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LONG STRING DYNAMICS IN PURE GRAVITY ON AdS_3

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We study the classical dynamics of a completion of pure AdS_3 gravity, whose only degrees of freedom are boundary gravitons and long strings. We argue that the best regime for describing pure gravity is that of "heavy" strings, for which back-reaction effects on the metric must be taken into account. We show that once back-reaction is properly accounted for, regular finite-energy states are produced by heavy strings even in the infinite-tension limit. Such a process is similar to, but different from, nucleation of space out of a "bubble of nothing".

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1. INTRODUCTION AND SUMMARY

This paper is dedicated to Valery Rubakov on the occasion of his 60th birthday. Valery has been a pioneer and a master in understanding the role of nonperturbative solutions of field equations in quantum field theory. This paper is devoted to a particular case of soliton dynamics. Although limited in scope, we believe that it contains some results worth reporting. We hope that its readers will consider it also a worthy tribute to Valery's work.

Pure gravity in three dimensions does not propagate local degrees of freedom, as a simple counting argument shows: six of the 12 Hamiltonian degrees of freedom of the 3D graviton $g_{\mu\nu}$ are removed by gauge invariances and the remaining ones are removed by 3+3 constraints that follow from Einstein's equations. Hence, 3D gravity does not propagate gravitational waves. In the presence of a negative cosmological constant, pure gravity still exhibits a nontrivial dynamics, because there exist boundary gravitons [1] and black hole solutions [2]. The Einstein–Hilbert action of pure gravity with a negative cosmological constant $-1/l^2$ is

$$S_{EH} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R+\frac{2}{l^2}\right). \tag{1}$$

Boundary gravitons exist because the asymptotic metric of 3D anti de Sitter space (AdS) is preserved by a set of diffeomorphisms that act nontrivially on the boundary. Specifically, the condition of being asymptotically AdS_3 means that the metric has the form [1]

$$\begin{gathered} g_{tt} = -r^2/l^2 + O(1), \quad g_{t\phi} = O(1),\\ g_{tr} = O(r^{-3}), \quad g_{rr} = l^2/r^2 + O(r^{-4}),\\ g_{r\phi} = O(r^{-3}), \quad g_{\phi\phi} = r^2 + O(1). \end{gathered} \tag{2}$$

These boundary conditions are preserved by diffeomorphisms with the asymptotic form

$$\begin{gathered} \zeta^t = l[f(x^+) + g(x^-)] + \\ + \frac{l^3}{2r^2}[\partial_+^2 f^+(x^+) + \partial_-^2 g(x^-)] + O(r^{-4}),\\ \zeta^\phi = [f(x^+) - g(x^-)] - \\ - \frac{l^2}{2r^2}[\partial_+^2 f^+(x^+) - \partial_-^2 g(x^-)] + O(r^{-4}),\\ \zeta^r = -r[\partial_+ f(x^+) + \partial_- g(x^-)] + O(r^{-1}). \end{gathered} \tag{3}$$

The allowed diffeomorphisms are parameterized by two arbitrary functions $f(x^+)$ and $g(x^-)$, each depending on only one of the two boundary light-cone coordinates ($x^\pm = t/l \pm \phi$). The time t and the angular coordinate $\phi \sim \phi + 2\pi$ parameterize the AdS_3 boundary, while r is its radial coordinate. The boundary is at $r=\infty$ and $2\partial_\pm = l\partial/\partial t \pm \partial/\partial\phi$.

The classical Poisson brackets associated with asymptotic diffeomorphisms (3) define two Virasoro algebras with the same central charge $c = 3l/2G$ [1]; therefore, after quantization, the Hilbert space of any quantum gravity with the same asymptotics (whether

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pure or with matter) must fall into unitary representations of the Virasoro algebras. This purely kinematical fact has a deep consequence if we further assume that quantum gravity on AdS_3 is dual to a 2D conformal field theory (CFT) [3]. Modular invariance of the CFT, discreteness of the spectrum, and the existence of an $Sl(2,C)$-invariant state with conformal weights $\Delta = \bar{\Delta} = 0$ then imply that the asymptotic density of states at levels $(\Delta, \bar{\Delta})$ is [4]

$$d(\Delta,\bar{\Delta}) \equiv e^S = \exp\left(2\pi\sqrt{c\Delta/6} + 2\pi\sqrt{c\bar{\Delta}/6}\right). \quad (4)$$

Rotating black-hole solutions for pure 3D AdS gravity (2) do exist [2]. Their metric depends on two parameters: the mass M and the angular momentum J. The metric is [2]

$$\begin{gathered} ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(N^\phi dt + d\phi)^2, \\ N^2 = -8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}, \quad N^\phi = \frac{4GJ}{r^2}. \end{gathered} \quad (5)$$

After the identification

$$\Delta + \frac{c}{24} = \frac{Ml+J}{2}, \quad \bar{\Delta} + \frac{c}{24} = \frac{Ml-J}{2},$$

Cardy formula (4) matches the Bekenstein–Hawking formula for the entropy of rotating black holes [5]

$$\begin{gathered} S = S_{BH} = 2\pi r_h/4G, \\ r_h = l\sqrt{4GM + 4G\sqrt{M^2 - J^2/l^2}}. \end{gathered} \quad (6)$$

The result in Ref. [5] is general. In particular, it does not depend on the matter content of the AdS_3 bulk theory. Amusingly, pure gravity seems to defy general formulas (4) and (6). Indeed, as noticed in [6], the asymptotic dynamics of Eq. (1) is described by a Liouville action. Upon quantization, the Liouville theory becomes an unusual conformal field theory because of two features. The first is that its spectrum does not include an $Sl(2,C)$ invariant state. Instead, physical states obey the "Seiberg bound" $\Delta, \bar{\Delta} > (c-1)/24$ [7]. The second is that physical states are only plane-wave normalizable, because the spectrum of the Liouville theory is continuous. These properties are well established in consistent quantizations of Liouville theory at $c > 1$ [8].

The reduction of pure gravity to a boundary Liouville theory is most easily proven by writing Einstein–Hilbert action (1) in terms of two $Sl(2,R)$ Chern–Simons theories [9]

$$S_{EH} = S_{CS,k}[A] - S_{CS,k}[\tilde{A}], \quad k = l/4G. \quad (7)$$

With t^a denoting the three $Sl(2,R)$ generators in the fundamental representation, the Chern–Simons action is

$$S_{CS,k}[A] = \frac{k}{4\pi}\mathrm{Tr}\int_M \left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) + \\ + \text{boundary terms}. \quad (8)$$

The gauge potentials A and $\tilde{A}$ are related to the dreibein e^a and spin connection ω^a by

$$A^a = \omega^a + \frac{e^a}{l}, \quad \tilde{A}^a = \omega^a - \frac{e^a}{l}, \quad A = A^a t^a. \quad (9)$$

Some of the equations of motion derived from (7) are constraints. In the gauge $A_- = \tilde{A}_+ = 0$, when the 3D space is topologically the product of a 2D disc D and the real line R, they imply that $A_r = U^{-1}dU$ and $\tilde{A} = V^{-1}dV$, with U (V) being an $Sl(2,R)$-valued function of r, x^+ (r, x^-). With the solution of the constraints substituted in the Chern–Simons action, the bulk terms disappear and the action reduces to a boundary term. This term is the 2D chiral Wess–Zumino action [10, 11]. Further constraints, following from the requirement that A and $\tilde{A}$ give an asymptotically AdS metric, reduce the Wess–Zumino action to a Liouville action [6].

An attentive reader should have noticed an unwarranted assumption here. We assumed that the 3D space was topologically global AdS_3 to arrive at a Liouville action. In the presence of black holes, i. e., horizons, or of time-like singularities associated with point-like particles in the bulk, the action at the $r = \infty$ boundary must be supplemented with other terms at the inner boundary/horizon. A possible interpretation of these terms is that they describe the states of the AdS_3 quantum gravity, more precisely, the primary states in each irreducible representation (irrep) of the $Virasoro \times Virasoro$ algebra acting on the Hilbert space of quantum AdS_3 gravity[1)]. The role of the boundary Liouville theory would then be simply to describe the Virasoro descendants in each irrep (cf. [12]). In this interpretation, other information is needed to determine the spectrum of primary operators.

One hint that pure gravity could nevertheless have the same spectrum of primaries as the Liouville theory comes from canonical quantization of pure gravity.

[1)] The "constrain first" Hamiltonian formalism was used in [11] to study the effect of point-like insertions and nontrivial topology for compact-group Chern–Simons theories.

Already in the 1990s, it was shown that the wave functions obtained by quantizing the $Sl(2,R)$ Chern–Simons theory are Virasoro conformal blocks [13]. Two $Sl(2,R)$ Chern–Simons actions are combined into the action of pure gravity, and hence the Hilbert space of pure gravity must be (a subspace of) the product of each Chern–Simons Hilbert space. In a forthcoming publication, we will argue that the pure-gravity Hilbert space is the target space of conformal field theories with continuous spectrum and obeying the Seiberg bound [14] (cf. [15]). Assuming from now on that this result holds, we conclude that pure gravity in AdS_3 should contain states that can reach the boundary at a finite cost in energy, since states confined to the interior of the AdS space have a discrete spectrum. Then one natural question to ask is: what are those states?

The mass of such states must be large in AdS units: $Ml \gg 1$, otherwise gravity could not be called "pure" in any sense. The states cannot be massive particles, which cannot reach the AdS boundary. Indeed, there is only one natural candidate for such states: they must be long strings. These states have already been invoked as a possible solution to certain problems of the partition function of Euclidean pure gravity in [16].

The rest of this paper is devoted to studying the effect of long strings in AdS_3 gravity. Section 2 summarizes known features of long strings in the probe approximation, which holds when back-reaction on the metric and quantum string dynamics effect can both be neglected. This happens when the string tension T is in the range $l^{-2} \ll T \ll G^{-1}l^{-1}$. Section 3 describes the case of "light" strings, which were studied in detail in [17]: $T \lesssim l^{-2}$. It is a regime where back-reaction can be neglected, but quantum effects cannot. This is an interesting case, but far from pure gravity, as we will argue using some results in Ref. [17]. Section 4 studies the "heavy" string case, $T \gtrsim G^{-1}l^{-1}$, when back-reaction cannot be neglected. We argue that this regime is best suited to describe a pure gravity theory containing BTZ black holes and no states below the Seiberg bound. We further show that in order to recover the mass gap predicted by the Seiberg bound, the string tension must be Planckian, $T = O(G^{-2}) \gg G^{-1}l^{-1}$. This is the limit $T \to \infty$, which is nonsingular thanks to back-reaction effects. Finite-mass BTZ states arise via a process similar to nucleation of the universe out of a "bubble of nothing"[2)].

[2)] Differently from the quantum nucleation case, the process under consideration here is a classical one, in which the initial state contains a long string approaching the boundary in the remote past. This is good, because up-tunnelling from a bubble of nothing [18] is forbidden in the AdS space [19].

2. LONG STRINGS IN THE PROBE APPROXIMATION

If short-string dynamics and back-reaction are negligible, as it happens when the string tension is in the intermediate range $l^{-2} \ll T \ll G^{-1}l^{-1}$, the effects of long strings can be described in the probe approximation. The long string probe is located at the radial position $r = R(\phi)$ and its classical action is made of two terms [20]. One is proportional to the area spanned by the string world-sheet Σ, and the other is proportional to the volume enclosed by the world-sheet:

$$S = TA(\Sigma) - \frac{q}{l^3}V(\Sigma). \qquad (10)$$

The second term requires coupling the string to an antisymmetric two-form. The world-sheet action of the string thus acquires a term

$$S = \ldots + q \int_\Sigma dX^\mu \wedge dX^\nu B_{\mu\nu}. \qquad (11)$$

The two-form B is analogous to the Kalb–Ramond form of fundamental strings. It has the gauge invariance $B \to B + d\Lambda$ and its bulk action is

$$S_B = -1/12 \int_{AdS_3} H \wedge *H, \quad H = dB. \qquad (12)$$

This action does not propagate any degree of freedom in three dimensions. Hence, the bulk theory in the presence of the form B is still pure gravity, but with a cosmological constant that depends on the value of the field strength H. The field strength is quantized in units of q, the two-form charge of the string [21, 22].

The asymptotic value of string action (10) is best written in terms of a redefined radial coordinate φ, the induced world-sheet metric h, and the world-sheet scalar curvature R [20] as

$$S = Tl^2 \int_\Sigma d^2\sigma\sqrt{h} \times \\ \times \left[\frac{(1-q)e^{2\varphi}}{4} + \frac{(\partial\varphi)^2}{2} + \frac{\varphi R}{2} - \frac{R}{4} + O(e^{-2\varphi})\right], \qquad (13)$$

$$r/l = e^\varphi + e^{-\varphi}\varphi R l^2 + O(e^{-2\varphi}). \qquad (14)$$

To reach the boundary with finite energy, we must set $q = 1$. At $q = 1$, Eq. (13) becomes the Liouville action. Its central charge is $c_L = 1 + 12\pi Tl^2$. Quantum effects can be neglected in the semiclassical regime for the Liouville theory, that is, when $c_L \gg 1$, and hence when

$T \gg l^{-2}$. Crossing the brane, the cosmological constant changes and so does the central charge $c = 3l/2G$. If we call l_+ the AdS radius outside the brane and l_- the radius inside, the central-charge change is

$$\frac{3l_+}{2G} - \frac{3l_-}{2G} \equiv \Delta c = c_L \approx 12\pi T l^2. \qquad (15)$$

Back-reaction effect can be neglected when $\Delta c/c \ll 1$, and hence when $T \ll G^{-1}l^{-1}$. This inequality on the other hand implies that the energy gap between the vacuum and the long-string states, given by the Seiberg bound with $c = c_L$, is

$$E = \frac{c_L - 1}{12} = \pi T l^2 \ll \frac{c-1}{12}.$$

Therefore, the theory contains states with energy well below the BTZ black hole threshold. It is therefore doubtful whether we can call gravity plus strings in the regime $l^{-2} \ll T \ll G^{-1}l^{-1}$ "pure".

The most obvious method for increasing the gap is to make $T \gtrsim G^{-1}l^{-1}$ and take full account of the back-reaction. This is done in Sec. 4. In the next section, we examine a more exotic possibility. Namely, we study the dynamics of light strings with the tension $T \ll l^{-2}$. Although a theory with strings of tension smaller than the AdS scale contain a large number of light states, maybe it could still bear resemblance to pure gravity if these states decouple in the limit as the string coupling constant goes to zero. In next section, we use the results in [17] to argue against this possibility.

3. LIGHT STRINGS AND THE ABSENCE OF BTZ STATES

Strings in AdS_3 with background NS forms can be studied to all orders in $\alpha' = l_s^2 = 1/2\pi T$. In particular, exact expressions for the generators of the target-space Virasoro algebras can be found [23]. The low-tension region $l_s \gtrsim l$ may seem quite the opposite of pure gravity, since it contains an abundance of light degrees of freedom. One exotic possibility is to decouple all the unwanted states by sending the string coupling constant g_s to zero. Because $g_s^2 = G/l_s$ [23], decoupling means that we are sending $l_s \to \infty$ while keeping G and l finite with $l/G \gg 1$. The last condition guarantees that the AdS space is still macroscopic and concepts such as a black hole, metric, and so on are meaningful. The first condition may decouple stringy excitations, leaving only BTZ states.

To check if decoupling is actually possible, we must parameterize our theory in terms of quantities that remains valid beyond the point-particle limit. Therefore, instead of the ratio l/l_s we should use the level k of the $Sl(2,R)$ world-sheet current algebra and use the target-space central charge c instead of l/G. In the semiclassical, point-particle limit, $k = l^2/l_s^2$ and $c = 3l/2G$.

It was argued in [17] that a sharp phase transition occurs at $k = 1$. For $k > 1$, the asymptotic density of states at high energy is dominated by BTZ black holes and the target-space theory has an $Sl(2,R) \times Sl(2,R)$ invariant vacuum. For $k < 1$, neither the vacuum nor the BTZ black hole states are normalizable. The asymptotic density of states is dominated by weakly coupled long strings. The first property agrees with expectations from canonical quantization of pure gravity and its similarities with the Liouville theory. The second property seems to contradict the fact that BTZ black holes are the only primary states in pure gravity. Nevertheless, it could be that at $k < 1$, weakly coupled long strings are just BTZ states in disguise.

The last possibility seems unlikely and in any case a better argument exists against the decoupling limit. The problem arises because in a conformal field theory where the lowest conformal weight of a physical primary operator is not $\Delta = 0$, but some $\Delta_m > 0$, the effective central charge appearing in Cardy's formula (4) is $c_{eff} = c - 24\Delta_m$ [4]. The Seiberg bound $\Delta \geq (c-1)/24$ then tells us that in "Liouville-like" pure gravity, $c_{eff} = 1$.

On the other hand, it was found in [17] that the effective target-space central charge for the long string gas is[3)]

$$c_{eff} = \\ = \begin{cases} 6g_s^{-2}(2 - 1/k) & \text{for type-II superstings,} \\ 6g_s^{-2}(4 - 1/k) & \text{for bosonic strings.} \end{cases} \qquad (16)$$

Setting $c_{eff} = 1$, we have

$$k = \begin{cases} 1/2 + O(g^2) & \text{for type-II superstings,} \\ 1/4 + O(g^2) & \text{for bosonic strings.} \end{cases} \qquad (17)$$

On the other hand, the target-space central charge is [17]

$$c = \begin{cases} 6g_s^{-2}k & \text{for type-II superstings,} \\ 6g_s^{-2}(k+2) & \text{for bosonic strings.} \end{cases} \qquad (18)$$

Hence, in both cases $g_s \to 0$ implies $c \to \infty$, while we want to keep $c \gg 1$ but finite.

[3)] Type-II and heterotic superstings were studied in [17]. Formulas for the bosonic string are the same as those for the right-moving sector of the heterotic string.

If we had tried to keep c finite in the limit $g_s \to 0$, we would have also run into a contradiction because c_{eff} would have become either negative or larger than c. Both these possibilities are forbidden in unitary CFTs.

4. HEAVY LONG STRINGS

In the regime $T \gtrsim G^{-1}l^{-1}$, the metric is deformed by the back-reaction of the string. The process that can lead to formation of massive point-like particles or a BTZ black hole is the collapse of a long string arbitrarily close to the boundary of AdS_3 in the remote past. This is what we examine at the classical level in this section. The collapse of shells of matter with various equations of state was considered in [24].

We first consider the collapse of a shell of matter with rotational symmetry (which is a closed string in two space dimensions) arriving from a radial position arbitrarily close to the boundary of an asymptotically AdS space in the remote past. Inside the shell, the metric is pure AdS_3 and outside is that of a nonrotating BTZ black hole. The metrics inside and outside a shell with a world-sheet Σ are

$$\begin{aligned} ds_-^2 &= -\left(1+\frac{r_-^2}{l_-^2}\right) dt_-^2 + \left(1+\frac{r_-^2}{l_-^2}\right)^{-1} dr_-^2 + d\phi_-^2, \\ ds_+^2 &= -\left(-8GM + \frac{r_+^2}{l_+^2}\right) dt_+^2 + \\ &\quad + \left(-8GM + \frac{r_+^2}{l_+^2}\right)^{-1} dr_+^2 + d\phi_+^2. \end{aligned} \tag{19}$$

Here, the subscript $-$ is used for variables defined inside the shell and $+$ for those outside it. If the string has no angular motion, we can define the "proper time" by moving on the world-sheet at a fixed ϕ and parameterize Σ as

$$ds_\Sigma^2 = -d\tau^2 + (R(\tau))^2\, d\phi^2, \tag{20}$$

where $R(\tau)$ is the radius of the string.

The discontinuity of the extrinsic curvature $K_{ij}^{\pm}$ across the shell is related to the stress–energy tensor of the string S_{ij} on Σ by the so-called Israel boundary conditions; precisely,

$$\gamma_{ij}^+ - \gamma_{ij}^- = 8\pi G S_{ij}, \quad \gamma_{ij}^{\pm} = K_{ij}^{\pm} - g_{ij}K^{\pm}. \tag{21}$$

It is convenient to study the string dynamics using a comoving frame, spanned by the proper time and $(1/R)\partial_\phi$. In such a frame, $S_{ij} = \mathrm{diag}(T, -T)$, and the discontinuity in $\gamma_{ij}^{\pm}$, which we call γ_{ij}, is

$$\gamma_{\tau\tau} = -\frac{1}{R}(\beta_+ - \beta_-), \quad \gamma_{\theta\theta} = \frac{d}{dR}(\beta_+ - \beta_-). \tag{22}$$

Here,

$$\beta_\pm = \sqrt{\dot{R}^2 - 8GM_\pm + R^2/l_\pm^2},$$

$R(\tau)$ is the position of the string, and $M_- = -1/8G$.

Although we can easily solve the single equation obtained from (21) and (22) exactly, examining its asymptotic behavior is sufficient for our purpose:

$$\begin{aligned} 8\pi GTR &= \sqrt{\dot{R}^2+1+\frac{R^2}{l_-^2}} - \sqrt{\dot{R}^2-8GM+\frac{R^2}{l_+^2}} = \\ &= \left(\frac{1}{l_-}-\frac{1}{l_+}\right) R + \frac{1}{2R}\left[(l_-+8GMl_+)+\dot{R}^2(l_--l_+)\right] + \\ &\quad + O(R^{-3}). \end{aligned} \tag{23}$$

The leading-order term tells us that the string tension should be

$$8\pi GT = \frac{1}{l_-} - \frac{1}{l_+}. \tag{24}$$

If the tension differs from this value, the string either cannot reach the boundary or reaches it with infinite radial speed. Equation (24) is the generalization to heavy strings of the condition $q = 1$ in Sec. 2. We call a T obeying Eq. (24) the critical tension and the string with such tension, a critical string.

For a critical string to exist, we must have $l_+ > l_-$, since T is positive. Then the subleading-order term in asymptotic behavior (23) gives us an interesting bound on the mass of the collapsing string:

$$8GM \geq -\frac{1}{1+8\pi GTl_+}. \tag{25}$$

We can understand this mass bound better by expressing it in AdS units. At this point, we have two length scales, l_+ and l_-, both of which can be used to convert energies into dimensionless quantities. The radius of the asymptotically AdS metric outside the shell is l_+, and therefore, if we want to relate bulk energies to CFT weights, we have to use l_+ as our unit of lenght.

To compare with CFT and with Sec. 2, it is convenient to redefine the AdS_3 energy as $E' = E + 1/8G$. The vacuum energy then vanishes, all masses are positive, and the mass bound becomes

$$M'l_+ \geq \frac{l_+}{8G}\frac{8\pi GTl_+}{1+8\pi GTl_+} = \frac{c_+}{12}\frac{8\pi GTl_+}{1+8\pi GTl_+}. \tag{26}$$

The mass bound approaches zero as we send $8GTl_+$ to zero, and therefore the tensionless limit cannot be related to the boundary Liouville theory obtained in Sec. 2.

If $8GM'l_+ \gg 1$, on the other hand, we have a finite mass gap

$$M'l_+ \gtrsim \frac{c_+}{12}, \quad (27)$$

where $c_+ = 3l_+/2G$. This agrees with the Seiberg bound in a theory with a large central charge $c \gg 1$, which is needed for classical geometry to make sense.

If we insist that this mass bound be equal to the Seiberg bound, we find $8\pi GTl_+ = c_+ - 1$. This implies that the tension is of the order of unity in Planck units: $TG^2 \sim 1$.

We also have $l_- \sim G$ from critical tension condition (24). Therefore, we are considering a process where a long string with large tension is nucleated at the boundary from an AdS_3 with the Planckian curvature. Although similar to the nucleation of an AdS "bubble of nothing" [18, 19], the process is different. It is not a quantum transition but a classical process: the collapse of a long string located at the boundary at past infinity. It is only thanks to the back-reaction effects that the two "infinities" involved in the process, $1/l_- \sim 1/G$ and $T \sim 1/G^2$, cancel to give a finite result.

Therefore, for $TG^2 \sim 1$, long strings can produce the right mass spectrum, consisting only of BTZ black holes. Moreover, the large tension ensures that no unwanted low-energy states are added to "pure" gravity.

We argued that long strings could account for BTZ black holes, but our attention was limited to nonrotating ones. We want to conclude this section with some comment on the rotating BTZ case. Inspired by the preceding consideration, we are tempted to use long strings to explain rotating BTZ black holes through the collapse of a shell formed by a rotating long string. We next show that this is impossible, as long as the world-sheet stress energy tensor S_{ij} is diagonal. The simplest case to analyze is the rotating BTZ black hole with a string rotating at a constant angular velocity and a fixed radius R. This case suffices to show the general problem that one encounters even in a more general setting.

We suppose that inside the shell, we have pure AdS_3 as before, but the outside metric is

$$\begin{gathered} ds_+^2 = -N^2 dt_+^2 + N^{-2} dr_+^2 + r_+^2 (N^\phi dt_+ + d\phi_+)^2, \\ N^2 = -8GM + \frac{r_+^2}{l_+^2} + \frac{16G^2J^2}{r_+^2}, \quad N^\phi = \frac{4GJ}{r_+^2}. \end{gathered} \quad (28)$$

We note that ds_-^2 is diagonal, while ds_+^2 is not. To compute γ_{ij}, however, the induced metrics on the world-sheet Σ must be the same, i.e., $(ds_-^2)|_\Sigma = (ds_+^2)|_\Sigma$. One way to accomplish this is to use the coordinate system spanned by

$$\begin{gathered} (e^-)_\tau = (\dot t_-, 0, 0), \quad (e^-)_\theta = (0,0,1/R), \\ (e^+)_\tau = (\dot t_+, 0, \dot\phi_+), \quad (e^+)_\theta = (0,0,1/R), \end{gathered} \quad (29)$$

where R is the radius of the string and $\dot x$ denotes the derivative of x with respect to the proper time τ.

This means that outside the world-sheet, we use a rotating frame with a constant angular velocity $\omega = \dot\phi_+/\dot t_+$, which, in general, may be different from that of the string. Both bases given in Eq. (29) are orthonormal if

$$\dot t_- = \frac{1}{\tilde\beta_-}, \quad \dot t_+ = \frac{1}{\tilde\beta_+}, \quad \omega = \frac{4GJ}{R^2}. \quad (30)$$

Here,

$$\tilde\beta_\pm = \sqrt{-8GM_\pm + \frac{R^2}{l_\pm^2} + \frac{16G^2J_\pm^2}{R^2}}$$

with $M_- = -1/8G$, $J_- = 0$, and $J_+ = J$. In these coordinate systems, we find that the extrinsic curvatures are

$$\begin{gathered} K_{\tau\tau}^\pm = \frac{d}{dR}\tilde\beta_\pm, \quad K_{\tau\theta}^\pm = -\frac{4GJ_\pm}{R^2}, \\ K_{\theta\theta}^\pm = -\frac{1}{R}\tilde\beta_\pm. \end{gathered} \quad (31)$$

We note that these equations give $\gamma_{\tau\theta} = -4GJ/R^2$. Since the origin of this term is the angular momentum of the BTZ black hole, we cannot make it vanish by giving radial dynamics to the string. As long as we consider a physical configuration with angular symmetry, radial dynamics and rotations are the only motions we can introduce at the classical level. This suggests, therefore, that we have to relax the string equation of state $\rho = -p = T$ to explain rotating BTZ states. In fact, when the equation of state is $p = -\rho$, the string stress–energy tensor S^i_j is diagonal in any coordinate frame, whether rotating or not.

One way to set $p \neq -\rho$ is by exciting degrees of freedom on the string. One such degree of freedom, the radial coordinate R, is always present, but others may exist, as they do in fundamental strings.

One amusing agreement between long strings with the equation of state $p = -\rho$ and the Liouville theory is that the latter contains only primaries with equal left and right conformal weights $\Delta = \bar\Delta$ [8]. Since BTZ states must be primaries of the would-be CFT dual, such equality implies the vanishing of the BTZ angular momentum.

At this point, the relation between long strings and rotating black holes is still unclear. It is possible that we would need a completely different description for states giving rise to rotating black holes by gravitational collapse. However, nothing so far seems to forbid excited strings to produce rotating BTZ black holes. In any case, production of BTZ black holes by long-string collapse already showed intriguing features and it is well worth more study.

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