}^N$ of a quantum characteristic function corresponding to a particular ordering of bosonic operators [21]. It is well known that negative values of the singular quasidistribution (i.e., the Sudarshan–Glauber function) for

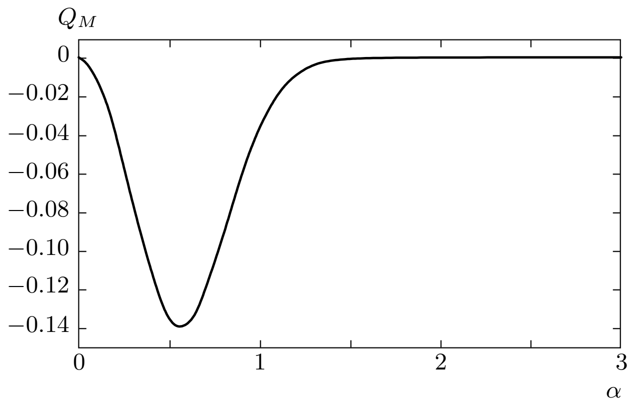

Fig. 2. Mandel Q_M parameter for $|\mathrm{HCS}_N^{\pm}(\alpha)\rangle$ for $|\alpha| \in$ $\in [0, 3]$

a given quantum state ρ indicate nonclassical photon statistics, i. e., indicate that the photon number distribution is not Poissonian [22]. In a similar spirit, the existence of negative values of the Wigner function for a given state indicates non-Gaussian quadrature correlations. The explicit form of the Wigner function of $|\mathrm{HCS}_N^{+}(\alpha)\rangle$ was shown in Ref. [12], but the analytic expression is only useful for technical purposes. The important point is that the Wigner function of a photonic state has an interpretation as a continuous set of interference experiments. This is clear from the definition of the single-mode Wigner function

$$W(\gamma) = (2/\pi^2)\langle D(\gamma)e^{i\pi a^\dagger a}D(-\gamma)\rangle;$$

it shows how a quantum state changes when its coherent-state components are displaced by $-\gamma$ in phase space, are then reflected in phase space, and are displaced again opposite to $-\gamma$. Coherence between coherent-state components appears as fringes in the Wigner function. Accordingly, the states $|\mathrm{HCS}_2^{\pm}(\alpha)\rangle$ exhibit two sources of interference: 1) from the coherence in each branch $|\psi_\pm\rangle^{\otimes 2}$, and 2) from the coherence between the branches. These two sources of coherence are not immediately visible from the expression for $|\mathrm{HCS}_2^{\pm}(\alpha)\rangle$ as an unequal superposition of tensor products of coherent states.

The existence of zeros of the Q-function of a given quantum state indicates that the singular quasidistribution for this state takes negative values and, hence, exhibits nonclassical features. The Q-function of $|\mathrm{HCS}_N^{+}(\alpha)\rangle$, which is a true probability distribution on $\mathbb{C}^N$, is given by

$$Q(|\mathrm{HCS}_N^{+}(\alpha)\rangle) = \frac{1}{\pi}|\left(\otimes_{k=1}^{N}|\beta_k\rangle, |\mathrm{HCS}_N^{+}(\alpha)\rangle\right)|^2.$$

For $N = 2$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned} Q_{|\mathrm{HCS}_2^{+}(\alpha)\rangle}(\beta_1, \beta_2) = \frac{\pi}{2}\,(Q_+(\beta_1)Q_+(\beta_2) + \\ + Q_-(\beta_1)Q_-(\beta_2)) + \frac{\exp\left\{-(|\beta_1|^2 + |\beta_2|^2)\right\}}{2\pi\,\mathrm{sh}(2|\alpha|^2)} \times \\ \times (\mathrm{sh}(2\,\mathrm{Re}(\beta_1\alpha))\,\mathrm{sh}(2\,\mathrm{Re}(\beta_2\alpha)) - \\ - \sin(2\,\mathrm{Im}(\beta_1\alpha))\sin(2\,\mathrm{Im}(\beta_2\alpha)))\,, \end{aligned} \quad (10)$$

where $Q_{+(-)}(z)$ is the Q-function of the even (odd) coherent state [14]. The Q-function of $|\mathrm{HCS}_2^{+}(\alpha)\rangle$ vanishes if and only if each of the terms vanishes. We take β_1, $\beta_2 \in \mathbb{C}$ such that $\mathrm{Re}(\beta_1\alpha) = \mathrm{Re}(\beta_2\alpha) = 0$. In addition, we require that β_1 and β_2 satisfy: 1) $\mathrm{Im}(\beta_1\alpha) = (2k+1)\pi/2$, where $k \in \mathbb{Z}$, and 2) $\mathrm{Im}(\beta_2\alpha) = m\pi$, where $m \in \mathbb{Z}$. Under these constraints, $Q_+(\beta_1) = Q_-(\beta_2) = 0$. For these values,

$$\begin{aligned} \mathrm{sh}(2\,\mathrm{Re}(\beta_1\alpha))\,\mathrm{sh}(2\,\mathrm{Re}(\beta_2\alpha)) - \\ - \sin(2\,\mathrm{Im}(\beta_1\alpha))\sin(2\,\mathrm{Im}(\beta_2\alpha)) = 0, \end{aligned}$$

and hence the Q-function is zero at these points of $\mathbb{C} \times \mathbb{C}$.

In addition to the quasiprobability distributions, the quadrature distribution, calculated as the squared modulus of the Schrödinger wavefunction, is an especially useful true probability distribution for systems of oscillators. However, the quadrature distribution is quite specific; all that is needed is a representation of the pure state $|\mathrm{HCS}_2^{\pm}(\alpha)\rangle$ in a functional Hilbert space. We choose the Bargmann representation [23] because relevant quantities such as the Schrödinger wavefunction and the singular quasidistribution of a pure state can be derived from the Bargmann representation by the use of variants of the Segal–Bargmann transformation. As an analytic function $f_{|\mathrm{HCS}_2^{\pm}(\alpha)\rangle}$ on $\mathbb{C} \times \mathbb{C}$, the state $|\mathrm{HCS}_2^{\pm}(\alpha)\rangle$ is represented by

$$\begin{aligned} f_{|\mathrm{HCS}_2^{\pm}(\alpha)\rangle}(z, w) = \sqrt{2}e^{-|\alpha|^2} \times \\ \times \frac{\mathrm{ch}(\alpha(z+w)) \mp e^{-2|\alpha|^2}\,\mathrm{ch}(\alpha(z-w))}{1 - e^{-4|\alpha|^2}}. \end{aligned} \quad (11)$$

That $f_{|\mathrm{HCS}_2^{\pm}(\alpha)\rangle}(z, w)$ takes the form of a sum of unequally weighted functions is a consequence of the fact that $|\mathrm{HCS}_2^{\pm}(\alpha)\rangle$ is an unequally weighted superposition of tensor products of coherent states.

The subspace $\mathcal{K} \cong \mathbb{C}^2$ of $\ell^2(\mathbb{C})$ spanned by the even and odd coherent states $|\psi_\pm\rangle$ has the property that certain photonic operators carry out equivalent operations as the Pauli matrices in this subspace. This allows quantum operations of a two-level system to be interpreted as photonic operations compressed to this subspace. For example, keeping in mind the action of

$$\sigma_x = |\psi_+\rangle\langle\psi_-| + |\psi_-\rangle\langle\psi_+|$$

in $\mathcal{K}$, we note that

$$a|\psi_+\rangle = \alpha\sqrt{\tanh|\alpha|^2}|\psi_-\rangle = \alpha\sqrt{\tanh|\alpha|^2}\sigma_x|\psi_+\rangle.$$

Considering the Pauli matrices as observables of a spin-1/2 particle, we find the following expressions in terms of self-adjoint photonic observables:

$$\begin{aligned}\sigma_x &= \frac{e^{-|\alpha|^2}\sqrt{\mathrm{sh}(2|\alpha|^2)}}{|\alpha|} P_{\mathcal{K}} x^{(\mathrm{Arg}\,\alpha)} P_{\mathcal{K}},\\ \sigma_y &= \frac{e^{|\alpha|^2}\sqrt{\mathrm{sh}(2|\alpha|^2)}}{|\alpha|} P_{\mathcal{K}} x^{(\pi/2+\mathrm{Arg}\,\alpha)} P_{\mathcal{K}},\\ \sigma_z &= \frac{1}{2\,\mathrm{Re}(\alpha^2)} P_{\mathcal{K}}(e^{i\pi a^\dagger a}a^2 + a^{\dagger 2}e^{-i\pi a^\dagger a})P_{\mathcal{K}},\end{aligned} \tag{12}$$

where $P_{\mathcal{K}}$ is the projection on $\mathcal{K}$. These expressions for Pauli matrices show a duality between quadratures and "magnetization" in the subspace $\mathcal{K}$. For example, if $|\alpha|$ is sufficiently large, we have

$$P_{\mathcal{K}} x^{(\mathrm{Arg}\,\alpha)} P_{\mathcal{K}} = \sqrt{2}|\alpha|\sigma_x,$$

i. e., the interaction picture dynamics of a quantum oscillator (restricted to $\mathcal{K}$) driven with unit amplitude is equivalent to a spin-1/2 particle with the magnetic moment $\sqrt{2}|\alpha|$ in a unit magnetic field along the x axis. The Pauli matrices in (12) do not have unique expressions in terms of products of photonic operators and $P_{\mathcal{K}}$, because we can rewrite $P_{\mathcal{K}}$ as

$$\frac{1}{2|\alpha|^2} P_{\mathcal{K}}(a^2 e^{-2i\mathrm{Arg}\,\alpha} + a^{\dagger 2}e^{2i\mathrm{Arg}\,\alpha})P_{\mathcal{K}}.$$

Some of these alternative expressions can be instructive; for example, we can write

$$\sigma_z = P_{\mathcal{K}}\cos(\pi a^\dagger a)P_{\mathcal{K}}.$$

A similar duality can be derived for $\mathfrak{su}(4)$ observables in terms of projectors in the subspace spanned by states (8) and photonic operations.

3. QUANTUM RESOURCES OF HCS

3.1. Metrological usefulness

We begin this section by recalling the main problem of quantum metrology and how certain quantum states can be used for estimation of dynamical parameters at higher precision than any classical states. Given a smooth manifold M, let quantum states be encoded by a differentiable map specified by $\lambda \mapsto \rho_\lambda$ for $\lambda \in M$. The goal is to estimate the parameter λ with greatest possible precision by using an optimal quantum measurement and optimal classical post-processing of the measurement results.

In this section, we are concerned with the special case of estimation of a displacement parameter. In this case, the parameter manifold is a line with a real coordinate $x \in \mathbb{R}$ and the state

$$\rho_x := e^{-ixH}\rho_0 e^{ixH}$$

lies on a path parameterized by x and generated by the self-adjoint x-dependent operator H. If $\{M(dx)\}$ is a positive operator-valued measure (we refer to $\{M(dx)\}$ as a "quantum measurement" or, simply, "measurement" from now on), which is an unbiased estimator of x, i. e.,

$$x = \int_{\mathbb{R}} x'\,\mathrm{tr}(\rho_x M(dx')),$$

then the quantum Cramér–Rao theorem [24, 25] states that

$$\langle(\delta\hat{x}_M)^2\rangle \ge \frac{1}{\mathrm{tr}(\rho_0 L^2)}, \tag{13}$$

where

$$\langle(\delta\hat{x}_M)^2\rangle := \int_{\mathbb{R}} (x'-x)^2\,\mathrm{tr}(\rho_x M(dx'))$$

is a general expression for the variance of the quantum measurement and $L = L^\dagger$ is the symmetric logarithmic derivative operator defined by the equation

$$\frac{d\rho}{dx} = -i[H,\rho] = \frac{1}{2}(L\rho + \rho L).$$

The quantity $\mathrm{tr}(\rho_0 L^2)$ is called the quantum Fisher information of ρ_0 and is constant on the unitary path generated by H [26]. If $\rho_0 = |\psi_0\rangle\langle\psi_0|$ is pure, then

$$\mathrm{tr}(\rho_0 L^2) = 4(\mathrm{tr}(\rho_0 H^2) - \mathrm{tr}(\rho_0 H)^2).$$

Hence, if an experimenter has unconstrained access to measurements saturating inequality (13), then a quantum state with larger quantum Fisher information with respect to H can be considered a more useful resource for estimating the displacement parameter x. In this section, we focus on using certain multimode pure states $|\psi_0\rangle \in \mathcal{H}^{\otimes N}$ as probes for displacement metrology for paths generated by 1-local Hamiltonians, i. e., H of the form

$$H = \sum_{j=1} H^{(j)} \otimes \mathbb{I}_{N-1}.$$

Specifically, the metrological problem at hand consists of: 1) preparation of N oscillator modes in the probe state $|\mathrm{ECS}_N^+(\alpha)\rangle$ or $|\mathrm{HCS}_N^+(\alpha)\rangle$, 2) application of a global unitary operator $\otimes_{j=1}^N \exp(ixH_j)$ with $H_j = H$ being an oscillator Hamiltonian and $x \in \mathbb{R}$, and 3) estimation of x by an optimal separable measurement on the N modes. It is important to note that determination of the optimal separable measurement corresponding to the probe state and the Hamiltonian H requires methods of quantum estimation theory; in particular, the optimal measurement does not necessarily correspond with traditional methods of oscillator signal detection such has homodyne detection. In this section, we assume that the optimal measurement can be performed for any H and we determine the set of H for which $|\mathrm{ECS}_N^+(\alpha)\rangle$ and $|\mathrm{HCS}_N^+(\alpha)\rangle$ allow a greater precision in the determination of x than the respective tensor product branch states $|\pm\alpha\rangle^{\otimes N}$ and $|\psi_\pm\rangle^{\otimes N}$.

As an example of displacement estimation in a finite dimensional Hilbert space, we can consider the problem of estimation of a phase parameter θ imprinted on a quantum state $\rho_\theta = e^{-i\theta H}\rho_0 e^{i\theta H}$. We take

$$H = \sum_{j=1}^N \sigma_z^{(j)} \otimes \mathbb{I}_{N-1}$$

and take ρ_0 to correspond to the GHZ state $|0\rangle^{\otimes N} + |1\rangle^{\otimes N}/\sqrt{2}$. The quantum Fisher information of ρ_0 with respect to H is $4N^2$; in fact, this is the maximum possible value of the quantum Fisher information in $(\mathbb{C}^2)^{\otimes N}$ with respect to 1-local Hamiltonians of unit operator norm [3]. In contrast, any product state has the maximal quantum Fisher information of order N over the set of such Hamiltonians. This fact suggests an ordering of superposition states based on their maximal usefulness for quantum metrology as compared to the maximal usefulness of the individual pure states that comprise the superposition. The following definition serves to characterize the multimode, equal-weight superposition states in a separable Hilbert space as "metrologically useful"; such states are extensively more useful for displacement estimation of a predefined set of self-adjoint generators than the component branches.

Definition 1. An equal-weight quantum superposition of q linearly independent pure states,

$$|\omega\rangle \propto \sum_{j=1}^q |\psi_j\rangle \in (\ell^2(\mathbb{C}))^{\otimes N},$$

is considered metrologically useful when the following condition on the quantity $N^{rF}(|\omega\rangle)$ is satisfied:

$$N^{rF}(|\omega\rangle) := \frac{\max_{H\in\mathcal{A}_{1\text{-}loc}}\langle\omega|(\Delta H)^2|\omega\rangle}{\frac{1}{q}\sum_{j=1}^q \max\limits_{H\in\mathcal{A}_{1\text{-}loc}}\langle\psi_j|(\Delta H)^2|\psi_j\rangle} \in \mathcal{O}(n_{tot}), \quad (14)$$

where

$$\langle\cdot|(\Delta H)^2|\cdot\rangle := \langle\cdot|H^2|\cdot\rangle - \langle\cdot|H|\cdot\rangle^2,$$

$$n_{tot} = \langle\omega|\sum_{j=1}^N a_j^\dagger a_j \otimes \mathbb{I}_{N-1}|\omega\rangle$$

is the expected total photon number, $\mathcal{A}$ is an algebra of observables on $\ell^2(\mathbb{C})$, and $\mathcal{A}_{1\text{-}loc}$ is the linear subspace of $\mathcal{A}^{\otimes N}$ in which each element is "1-local," i. e., has the form $\sum_{j=1}^N x_j \otimes \mathbb{I}_{N-1}$ for $x_j \in \mathcal{A}$.

The set $\mathcal{A}_{1\text{-}loc}$ should be such that the denominator in Eq. (14) is nonzero. The restriction to 1-local observables in Definition 1 allows using product states as a scaling standard. Specifically, given $H = \sum_{j=1}^N x_j \in \mathcal{A}_{1\text{-}loc}$ (here, we have omitted the identity operators for clarity), and a product state $|\psi\rangle = |\psi^{(1)}\rangle \otimes \cdots \otimes |\psi^{(N)}\rangle$, it follows that

$$\langle\psi|(\Delta H)^2|\psi\rangle \le N\max_j \langle\psi^{(j)}|(\Delta x^{(j)})^2|\psi^{(j)}\rangle. \quad (15)$$

Hence, the variance of a measurement of a 1-local observable always scales linearly in the number of modes when the system is in a product state. A pure state $|\omega\rangle$ having the above form is metrologically useful if there exists H (having the 1-local form above) such that

$$\langle\omega|(\Delta H)^2|\omega\rangle \in \mathcal{O}(Nn_{tot}\max_{j,k}\langle\psi_j^{(k)}|(\Delta x^{(k)})^2|\psi_j^{(k)}\rangle).$$

The quantity N^{rF} was originally introduced as a measure of macroscopicity for quantum superpositions in $(\mathbb{C}^2)^{\otimes N}$ [27]; in that context, $\mathcal{A} = \mathfrak{su}(2,\mathbb{C})$ (represented by the Pauli matrices) and n_{tot} is taken to be equal to the number of modes N. The notion of metrological usefulness in Definition 1 refers to the greater ultimate precision achievable in the quantum Cramér–Rao bound when the displacement parameter is encoded in the equal-weight quantum superposition state $|\omega\rangle$ compared to the ultimate precision achievable when the displacement parameter is encoded in branches $\{|\psi_m\rangle\}_{m=1}^q$ comprising $|\omega\rangle$. It should be noted that we can speak of a superposition state as being metrologically useful only if the algebra $\mathcal{A}$ is specified. In addition, there may be many ways to write $|\omega\rangle$ as an equal-weight superposition of pure states; in that case, Definition 1 clearly refers to the metrological usefulness of $|\omega\rangle$ relative to a given decomposition of $|\omega\rangle$ into branches. In realistic parameter estimation protocols, the branch decomposition could be imposed by the preferred basis of an experiment.

In the case of a separable Hilbert space $\mathcal{H}$, the algebra $\mathcal{A}$ does not have to be represented in the von Neumann algebra $B(\mathcal{H})$. Many observables of interest, e.g., the quadrature operators and the photon number operator, are unbounded on $\mathcal{H}$ but appear in quantum optical Hamiltonians of interest to quantum metrology. However, for most quantum optical states of interest, these unbounded operators have a finite second moment [25] and therefore have a bounded variance in these states. In particular, if an unbounded, essentially self-adjoint operator $x = x^\dagger$ satisfies $\langle\omega|x^2|\omega\rangle < \infty$ for a normalized superposition state $|\omega\rangle = \sum_j |\psi_j\rangle$, then $\langle\psi_j|x^2|\psi_j\rangle < \infty$ for all j. This feature can be used to introduce a Lie algebra for which the state ω is metrologically useful [19]. $\mathcal{A}$ is then formed by taking the 1-local sums of essentially self-adjoint elements of this Lie algebra (we have here assumed a representation on $\mathcal{H}$). In order for the denominator of the expression for N^{rF} to be well-defined, at least one branch $|\psi_j\rangle$ of $|\omega\rangle$ must not be an eigenvector of all essentially self-adjoint elements of the Lie algebra.

A simple nontrivial example shows that $|\mathrm{ECS}_N^+(\alpha)\rangle$ is metrologically useful when $\mathcal{A}_{1\text{-}loc}$ is formed from observables of the Lie algebra $\mathfrak{h}_3 = (\mathrm{span}\{a^\dagger, a, \mathbb{I}\}, [\cdot,\cdot])$, represented as linear operators on $\ell^2(\mathbb{C})$ in the usual way. Given $\alpha \in \mathbb{C}$, the even and odd coherent states $|\psi_\pm\rangle$ (which coincide with $|\mathrm{ECS}_{N=1}^\pm(\alpha)\rangle$) exhibit an order-$|\alpha|^2$ variance for measurements of the quadrature corresponding to the direction $\mathrm{Arg}(\alpha)$ and exhibit squeezing in the variance of measurements of the conjugate quadrature corresponding to $\mathrm{Arg}(\alpha) + \pi/2$ [28]. Physically, this is because the quantized electric field is π phase-shifted (in expectation) between the $|\alpha\rangle$ and $|-\alpha\rangle$ coherent states. Explicitly, for $\alpha \in \mathbb{R}$ and the quadratures $x^{(\theta)}$ as above,

$$\langle\psi_+|(\Delta x^{(0)})^2|\psi_+\rangle = \alpha^2(1 + \tanh\alpha^2) + \frac{1}{2}$$

while

$$\langle\psi_+|(\Delta x^{(\pi/2)})^2|\psi_+\rangle = \frac{1}{2} - \alpha^2(1 - \tanh\alpha^2).$$

Thus, if α is purely real, the $\theta = 0$ quadrature exhibits large fluctuations, while the conjugate quadrature fluctuates just below the vacuum level. Since the only observables arising from $\mathfrak{h}_3$ are the oscillator quadratures and the identity, it is clear that an observable $\sum_{j=1}^N z_j a_j^\dagger + \overline{z}_j a_j$ exists that exhibits variance of the order of $N^2\alpha^2 \max_j |z_j|^2$ in $|\mathrm{ECS}_N^+(\alpha)\rangle$. On the other hand, since every quadrature $x^{(\theta)}$ exhibits variance $1/2$ in the coherent state $|\pm\alpha\rangle$, any 1-local observable $\sum_{j=1}^N z_j a_j^\dagger + \overline{z}_j a_j$ has a variance of the order of $N \max_j |z_j|^2$ in $|\pm\alpha\rangle^{\otimes N}$. Using Definition 1, we see that taking $\mathcal{A}_{1\text{-}loc}$ to be composed of observables from $\mathfrak{h}_3$ allows $|\mathrm{ECS}_N^\pm(\alpha)\rangle$ to be considered metrologically useful. In particular, by taking $z_j = x \in \mathbb{R}$, $|\mathrm{ECS}_N^\pm(\alpha)\rangle$ are metrologically useful for estimation of global amplitude displacements $\bigotimes_{j=1}^N D_j(x)$ of N-mode oscillators. The estimation of arbitrary local displacements in the complex plane comprises a multiparameter ($2N$ real parameters) estimation task [29]. It is an interesting problem whether a measure analogous to N^{rF} can be used to identify Schrödinger-cat states as a resource for parameter estimation of more general quantum dynamics.

It should be noted that $|\mathrm{ECS}_N^\pm(\alpha)\rangle$ is not metrologically useful when $\mathcal{A}_{1\text{-}loc}$ is composed of observables from the oscillator Lie algebra

$$\mathfrak{h}_4 = (\mathrm{span}\{a^\dagger a, a^\dagger, a, \mathbb{I}\}, [\cdot,\cdot])$$

instead of $\mathfrak{h}_3$. This is because the 1-local photon number operator $\sum_{j=1}^N a_j^\dagger a_j$ exhibits the extensive variance $N|\alpha|^2$ in the coherent states $|\pm\alpha\rangle^{\otimes N}$, and therefore the ratio in Definition 1 exhibits linear scaling with N, the number of modes, and not the total number of photons.

We now detail the argument that $|\mathrm{HCS}_N^\pm(\alpha)\rangle$ are metrologically useful when the algebra $\mathcal{A}$ is the Lie algebra of observables of $\mathfrak{sl}(2,\mathbb{C})$. Algebraically, it is simpler to show this fact for a closely related hierarchical cat state. By returning to (2) and taking $|\phi\rangle = |\psi_+\rangle$ and

$$U = e^{i\pi a^\dagger a/2}(|\psi_+\rangle\langle\psi_-| + |\psi_-\rangle\langle\psi_+|),$$

the following state is produced:

$$|\Omega(\alpha)\rangle = \frac{1}{\sqrt{2}}\left(\left(\frac{|\alpha\rangle + |-\alpha\rangle}{\sqrt{2 + 2e^{-2|\alpha|^2}}}\right)^{\otimes N} + \right.$$
$$\left. + \left(\frac{|i\alpha\rangle - |-i\alpha\rangle}{\sqrt{2 - 2e^{-2|\alpha|^2}}}\right)^{\otimes N}\right). \quad (16)$$

We consider the 1-local Hamiltonian

$$\sum_{j=1}^N (\overline{z}a^{(j)2} + za^{\dagger(j)2}) \otimes \mathbb{I}_{N-1}$$

as would describe two-photon parametric downconversion into N modes, each with a classical pumping amplitude $z \in \mathbb{C}$. Because

$$a^2|\psi_+\rangle = \alpha^2|\psi_+\rangle, \quad a^2 e^{i\pi a^\dagger a/2}|\psi_-\rangle = -\alpha^2 e^{i\pi a^\dagger a/2}|\psi_-\rangle,$$

it is clear that for any states

$$|\xi_1\rangle, |\xi_2\rangle \in \mathrm{span}_C\{|\psi_+\rangle, e^{i\pi a^\dagger a/2}|\psi_-\rangle\},$$

the following Pauli matrix/two-photon quadrature duality holds:

$$\langle\xi_1|\sigma_z|\xi_2\rangle = \langle\xi_1|\frac{1}{\alpha^2}(a^2 + a^{\dagger 2})|\xi_2\rangle, \tag{17}$$

where

$$\sigma_z = |\psi_+\rangle\langle\psi_+| - e^{i\pi a^\dagger a/2}|\psi_-\rangle\langle\psi_-|e^{-i\pi a^\dagger a/2}$$

is the appropriate Pauli matrix in $\mathrm{span}_C\{|\psi_+\rangle, \exp(i\pi a^\dagger a/2)|\psi_-\rangle\}$. From this, it is clear that the variance of

$$\sum_{j=1}^{N}(\overline{z}a^{(j)2} + za^{\dagger(j)2}) \otimes \mathbb{I}_{N-1}$$

in $|\Omega(\alpha)\rangle$ should be of the order of $N^2|z|^2|\alpha|^4$. In fact, for $\alpha \in \mathbb{R}$, the variance is

$$4N^2\,\mathrm{Re}(\overline{z}\alpha^2)^2 + \frac{N}{2}(4\,\mathrm{Re}(\overline{z}^2\alpha^4) - 8\,\mathrm{Re}(\overline{z}\alpha^2)^2 + \\ + 4|z|^2|\alpha|^2(\tanh\alpha^2 + \coth\alpha^2) + 2|z|^2), \tag{18}$$

which is of the order of $N^2|z|^2|\alpha|^4$ for $\mathrm{Arg}\,z = 2\mathrm{Arg}\,\alpha$. In addition, the variance of $\sum_{j=1}^{N}(\overline{z}a^{(j)2} + za^{\dagger(j)2}) \otimes \mathbb{I}_{N-1}$ in the product states $|\psi_+\rangle^{\otimes N}$ or $(\exp(i\pi a^\dagger a/2)|\psi_-\rangle)^{\otimes N}$ is at most of the order of $N|z|^2|\alpha|^2$, as can easily be verified. The final step in finding a minimal algebra $\mathcal{A}$ that allows $|\Omega(\alpha)\rangle$ to be metrologically useful is to append the element $(1/2)a^\dagger a + 1/4$ to the set $\{a^{\dagger 2}/2, a^2/2\}$ and check that the 1-local observable given by, e.g., $\sum_{j=1}^{N} a_i^\dagger a_i$ does not exhibit fluctuations in either of the branch states $|\psi_+\rangle^{\otimes N}$ or $(\exp(i\pi a^\dagger a/2)|\psi_-\rangle)^{\otimes N}$ scaling as $|\alpha|^4$. If the 1-local photon number operator were to exhibit such fluctuations, then the ratio in the left-hand side of Eq. (14) would lose the property of scaling with the total expected number of photons in $|\Omega(\alpha)\rangle$. It is easy to verify that the 1-local photon number operator exhibits variance of the order of $N|\alpha|^2$ in these product states, and hence $|\Omega(\alpha)\rangle$ is metrologically useful when

$$\mathcal{A} = \mathfrak{sl}(2,\mathbb{C}) := \left(\mathrm{span}_{\mathbb{C}}\left\{\frac{1}{2}a^\dagger a + \frac{1}{4}, \frac{a^{\dagger 2}}{2}, \frac{a^2}{2}\right\}, [\cdot,\cdot]\right).$$

In particular, when $z \in \mathbb{R}$, the calculation above shows that $|\mathrm{HCS}_N^{\pm}(\alpha)\rangle$ are metrologically useful for displacement estimation, where the displacement parameter now corresponds to the global squeezing amplitude $z \in \mathbb{R}$. It is intriguing that while some superpositions of the $\mathfrak{sp}(N,\mathbb{C})$ Barut–Girardello coherent states do exhibit squeezing, the $|\mathrm{HCS}_N^{+}(\alpha)\rangle$ state does not; in addition, the product states comprising each of the branches of $|\mathrm{HCS}_N^{+}(\alpha)\rangle$ exhibit negligible squeezing if $|\alpha|^2 > 1$.

However, if squeezed states and their superpositions are available, one may wonder if there exist other types of hierarchical cat states having N^{rF} scaling exponentially in a squeezing parameter when the observables of $\mathcal{A}_{1\text{-}loc}$ are taken from $\mathfrak{sl}(2,\mathbb{C})$. Indeed, it is known that squeezed states provide a higher precision in the estimation of a single-mode squeezing parameter than coherent states do [30]. The following hierarchical cat state, having branches composed of superpositions of ideal squeezed states, allows such scaling:

$$\frac{1}{\sqrt{2}}\left(\left(\frac{(D(\alpha)+D(-\alpha))S(w)}{\sqrt{2+2\exp(-2\alpha^2e^{2w})}}|0\rangle\right)^{\otimes N} + \\ + \left(\frac{(D(i\alpha)-D(-i\alpha))S(w)}{\sqrt{2-2\exp(-2\alpha^2e^{2w})}}|0\rangle\right)^{\otimes N}\right), \tag{19}$$

where we assume that $\alpha, w \in \mathbb{R}_{>0}$ and take

$$S(w) := \exp\left\{\frac{1}{2}(\overline{w}a^2 - wa^{\dagger 2})\right\}$$

as the unitary squeezing operator. For such α and w, the identity

$$D(\alpha)S(w) = S(w)D(\alpha e^w)$$

holds and hence the above hierarchical cat state can be rewritten as $S(w)^{\otimes N}|\Omega(\alpha e^w)\rangle$. The N^{rF} value of $S(w)^{\otimes N}|\Omega(\alpha e^w)\rangle$ exhibits the same scaling as the N^{rF} value for $|\Omega(\alpha e^w)\rangle$, i.e., of the order of $N\alpha^2e^{2w}$, because $S(w)$ acts on $\mathfrak{sl}(2,\mathbb{C})$ by the adjoint action. As a final remark, we point out that the coherent states $\pm|\alpha\rangle$ are minimum-uncertainty states for the Heisenberg uncertainty relation for observables of $\mathfrak{h}_3$ while the even/odd coherent states $|\psi_\pm\rangle$ are minimum-uncertainty states for the generalized uncertainty relation for observables of $\mathfrak{su}(2,\mathbb{C})$ [31]. This is not a surprising coincidence because the definition of metrological usefulness (Definition 1) requires that the maximum uncertainty in the product states comprising the branches of a state having form (1) be extensively smaller than the maximum uncertainty in the multimode superposition state.

Thus far, the discussion of Definition 1 has been mainly mathematical. It is useful to mention that from a basic physical perspective, the problem of determining the precision of optimal estimation of a global real displacement parameter is equivalent to determing the energy–time uncertainty in a given quantum state. Therefore, Definition 1 can be reinterpreted from a physical perspective by stating that $|\omega\rangle$ is metrologically useful with respect to Hamiltonians $H \in \mathcal{A}_{1\text{-}loc}$ if its maximal decay rate (i.e., the minimal time t for which $e^{-iHt}|\omega\rangle$ becomes distinguishable from $|\omega\rangle$) is extensively greater than the maximal decay rates of the branch states $\{|\psi_j\rangle\}_{j=1}^{q}$, i.e., if $|\omega\rangle$ is extensively more sensitive to evolution generated by H as compared to the branches. In the particular case of $|\mathrm{HCS}_N^+(\alpha)\rangle$ ($|\mathrm{ECS}_N^+(\alpha)\rangle$), we can say qualitatively that its metrological usefulness arises simply because its intermode quantum coherence causes the squeezing operation (displacement operation) to change it more drastically than the product states $|\psi_\pm\rangle^{\otimes N}$ ($|\pm\alpha\rangle^{\otimes N}$) considered independently.

3.2. Entanglement entropy

Because of the orthogonality of the branches, the mode entanglement structure of $|\mathrm{HCS}_2^\pm\rangle$ is the same as that of the GHZ states in $(\mathbb{C}^2)^{\otimes 2}$. Hence, $|\mathrm{HCS}_2^\pm\rangle$ exhibits the maximal entanglement entropy in the $(\mathbb{C}^2)^{\otimes 2}$ subspace spanned by $\{|e_i\rangle \otimes |e_j\rangle\}_{i,j=1,2}$ introduced in Sec. 1. In particular, $|\mathrm{HCS}_2^\pm(\alpha)\rangle$ exhibits greater mode entanglement than the subset of entangled coherent states that cannot be expressed in form (2). However, the hierarchical photonic superpositions are not maximally entangled states in $\ell^2(\mathbb{C})\otimes\ell^2(\mathbb{C})$. The dissipative dynamics of the entanglement entropy of entangled coherent states was studied in Refs. [18, 32, 33].

It is known that a nonclassical product state incident on a beam splitter does not necessarily generate entanglement between the output modes [34]. In fact, a 50:50 beam splitter destroys the entanglement of a two-mode squeezed state [35]. It is easy to see that a beam splitter described by the unitary operation

$$B(\theta) = \exp\left\{\frac{i\theta}{2}(a_1^\dagger a_2 + a_2^\dagger a_1)\right\}$$

acting on two input photonic modes maps $|\mathrm{HCS}_2^+(\alpha)\rangle$ into the state

$$\begin{aligned}\frac{1}{\sqrt{2}}\Bigg(&\frac{1}{1-e^{-4|\alpha|^2}}(|\alpha e^{i\theta/2}\rangle \otimes |\alpha e^{i\theta/2}\rangle + \\ &+ |-\alpha e^{i\theta/2}\rangle \otimes |-\alpha e^{i\theta/2}\rangle) - \\ &- \frac{1}{2\,\mathrm{sh}(2|\alpha|^2)}(|\alpha e^{-i\theta/2}\rangle \otimes |-\alpha e^{-i\theta/2}\rangle + \\ &+ |-\alpha e^{-i\theta/2}\rangle \otimes |\alpha e^{-i\theta/2}\rangle)\Bigg). \end{aligned} \tag{20}$$

For moderately large $|\alpha|$, the exponentially decaying term becomes negligible and we are left with an entangled coherent state in the output modes of the beam splitter. Hence, the beam splitter does not destroy the entanglement of $|\mathrm{HCS}_2^+(\alpha)\rangle$ for any values of the transmission and reflection amplitudes. The exact entanglement entropy of $B(\theta)|\mathrm{HCS}_2^+(\alpha)\rangle$, calculcated as the von Neumann entropy of the reduced density matrix, is shown in Fig. 3 for a range of real α and θ. Except for low-power ($\alpha \lesssim 1$) $|\mathrm{HCS}_2^+(\alpha)\rangle$ states, the maximum entanglement entropy is maintained throughout the range of transmission amplitudes of the beam splitter.

The quantification of entanglement in terms of an entropic quantity naturally leads to questions about its fluctuations. Entanglement fluctuations can be interpreted as the root variance of a measurement of the entanglement Hamiltonian [36]: in terms of the reduced density matrix ρ_A of a pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$, it is given by the expression

$$\Delta S_E := \sqrt{\mathrm{tr}(\rho_A H_E^2) - \mathrm{tr}(\rho_A H_E)^2},$$

where $H_E := -\log_2 \rho_A$. We show the entanglement fluctuation of $B(\theta)|\mathrm{HCS}_2^+(\alpha)\rangle$ in Fig. 4.

We now show that in the Bell basis

$$\{|\mathrm{HCS}_2^+(\alpha)\rangle, |\mathrm{HCS}_2^-(\alpha)\rangle, |\mathrm{ECS}_2^-(\alpha)\rangle,\\ (\exp(i\pi a^\dagger a)\otimes\mathbb{I})|\mathrm{ECS}_2^-(\alpha)\rangle\},$$

the hierarchical cat states, in some sense, comprise the most stable entanglement resource. We consider each mode coupled independently to a zero-temperature bath of photons, each bath having an absorption rate Γ, with the non-Hamiltonian part of the evolution given by

$$\rho' = \frac{\Gamma}{2}\sum_{j=1}^{2}[a, \rho(t)a^\dagger] + [a\rho, a^\dagger]. \tag{21}$$

This is the case of (independent) Lindbladian amplitude damping. For an initial state $\rho(t=0)$ given by an entangled coherent state, it follows from the well-known solution of the amplitude damping master equation [37] that the $t \to \infty$ asymptotic state is unentangled. In contrast, $|\mathrm{HCS}_2^\pm(\alpha)\rangle$ maintain a nearly maximal entanglement entropy S_E throughout the non-Hamiltonian evolution as long as $|\alpha|^2 \gtrsim 1$, as seen in Fig. 5. The entanglement entropy for this state was calculated from an analytic expression, which we omit. It should be noted that for $|\alpha| \lesssim 1$, the entanglement entropy decays with time, but is still substantial for $t \gtrsim \Gamma^{-1}$. The persistence of entanglement during the amplitude damping can be simply seen by considering a limit model. By taking the low-power limit $\alpha \to 0$, it is clear that

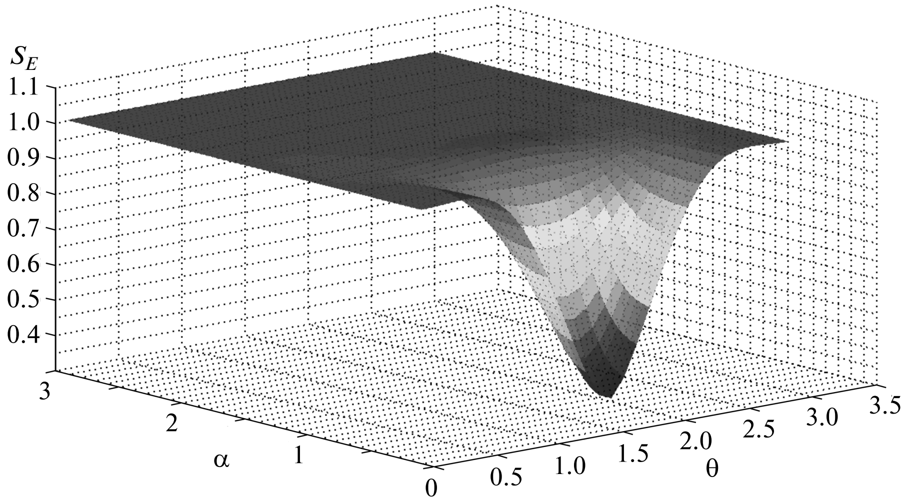

Fig. 3. The entanglement entropy of $U(\theta)|\mathrm{HCS}^+_{N=2}(\alpha)\rangle$, $\alpha \in [0.2, 3.0]$, and $\theta \in [0.1, \pi - 0.1]$

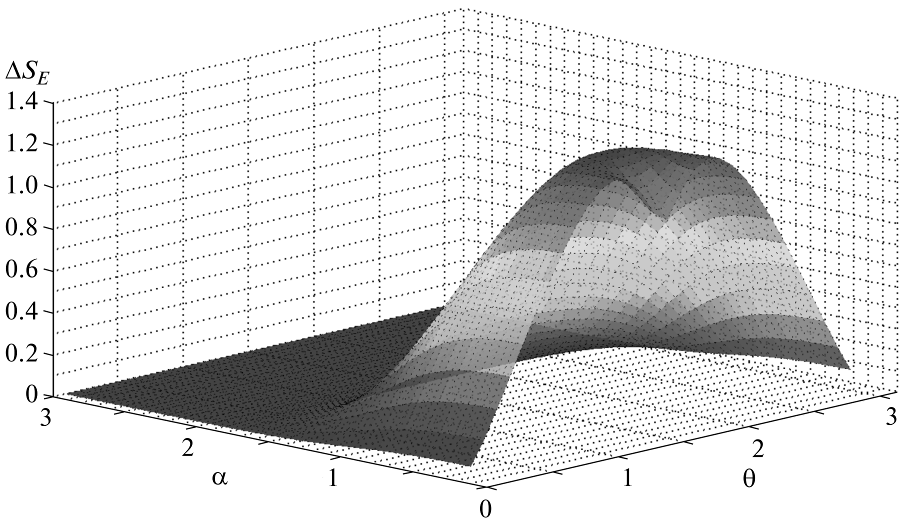

Fig. 4. The entanglement entropy fluctuations of $U(\theta)|\mathrm{HCS}^+_{N=2}(\alpha)\rangle$, $\alpha \in [0.2, 3.0]$ and $\theta \in [0.1, \pi - 0.1]$

$|\mathrm{HCS}_2^+(\alpha)\rangle$ exhibits an inner product of magnitude 1 with a state

$$\frac{1}{2}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}), \tag{22}$$

which is a superposition of the two-mode vacuum and the product Fock state $|1\rangle \otimes |1\rangle$. Under the amplitude damping map defined in Eq. (21), state (22) evolves to

$$\frac{1}{2}\left\{|0\rangle\langle 0| \otimes |0\rangle\langle 0| + e^{-2\Gamma t}(|0\rangle\langle 1| \otimes |0\rangle\langle 1| + \right. \\ + |1\rangle\langle 0| \otimes |1\rangle\langle 0|) + (e^{-2\Gamma t}|1\rangle\langle 1| + \\ + (1 - e^{-2\Gamma t})|0\rangle\langle 0|) \otimes (e^{-2\Gamma t}|1\rangle\langle 1| + \\ \left. + (1 - e^{-2\Gamma t})|0\rangle\langle 0|)\right\}. \tag{23}$$

Taking the partial trace to form $\rho_1(t)$, we compute

$$\lim_{t\to\infty} -\mathrm{tr}(\rho_1(t)\log_2 \rho_1(t)) = 1.$$

The robustness of the entanglement entropy of $|\mathrm{HCS}_2^+(\alpha)|$ under amplitude damping exhibited for large $|\alpha|$ is proved by considering the $|\alpha| \to \infty$ asymptotics regime. An explicit calculation shows that for any finite $|\alpha|$,

$$\lim_{t\to\infty} S_E = 0,$$

whereas

$$\lim_{t\to\infty} \lim_{|\alpha|\to\infty} S_E = 1.$$

The increased stability of the entanglement of hierarchical cat states to local amplitude damping (relative to

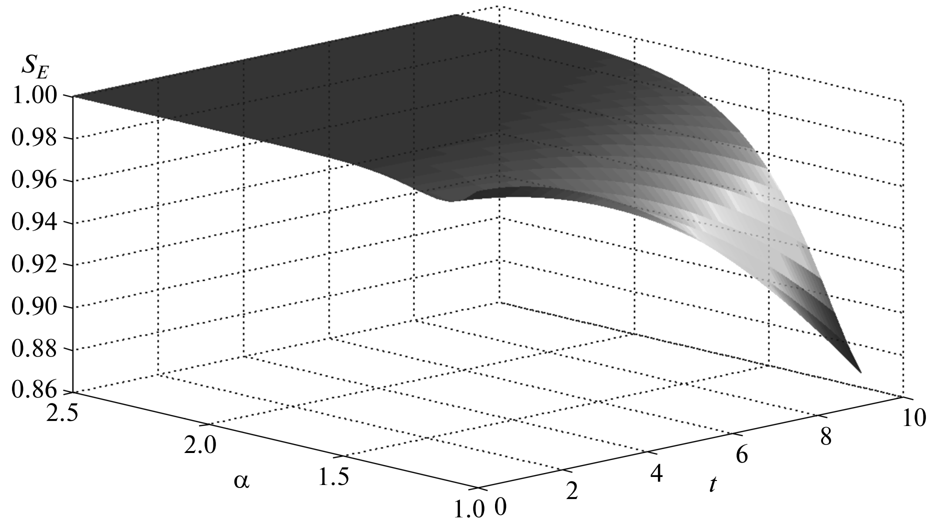

Fig. 5. The entanglement entropy of the image of $|\mathrm{HCS}^{+}_{N=2}(\alpha)\rangle$ under independent amplitude damping for $\Gamma = 0.1$, $\alpha \in [1, 2.5]$, and $t \in [0, 9]$

the entanglement of entangled coherent states) makes these states desirable targets for quantum optical state engineering and optical quantum communication.

4. GENERATION OF $|\mathrm{HCS}^{+}_{2}(\alpha)\rangle$

In this section, we limit ourselves to proposals for experimental generation of the two-mode hierarchical superposition state $|\mathrm{HCS}^{+}_{2}(\alpha)\rangle$ because the main difficulties are already present in this case. For all of the proposals we describe, a generalization to $N > 2$ requires the experimenter to overcome a linear (with N) increase in errors associated with imperfect implementation of the required unitary operations, in addition to the usual problem of decoherence due to photon losses.

If an experimenter has access to arbitrary unitary operations in the $\mathbb{C}^2$ sub-Hilbert space spanned by $|\pm\alpha\rangle$ over a range of amplitudes α, then $|\mathrm{HCS}^{+}_{2}(\alpha)\rangle$ can be readily generated. Specifically, one applies the "$\pi/2$" (or "50:50") beam splitter

$$U_{12}(\pi/2) = \exp\left\{\frac{1}{2}\,\frac{\pi}{2}(a_1^\dagger a_2 - a_2^\dagger a_1)\right\}$$

to the product state $\propto (|\sqrt{2}\alpha\rangle_1 - |-\sqrt{2}\alpha\rangle_1) \otimes |0\rangle_2$ to produce the state $\propto |\alpha\rangle_1|-\alpha\rangle_2 - |-\alpha\rangle_1|\alpha\rangle_2$. Applying the phase shift $\exp(i\pi a_2^\dagger a_2)$ produces the Bell state $(1/\sqrt{2})(|\psi_+\rangle|\psi_-\rangle + |\psi_-\rangle|\psi_+\rangle)$. Applying $\sigma_x = |\psi_+\rangle\langle\psi_-| + |\psi_-\rangle\langle\psi_+|$ to mode 2 produces $|\mathrm{HCS}^{+}_{2}(\alpha)\rangle$; alternatively, $|\mathrm{HCS}^{-}_{2}(\alpha)\rangle$ is produced (up to a global phase) conditional on the application of the annihilation operator $\mathbb{I} \otimes a_2$ to the above Bell state. Along these lines, the method in Ref. [38] for preparing entangled coherent states by a coherent photon loss may be modified in a simple way to produce the family of states

$$\frac{1}{\sqrt{2}}\left(|\psi_+\rangle^{\otimes 2} \pm e^{-i\theta}|\psi_-\rangle^{\otimes 2}\right). \tag{24}$$

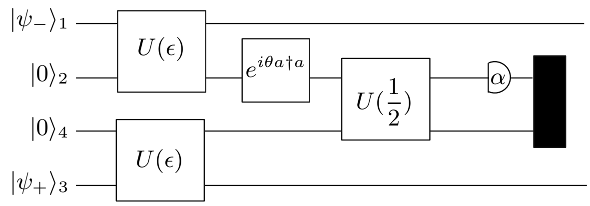

Fig. 6. Quantum circuit diagram for the transformation $|\psi_-\rangle_1 \otimes |0\rangle_2 \otimes |\psi_+\rangle_3 \otimes |0\rangle_4 \to \frac{1}{\sqrt{2}}(|\psi_+\rangle_1 \otimes |\psi_+\rangle_3 + e^{-i\theta}|\psi_-\rangle_1 \otimes |\psi_-\rangle_3)$

The method is based on the observation that a coherent photon loss can generate a photonic HCS from a product of single-mode Schrödinger-cat states. For example, the above HCS state is equivalent (in the projective Hilbert space) to $(a - be^{-i\theta})(|\psi_-\rangle_a \otimes |\psi_+\rangle_b)$. The implementation of the coherent photon loss via a linear quantum optical circuit is shown in Fig. 6. In detail, we append vacuum modes to the tensor product $|\psi_-\rangle \otimes |\psi_+\rangle$ to form the initial state $|\psi_-\rangle_1 \otimes |0\rangle_2 \otimes |\psi_+\rangle_3 \otimes |0\rangle_4$:

$$U_{ij}(\epsilon) := \exp\left(\frac{1}{2}\epsilon(a_i^\dagger a_j - a_j^\dagger a_i)\right)$$

is a 50:50 beam splitter with $\epsilon \ll 1$, i. e., the beam splitter is highly transmissive for mode i. Applying

$$U_{24}(\pi/2)\exp(i\phi a_2^\dagger a_2)U_{12}(\epsilon)U_{34}(\epsilon)$$

to the initial state produces

$$\begin{aligned}
&|\alpha\cos\epsilon\rangle_1 \otimes |-\alpha\frac{e^{i\phi}+1}{\sqrt{2}}\sin\epsilon\rangle_2 \otimes |\alpha\frac{e^{i\phi}-1}{\sqrt{2}}\sin\epsilon\rangle_4 \otimes \\
&\otimes |\alpha\cos\epsilon\rangle_3 + |\alpha\cos\epsilon\rangle_1 \otimes |-\alpha\frac{e^{i\phi}-1}{\sqrt{2}}\sin\epsilon\rangle_2 \otimes \\
&\otimes |\alpha\frac{e^{i\phi}+1}{\sqrt{2}}\sin\epsilon\rangle_4 \otimes |-\alpha\cos\epsilon\rangle_3 - |-\alpha\cos\epsilon\rangle_1 \otimes \\
&\otimes |\alpha\frac{e^{i\phi}-1}{\sqrt{2}}\sin\epsilon\rangle_2 \otimes |-\alpha\frac{e^{i\phi}+1}{\sqrt{2}}\sin\epsilon\rangle_4 \otimes |\alpha\cos\epsilon\rangle_3 - \\
&- |-\alpha\cos\epsilon\rangle_1 \otimes |\alpha\frac{e^{i\phi}+1}{\sqrt{2}}\sin\epsilon\rangle_2 \otimes \\
&\otimes |-\alpha\frac{e^{i\phi}-1}{\sqrt{2}}\sin\epsilon\rangle_4 \otimes |-\alpha\cos\epsilon\rangle_3. \qquad (25)
\end{aligned}$$

The coherent photon loss is now implemented by photodetection on mode 2, modeled by application of the annihilation operator a_2. In the final step, we trace over modes 2 and 4. In the $\epsilon \to 0$ limit, the "+" state in (24) is produced; if photodetection is carried out on mode 4 instead of mode 2, then the "−" state is produced. For the initial superpositions $|\psi_\pm\rangle$ with large $|\alpha|^2$, ϵ must be concomitantly decreased to maintain high fidelity of the output state to $|\mathrm{HCS}_2^\pm(\alpha)\rangle$. The decrease in ϵ necessarily increases noise in the photodetection process. In addition, for large $|\alpha|^2$, it is vital to generate the initial product state $|\psi_-\rangle_1 \otimes |\psi_+\rangle_3$ with high fidelity. The next method that we discuss readily satisfies this requirement.

The experimental generation of single-mode photonic Schrödinger-cat states $|\psi_\pm\rangle$ via dispersive interaction between the monochromatic electromagnetic field and a superconducting two-level system [39] or a Rydberg atom [40] provides some clues toward feasible methods for preparation of photonic HCSs. To extend these protocols to the many-mode case, one must effectively entangle the field states of spatially separated resonating cavities. For example, it has been proposed to generate entangled coherent states by sequential coupling of a Rydberg atom to two microwave cavities [41]. In general, proposals for creating entangled field states involve coupling the field modes to easily controllable, low-dimensional quantum systems.

A simple scheme for generating $|\mathrm{HCS}_2^+(\alpha)\rangle$ from a tensor product of even coherent states $|\psi_+\rangle \otimes |\psi_+\rangle$ is as follows:

$$\begin{aligned}
|\psi_+\rangle \otimes |\psi_+\rangle \xrightarrow{H\otimes\mathbb{I}} \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle) \otimes \\
\otimes |\psi_+\rangle \xrightarrow{\mathrm{CNOT}} |\mathrm{HCS}_2^+(\alpha)\rangle, \qquad (26)
\end{aligned}$$

where $H := (1/\sqrt{2})(\sigma_x + \sigma_z)$ is the Hadamard gate in the subspace $\mathcal{K}$ spanned by the orthonormal basis of even/odd coherent states and $\mathrm{CNOT} := |\psi_+\rangle \times \times \langle\psi_+|\otimes\mathbb{I} + |\psi_-\rangle\langle\psi_-|\otimes\sigma_x$ is the conditional σ_x operation on the second field mode. To implement the Hadamard operation, it is sufficient to generate the superposition

$$\begin{aligned}
H|\psi_+\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle) = \\
= \frac{\sqrt{1+e^{-2\alpha^2}} + \sqrt{1-e^{-2\alpha^2}}}{2\sqrt{1-e^{-4\alpha^2}}}|\alpha\rangle + \\
+ \frac{\sqrt{1+e^{-2\alpha^2}} - \sqrt{1-e^{-2\alpha^2}}}{2\sqrt{1-e^{-4\alpha^2}}}|-\alpha\rangle. \qquad (27)
\end{aligned}$$

Arbitrary superpositions of photonic coherent states $|\pm\alpha\rangle$ can be generated by a dispersive coupling between a coherent microwave field $|\alpha\rangle$ and a transmon qubit if the transmon qubit can be prepared in an arbitrary pure state in $\mathbb{C}^2$ [39]. In addition, it has been proposed to generate parametrically tuning states [42]. It is worth noting that $H|\psi_+\rangle$ is an eigenvector of the operator $|\alpha\rangle\langle\alpha| - |-\alpha\rangle\langle-\alpha|$, which is proportional to the observable corresponding to the measurement that optimally detects $|\alpha\rangle$ or $|-\alpha\rangle$ (in the sense of quantum binary distinguishability problem with equal *a priori* probabilities and Bayes' cost criterion [43]) with maximal probability of success. The pure states $H|\psi_\pm\rangle$ have been studied for their role in optimal detection of coherent states $|\pm\alpha\rangle$ (the "binary phase shift key") [44].

The CNOT gate in the scheme (26) is more difficult to engineer than the Hadamard gate because it requires not only a large intramode coherence time for the even and odd coherent states but also a large intermode coherence between two microwave cavities. However, if two transmon qubits can be prepared in a maximally entangled (i. e., GHZ) state in $(\mathbb{C}^2)^{\otimes 2}$ and independently coupled to spatially separated photonic modes of microwave cavities via a dispersive interaction, this CNOT gate can be implemented. We now provide the details for factoring the unitary operator corresponding to the CNOT gate into easily implementable unitary operations on the field/qubit and qubit/qubit subsystems.

We first note that we can factor the CNOT gate on

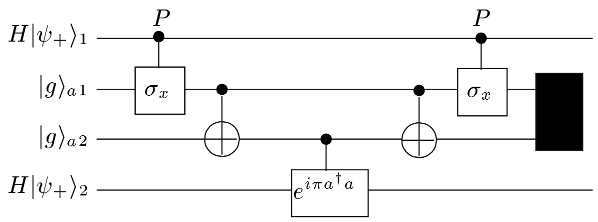

Fig. 7. Quantum circuit diagram for the transformation $H|\psi_+\rangle_1 \otimes H|\psi_+\rangle_2 \otimes |g\rangle_{a_1} \otimes |g\rangle_{a_1} \to |\mathrm{HCS}_2^+(\alpha)\rangle_{1,2}$

$\mathcal{K} \otimes \mathcal{K}$ into the following product of local Hadamard gates and the conditional σ_z gate:

$$\mathrm{CNOT} = (\mathbb{I} \otimes H)(|\psi_+\rangle\langle\psi_+| \otimes \otimes \mathbb{I} + |\psi_-\rangle\langle\psi_-| \otimes \sigma_z)(\mathbb{I} \otimes H). \quad (28)$$

We have already described the procedure for applying a Hadamard gate to the field via the local coupling of the field mode and transmon qubit; hence, we take the initial state to be $H|\psi_+\rangle_1 \otimes H|\psi_+\rangle_2 \otimes |g\rangle_{a_1} \otimes |g\rangle_{a_1}$ (where we now explicitly include the field mode labels 1, 2 and the transmon qubit mode labels a_1, a_2) and show how to implement the conditional σ_z gate. Let an orthonormal basis for a transmon qubit Hilbert space be taken as $\{|g\rangle, |e\rangle\}$. A quantum circuit diagram showing our method for indirectly performing the CNOT gate on the initial product state is shown in Fig. 7. In this circuit, the first field/qubit operation is a π rotation of the qubit a_1 conditioned on the parity of field mode 1 and is labeled in Fig. 7 by the operation with the P superscript. Explicitly, this unitary operation is given by

$$|\psi_+\rangle_1\langle\psi_+|_1 \otimes \mathbb{I} + |\psi_-\rangle_1\langle\psi_-|_1 \otimes \sigma_x. \quad (29)$$

A similar conditional transformation has been achieved experimentally in Ref. [39]. This transformation should be followed by a CNOT gate between the qubit modes a_1 and a_2, as shown; we assume that this gate is accessible with high fidelity by precise control of the qubit–qubit state. At this point, the full normalized state is given by

$$\frac{1}{2}\left((|\psi_+\rangle_1 \otimes |\psi_+\rangle_2 + |\psi_+\rangle_1 \otimes |\psi_-\rangle_2) \otimes |g\rangle_{a_1} \otimes |g\rangle_{a_2} + \right. \\ + (|\psi_-\rangle_1 \otimes |\psi_+\rangle_2 + |\psi_-\rangle_1 \otimes |\psi_-\rangle_2) \otimes \\ \left. \otimes |e\rangle_{a_1} \otimes |e\rangle_{a_2}\right). \quad (30)$$

The next step is a π rotation of field 2 conditioned on the state of the qubit a_2. This operation has been implemented in the experiment reported in Ref. [39]. We recall from Sec. 3 that the π phase rotation operator acts like σ_z in the subspace $\mathcal{K}$. Applying the CNOT gate between qubits a_1 and a_2 again, followed by the parity-conditioned π qubit rotation, gives the desired CNOT gate in $\mathcal{K} \otimes \mathcal{K}$. Finally, applying the local Hadamard operator $\mathbb{I} \otimes H$ produces the output state $|\mathrm{HCS}_2^+(\alpha)\rangle \otimes |g\rangle_{a_1} \otimes |g\rangle_{a_2}$.

The above method for generating $|\mathrm{HCS}_2^+(\alpha)\rangle$ is not the most efficient possible. It would be favorable to utilize a single qubit or few-level mode that can be sequentially entangled with both fields [45].

5. MORE EXOTIC HIERARCHICAL SUPERPOSITIONS

The notion of hierarchical cat states can be extended to deeper levels of the hierarchy. The principal motivation for an analysis of these states comes from the theory of quantum error correction, which uses encoded states to strengthen quantum information against unwanted decoherence. In Refs. [46, 47], a class of "concatenated" GHZ states of the form

$$|\text{C-GHZ}_{M,N}^+\rangle := \frac{1}{\sqrt{2}}(|\mathrm{GHZ}_N\rangle^{\otimes M} + |\mathrm{GHZ}_N\rangle^{\otimes M})$$

were introduced as entangled states that are relatively stable with respect to local noise compared to the full GHZ state $|\mathrm{GHZ}_{NM}^+\rangle$. An analog of the C-GHZ states in $(\mathbb{C}^2)^{\otimes MN}$ can be constructed from an HCS in $\ell^2(\mathbb{C})^{\otimes MN}$ by forming

$$|\text{C-HCS}_{M,N}^\pm\rangle := \\ := \frac{1}{\sqrt{2}}\left(|\mathrm{HCS}_N^+\rangle^{\otimes M} \pm |\mathrm{HCS}_N^-\rangle^{\otimes M}\right). \quad (31)$$

This state can retain coherence on the scale of N modes even after global coherence on the scale of all MN modes has been lost. The C-HCS states are expected to be useful as encoded photonic states for continuous-variable quantum error correction schemes. Of course, the entangled coherent states can be concatenated in a similar way:

$$|\text{C-ECS}_{M,N}^\pm(\alpha)\rangle := \frac{1}{\sqrt{2}} \times \\ \times \left(|\mathrm{ECS}_N^+(\alpha)\rangle^{\otimes M} \pm |\mathrm{ECS}_N^-(\alpha)\rangle^{\otimes M}\right). \quad (32)$$

It also follows from the basic theory of quantum binary distinguishability that the optimal projection-valued measurement for distinguishing $|\alpha\rangle^{\otimes N}$ from $|-\alpha\rangle^{\otimes N}$ has elements

$$\{|\text{C-ECS}^{+}_{1,N}(\alpha)\rangle\langle\text{C-ECS}^{+}_{1,N}(\alpha)|,\\ |\text{C-ECS}^{-}_{1,N}(\alpha)\rangle\langle\text{C-ECS}^{-}_{1,N}(\alpha)|\}.$$

The C-HCS and C-ECS states are robust quantum resources in the sense that if coherence is lost among the M blocks of N single-mode systems, a statistical mixture of N-mode entangled states remains. To lose all entanglement, the intermode coherence in $\mathcal{H}^{\otimes N}$ must subsequently be lost. In a higher-order hierarchical cat state, these "shells" of coherence degrade according to the strengths of local and nonlocal interactions. It has been suggested to generate $|\text{C-GHZ}^{+}_{M,N}\rangle$ in spin-1/2 chains by application of the 2-local Mølmer–Sorensen unitary gate to the NM-mode GHZ state $(1/\sqrt{2})(|0\rangle^{\otimes MN} + |1\rangle^{\otimes MN})$ [47]. Efficient preparation of hierarchically encoded entangled states in $(\ell^2(\mathbb{C}))^{\otimes NM}$ represents a great challenge for continuous-variable quantum information processing.

The author acknowledges support from the Howard W. Crandall Memorial Fund at the UC Berkeley.

REFERENCES

1. D. Greenberger, M. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, Kluwer, Dordrecht (1989), p. 69.
2. M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, Cambridge Univ. Press, Cambridge (2000).
3. P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, Phys. Rev. A **85**, 022321 (2012).
4. G. Tóth, Phys. Rev. A **85**, 022322 (2012).
5. V. Dodonov and V. Man'ko, *Theory of Nonclassical States of Light*, Taylor & Francis (2003).
6. B. Sanders, J. Phys. A: Math. Theor. **45**, 244002 (2012).
7. G. Najarbashi and Y. Maleki, Int. J. Theor. Phys. **50**, 2601 (2011).
8. P. Munhoz, F. L. Semião, A. Vidiella-Barranco, and J. Roversi, Phys. Lett. A **372**, 3580 (2008).
9. T. Ralph, A. Gilchrist, G. Milburn, W. Munro, and S. Glancy, Phys. Rev. A **68**, 042319 (2003).
10. A. Gilchrist, K. Nemoto, W. Munro, T. Ralph, S. Glancy, S. Braunstein, and G. Milburn, J. Opt. B: Quantum Semiclass. Opt. **6**, S828 (2004).
11. V. Dodonov, I. Malkin, and V. Man'ko, Physica **72**, 597 (1974).
12. T. Volkoff and K. Whaley, Phys. Rev. A **91**, 012122 (2014).
13. D. Trifonov, J. Phys. A: Math. Gen. **31**, 5673 (1998).
14. N. Ansari and V. Man'ko, Phys. Rev. A **50**, 1942 (1994).
15. V. Dodonov and L. de Souza, J. Russ. Laser Res. **28**, 453 (2007).
16. S. van Enk and O. Hirota, Phys. Rev. A **64**, 022313 (2001).
17. X. Wang, J. Phys. A: Math. Gen. **35**, 165 (2002).
18. H. Jeong, M. Kim, and J. Lee, Phys. Rev. A **64**, 052308 (2001).
19. T. Volkoff and K. Whaley, Phys. Rev. A **90**, 062122 (2014).
20. V. Bužek, J. Mod. Opt. **37**, 303 (1990).
21. K. Cahill and R. Glauber, Phys. Rev. **177**, 1882 (1969).
22. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, Cambridge Univ. Press, Cambridge (1995).
23. V. Bargmann, Comm. Pure Appl. Math. **14**, 187 (1961).
24. C. W. Helstrom, *Quantum Detection and Estimation Theory*, Acad. Press, New York (1976).
25. A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam (1982).
26. M. Paris, Int. J. Quant. Inf. **7**, 125 (2009).
27. F. Fröwis and W. Dür, New J. Phys. **14**, 093039 (2012).
28. V. Bužek and P. Knight, in *Progress in optics XXXIV*, Elsevier (1995), p. 1.
29. M. Genoni, M. Paris, G. Adesso, H. Nha, P. Knight, and M. Kim, Phys. Rev. A **87**, 012107 (2013).
30. G. Chiribella, G. D'Ariano, and M. Sacchi, Phys. Rev. A **73**, 062103 (2006).
31. C. Brif, Ann. Phys. (N.Y.) **251**, 180 (1996).
32. F. Lastra, G. Romero, C. López, N. Zagury, and J. Retamal, Opt. Comm. **283**, 3825 (2010).
33. H. Jeong and M. Kim, arXiv:0111015v2.
34. M. S. Kim, W. Son, V. Bužek, and P. L. Knight, Phys. Rev. A **65**, 032323 (2002).

35. G. Agarwal, *Quantum Optics*, Cambridge Univ. Press, Cambridge (2013).

36. E. Feld'man and M. Yurishchev, JETP Lett. **90**, 70 (2009).

37. S. Barnett and P. Radmore, *Methods in Theoretical Quantum Optics*, Oxford Univ. Press, New York (1997).

38. A. Ourjoumtsev, F. Ferreyrol, R. Tualle-Brouri, and P. Grangier, Nature Phys. **5**, 189 (2009).

39. B. Vlastakis, G. Kirchmair, Z. Leghtas, S. Nigg, L. Frunzio, S. Girvin, M. Mirrahimi, M. Devoret, and R. Schoelkopf, Science **342**, 607 (2013).

40. S. Deléglise, I. Dotsenko, C. Sayrin, J. Bernu, M. Brune, J.-M. Raimond, and S. Haroche, Nature **455**, 510 (2008).

41. P. Milman, A. Auffeves, F. Yamaguchi, M. Brune, J. M. Raimond, and S. Haroche, Eur. Phys. J. D **32**, 233 (2005).

42. V. Albert, S. Krastanov, C. Shen, R.-B. Liu, R. Schoelkopf, M. Mirrahimi, M. Devoret, and L. Jiang, arXiv: 1503.00194v2.

43. A. Holevo, *Quantum Systems, Channels, Information*, De Gruyter, Berlin (2012).

44. M. Sasaki, T. Usuda, O. Hirota, and A. Holevo, Phys. Rev. A **53**, 1273 (1996).

45. Q.-P. Su, C.-P. Yang, and S.-B. Zheng, Sci. Rep. **4**, 3898 (2014).

46. F. Fröwis and W. Dür, Phys. Rev. Lett. **106**, 110402 (2011).

47. F. Fröwis and W. Dür, Phys. Rev. Lett. **85**, 052329 (2012).