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# RESURGENCE, OPERATOR PRODUCT EXPANSION, AND REMARKS ON RENORMALONS IN SUPERSYMMETRIC YANG–MILLS THEORY

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We discuss similarities and differences between the resurgence program in quantum mechanics and the operator product expansion in strongly coupled Yang–Mills theories. In $\mathcal{N} = 1$ super-Yang–Mills theories, renormalons are peculiar and are not quite similar to renormalons in QCD.

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## 1. PREAMBLE

This paper is written for Valery Rubakov-60 Festschrift on the basis of my talk at CERN in the summer of 2014. I first met Valery around 1980, when he discovered the monopole catalysis of the proton decay, which later became known as the Callan–Rubakov effect. There is a beautiful paper of Edward Witten illustrating subtle points in this effect, which appeared shortly after Rubakov's publication. I remember Witten's seminar based on this paper delivered during his only visit to the USSR in the early 1980s.

After the Callan–Rubakov effect, Rubakov published many inspiring papers and raised two or three generations of bright students. These students, in turn, now inspire new young generations of theoretical physicists all over the world.

## 2. INTRODUCTION

The notion of resurgence and trans-series associated with it — a breakthrough discovery[1)] in constructive mathematics in the 1980s mostly associated with the name of Jean Ecalle — gradually spread in mathematical and theoretical physics. I was impressed by diverse and numerous applications of these ideas recently discussed by J. Zinn-Justin, M. Berry, U. Jentschura, G. Dunne, M. Beneke, and others. The issues to be discussed below are rather close to resurgence in quantum mechanics, although they go far beyond and are much more complicated, because I discuss strongly coupled field theories, such as quantum chromodynamics (QCD).

In quantum mechanics, the program of resurgence works well, and trans-series of the type

$$E(g^2) = E_{PT,\ regularized}(g^2) + \\ + \sum_{k=1}^{\infty}\sum_{l}\sum_{p=0}^{\infty} \underbrace{\left(\frac{1}{g^{2N+1}}\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k--instanton}} \left(\log\frac{c}{g^2}\right)^l \times \\ \times \underbrace{c_{k,l,p}\, g^{2p}}_{\text{regularized PT}} \qquad (1)$$

can be derived for all energy eigenvalues ($g^2$ is assumed to be small; the subscript PT stands for perturbation theory).

In weakly coupled field theories, trans-series could be perhaps constructed, although conclusive arguments have not yet been presented. One of my tasks is to explain why resurgence, being conceptually close to the operator product expansion (OPE), does not work in strongly coupled field theories, for instance, in QCD. It is worth noting that OPE existed in QCD from the mid-1970s, and in its general form, the late 1960s. It grew from a formalism that had been suggested by K. Wil-

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1) For a pedestrian review understandable to physicists (at least, in part) and an exhaustive list of references, see [1, 2].

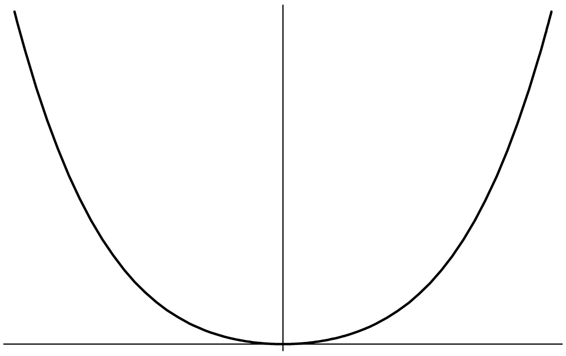

**Fig. 1.** $V(x)$ in the anharmonic oscillator problem (2)

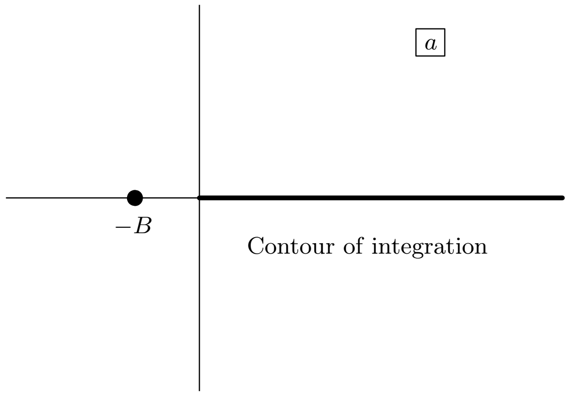


**Fig. 2.** The perturbative series in the anharmonic oscillator problem is Borel-summable. The $g^2$ series for $E_0$ is sign alternating; $f(a)$ has a singularity on the real negative semi-axis in the Borel parameter complex plane. $a$ is the Borel parameter

son before the advent of QCD. The first part of this paper is devoted to this issue.

In the second part, I focus on a more technical aspect: peculiarities of the factorial divergence of perturbation theory in $\mathcal{N}=1$ super-Yang–Mills (SYM). So far renormalons in SYM were scarcely discussed. No final conclusion was reached. To a large extent this question remains open.

## 3. THE SIMPLEST QUANTUM MECHANICAL EXAMPLES

### 3.1. Anharmonic oscillator

We consider a one-dimensional anharmonic oscillator,

$$\mathcal{H}=\frac{1}{2}p^2+\frac{1}{2}\omega^2x^2+g^2x^4 \tag{2}$$

(see Fig. 1).

For definiteness, we focus on the ground state energy $E_0$. There exists a well-defined procedure for constructing $E_0$ order by order in perturbation theory, to any finite order,

$$E_0=\frac{\omega}{2}\left(1+c_1g^2+c_2g^4+\ldots\right). \tag{3}$$

Nevertheless, Eq. (3) does not define the ground state energy. Indeed, the coefficients $c_k$ are factorially divergent at large $k$ [3],

$$c_k\sim(-1)^kB^{-k}k!,\quad k\gg 1, \tag{4}$$

where $B=\omega^3/3$ is the so-called bounce action[2]. Hence, the sum in (3) needs a regularization.

[2] Equation (4) is slightly simplified. For a more precise formula, see [3].

In the simplest case under consideration, an appropriate (and exhaustive) regularization is provided by the Borel transformation $\mathcal{B}$,

$$\mathcal{B}E_0\equiv\frac{\omega}{2}\left(1+\sum_{k=1}^{\infty}\frac{1}{k!}c_kg^{2k}\right)\equiv\frac{\omega}{2}\,f(g^2). \tag{5}$$

The Borel transformation introduces $1/k!$ in each term of series (3), rendering it convergent. Moreover, if the convergent series

$$1+\sum_{k=1}^{\infty}\frac{1}{k!}c_kg^{2k}\equiv f(g^2), \tag{6}$$

which defines the Borel function $f(g^2)$, has no singularities on the real positive semi-axis $g^2\geq 0$, then we can obtain the ground-state energy $E_0$ starting from the well-defined expression for $\mathcal{B}E_0$ and using the Laplace transformation,

$$E_0=L\left(\mathcal{B}E_0\right)\equiv\frac{\omega}{2}\int\limits_0^{\infty}da\,g^{-2}\exp\left(-\frac{a}{g^2}\right)f(a). \tag{7}$$

This procedure is usually referred to as the Borel summation. Thus, the perturbative expansion in the anharmonic oscillator is Borel-summable because the singularities of $f(a)$ are on the negative real semi-axis. Indeed, we assume that $f(a)$ has a pole at $a=-B$ (see Fig. 2), namely,

$$f(a)=\frac{B}{a+B},\quad B=\frac{\omega^3}{3}. \tag{8}$$

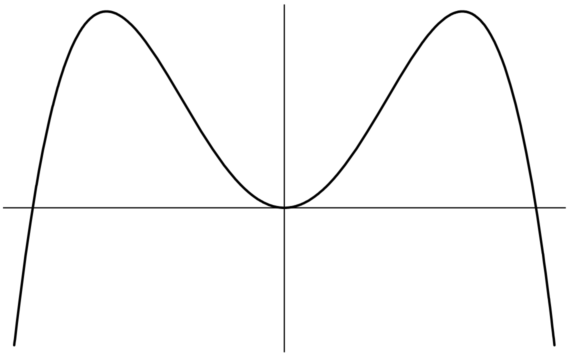

**Fig. 3.** The same potential with the replacement $g^2 \to -g^2$, to be denoted as $\tilde{V}(x)$

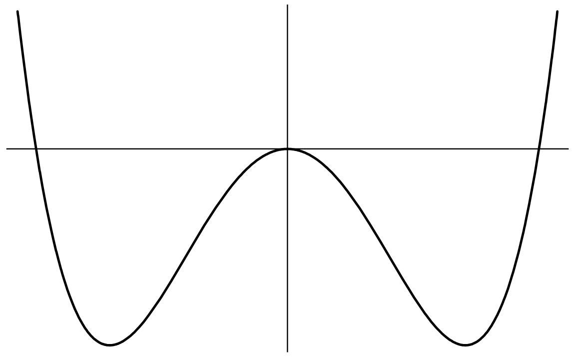

**Fig. 4.** An effective potential in Euclidean time. This potential is a sign reflection of that in Fig. 3, i.e., is $-\tilde{V}(x)$. It vanishes at $x = 0$ and at $x = \pm x_*$, where $x_* = \omega/\sqrt{2}\, g$

Then the integral (7) is well-defined. At the same time, expanding (8),

$$f(a) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{a}{B}\right)^k, \tag{9}$$

and substituting this series in (7), we immediately arrive at (4).

The fact that the position of the singularity in the $a$ plane is to the left of the origin and that the series is sign-alternating are in one-to-one correspondence with each other.

Exactly fifty years ago, Vainshtein identified [4] the physical meaning of the factorial growth of coefficients (4) and explained why the underlying singularity in the Borel parameter plane is on the negative semiaxis. Changing the sign of $g^2$ from positive to negative, $g^2 \to -g^2$, we convert a stable potential $V(x)$ in (2) into an unstable potential $\tilde{V}(x)$ presented in Fig. 3, allowing for the wave function to leak to large distances.

In the leaking potential $\tilde{V}$, the energy corresponding to the 0th eigenvalue acquires an imaginary part (as do other energy eigenvalues). This imaginary part can be easily determined. Indeed, after the Euclidean time rotation, the potential effectively changes as $\tilde{V}(x) \to -\tilde{V}(x)$, as shown in Fig. 4. Then the so-called bounce trajectory becomes classically accessible[3)]. The bounce trajectory starts at $x = 0$, slides to the right, bounces off at $x_* = \omega/\sqrt{2}\, g$, and then returns to the point $x = 0$. The Euclidean action on the bounce trajectory is readily calculable,

$$A_{bounce} = \frac{B}{g^2}, \tag{10}$$

where $B$ is defined in Eq. (8). In this way, we obtain that

$$\mathrm{Im}\, E_0 = \frac{\pi\omega}{2} \frac{B}{g^2} \exp\left(-\frac{B}{g^2}\right). \tag{11}$$

Now we can calculate the ground-state energy for the original potential in Fig. 1 by using (11) and a dispersion relation in the coupling constant [4],

$$E_0 = \frac{1}{\pi} \int_0^{\infty} d\tilde{g}^2 \frac{1}{g^2 + \tilde{g}^2} \mathrm{Im}\, E_0\left(\tilde{g}^2\right) = \\ = \frac{\omega}{2} \int dz \frac{1}{1 + (g^2/B)\, z}\, e^{-z}. \tag{12}$$

The last expression reproduces the series in (3) and (4) with its sign alternation and factorial divergence of the coefficients. Both features are explained by the imaginary part in (11) being proportional to $\exp(-B/g^2)$.

Summarizing, the perturbative expansion for the anharmonic oscillator is factorially divergent; however, the Borel summability allows finding the closed, well-defined, and exact expressions for the energy eigenvalues. The physical meaning of the factorial divergence, as well as the sign alternation, are fully understood. Now we pass to a more complicated but more interesting non-Borel-summable case.

### 3.2. Double-well potential

The double-well problem is described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}p^2 - \frac{1}{4}\omega^2 x^2 + g^2 x^4, \tag{13}$$

[3)] See, e.g., Chapter 7 in [5].

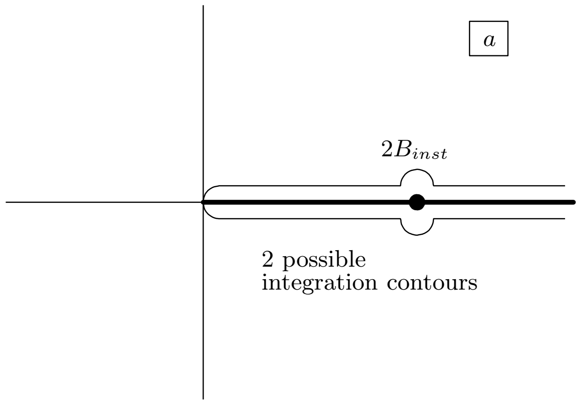


**Fig. 5.** The $g^2$ series in the double-well problem is not sign alternating; $f(a)$ has a singularity on the real negative semi-axis in the Borel parameter complex plane at $a = 2B_{inst}$, where $a$ is the Borel parameter

i. e., the sign of the $O(x^2)$ term is changed, and the point $x = 0$ becomes unstable. Instead, two stable minima develop at $x_* = \pm x_*$ with

$$x_* = \omega/2\sqrt{2}g.$$

The shape of the double-well potential is depicted in Fig. 4. Classically, each of the two minima $x = \pm\omega/2\sqrt{2}g$ presents a stable solution of the system. Quantum mechanically, zero-point oscillations about the minima occur. Taking the anharmonicity near the minima into account, we generate a perturbative series for the ground-state energy. This is in the perturbation theory. In fact, the two minima are connected by the tunneling trajectory (instanton) in Euclidean time. The instanton action is

$$S_{inst} = \frac{\omega^3}{12g^2} \tag{14}$$

(see, e. g., [6]). In what follows, it will be convenient to introduce

$$B_{inst} = g^2 S_{inst} = \frac{\omega^3}{12}. \tag{15}$$

At small $g^2$, the ground-state energy is close to $\omega/2$ plus corrections in $g^2$ and nonperturbative corrections of the type $\exp(-c/g^2)$. A crucial distinction from the anharmonic oscillator discussed in Sec. 3.1 is that the $g^2$ series in this case is not sign-alternating (although still factorially divergent), corresponding to a singularity in the Borel function at a real positive value $a = 2B_{inst}$, i. e., on the integration contour (see Fig. 5). Thus, we have to rethink the Borel summation procedure.

Equation (7) is replaced by

$$E_0 = L(\mathcal{B}E_0) \equiv \frac{\omega}{2}\int_0^\infty da\, g^{-2}\exp\left(-\frac{a}{g^2}\right) f(a), \tag{16}$$

where, roughly speaking,

$$f(a) = \frac{-2B_{inst}}{a - 2B_{inst}}. \tag{17}$$

Then, instead of Eq. (4), we obtain

$$c_k = k!\,(2B_{inst})^{-k}. \tag{18}$$

The perturbative series is not sign alternating, unlike the case of the anharmonic oscillator.

We pause here to take a closer look at the above results. In fact, integral (16) is undefined: the integration along the real positive semi-axis cannot be performed since we hit a singularity. It must be circumvented along either the upper or the lower small semicircles as shown in Fig. 5. Depending on whether we choose the upper or lower semicircle, we obtain an imaginary contribution

$$(\Delta E_0)_{Borel} = \pm\pi i\left(-\frac{2B_{inst}}{g^2}\right) \times \exp\left(-\frac{2B_{inst}}{g^2}\right). \tag{19}$$

However, in the case of the double-well potential, the system is stable and does not decay, implying that the ground-state energy must be strictly real; $(\Delta E_0)_{Borel}$ must be canceled by something, and, indeed, it is canceled by a contribution coming from the instanton–anti-instanton (IA) pair. The position of the singularity at $2B_{inst}$ in Fig. 5 prompts us that it is a pair of instantons which is important.

The IA pair is only an approximate saddle point. There is an attraction potential that is very shallow when they are far apart. As usual, approximate saddle points require a regularization. One of regularizations, which is very helpful at least for qualitative purposes, is to consider the IA pair at a finite (rather than zero) energy $E$, along the lines described, e. g., in [5], Secs. 23.2 and 23.3. Then the imaginary part of the IA contribution reduces to $\exp(-2S_{inst})$ with a known pre-exponential, and cancels the imaginary part in (19). The real part of the IA contribution is proportional to $\omega T_* \sim \log(S_{inst})(\omega/E)$, where $T_*$ is the critical IA separation and the value of $E$ relevant to the problem is $E \sim \omega$. Thus, the real part of the IA contribution reduces to $\log S_{inst}\exp(-2S_{inst})$ times a known power of $S_{inst}$ in the pre-exponential. For a more careful calculation, see [7–9].

If we write the ground-state energy in the form

$$E_0 = \frac{\omega}{2}\,\mathrm{P}\int_0^\infty da\, g^{-2}\exp\left(-\frac{a}{g^2}\right) f(a) + \\ + (S_{inst})^p(\log S_{inst})\exp(-2S_{inst}), \quad (20)$$

where P stands for the principal value of the integral, this expression is well defined and, being expanded, generates the perturbative series in its entirety[4)]. Strictly speaking, the second line is oversimplified, since the pre-exponential in the second line is also represented by an infinite $g^2$ series with factorially divergent coefficients. To amend this series, we have to include a 2I–2A contribution, and so on. We refer to this formula as the minimal Borel procedure (MBP). The MBP formula contains all information one can squeeze from perturbation theory. It still lacks something. In order to understand what this something is, we make a digression.

As is well known, perturbation theory (PT) describes fluctuations of a quantal system around classical minima of the potential. In the case at hand, we have two degenerate minima reflecting a $Z_2$ symmetry of the potential. We choose one of them for the "unperturbed" Hamiltonian, for instance,

$$H_0 = \frac{p^2}{2} + \frac{\omega^2}{2}(x - x_*)^2. \quad (21)$$

All cubic and quartic terms from the expansion of potential (13) are referred to $H_{int}$. The perturbation theory in $H_{int}$ is well defined in any order.

The Hamiltonian $H_0$ does not know about the second vacuum, but high-order corrections reflect the existence of the second vacuum indirectly, through the factorial divergence of the PT series. The perturbation theory in $H_{int}$ requires only the knowledge of the unperturbed eigenfunctions and eigenvalues (i. e., those of harmonic oscillator (21)). The eigenfunctions of $H_0$ should be square normalizable, and no other requirements are imposed.

Next, we define the sum of the factorially divergent series as MBP plus IA. Using this procedure, we would conclude that the system has two degenerate ground states: the $Z_2$ restoration in the vacuum is still absent.

This fact — restoration of $Z_2$ — does not ensue with necessity from the amended PT series. It presents an additional information on the global vacuum structure: a $Z_2$ order parameter drastically changes compared to its PT value, and the degeneracy of the ground state is lifted accordingly. This effect is proportional to $\exp(-S_{inst})$, as opposed to $\exp(-2S_{inst})$ reflecting the corresponding singularity in the Borel plane at $2B_{inst}$[5)].

Conceptually, this is similar to chiral symmetry breaking in the chiral limit in QCD. No matter what we do with the PT, we do not see any splitting between the axial and vector quark two-point functions. We have to infer the global vacuum structure of QCD from other sources.

## 4. ASYMPTOTICALLY FREE FIELD THEORIES

We arrived at a point where it would be natural to pass from quantum mechanics to asymptotically free field theories. Up to a certain point, we can proceed along the lines outlined in Sec. 3. There are two cases when we can go all the way up to complete resurgence: (a) if a given field theory is exactly solvable (in which case this is a triviality), or (b) if it is weakly coupled (perhaps, after a certain deformation) and hence can be treated semiclassically. In the latter case, complications that arise are of a technical nature. Today, we are aware of quite a few examples of this type that have been identified and studied in the past.

However, the most interesting theories are QCD and its relatives. They are special because QCD is the theory of Nature, describing the quark–gluon dynamics. They are strongly coupled in the infrared (IR) domain, where it is impossible to treat them semiclassically: the perturbation theory fails even qualitatively. It does not capture the drastic rearrangement of the vacuum structure related to confinement.

I would like to discuss the following question: how far can we go in the resurgence program in these theories? We see in what follows that a certain procedure suggested in the late 1960s [10] and implemented in QCD in the 1970s [11] allows advancing rather far, although, unfortunately, not to the very end. This is as good as it gets ...

The Lagrangian of QCD has the form (in the chiral limit)

$$\mathcal{L} = -\frac{1}{4g^2}G^a_{\mu\nu}G^{\mu\nu\,a} + \sum \bar\psi i \not{D}\psi, \quad (22)$$

[4)] Equation (20) is to be compared with the general trans-series formula (1).

[5)] The instanton can leak to another minimum and then an anti-instanton would return the system to the original minimum. That is the origin of the $\exp(-2S_{inst})$ factor. The splitting between the ground state and the first excitation is due to a single instanton that connects two "prevacua". This effect is proportional to $\exp(-S_{inst})$.

where the sum ranges over the massless quark flavors, and $\psi$ is the quark field in the fundamental representation of SU($N$). In actual world, $N = 3$, but in theoretical laboratory we are free to consider any value of $N$. If we drop the quark term, we are left with the $G^2_{\mu\nu}$ gluon term. This is a pure Yang–Mills theory. Moreover, $g^2$ in front of the gluon term is the asymptotically free gauge coupling.

As we know, this is a strongly coupled theory. The Lagrangian is defined at short distances in terms of gluons and quarks, while at large distances of the order of $\gtrsim \Lambda^{-1}_{QCD}$, we deal with hadrons, e. g., pions and protons. Certainly, the latter are connected with quarks and gluons in a divine way, but this connection is highly nonlinear, nonlocal, and is not amenable to analytic description at the moment. Moreover, the very existence of massless pions and massive protons is due to a dramatic restructuring of the QCD vacuum reflecting spontaneous breaking of the chiral symmetry. This phenomenon is only possible at a very strong coupling. Perhaps, in the future, string theory will be able to provide an adequate description, but as they say, "the future is not ours to see ... ".

Another (a much simpler) example is the two-dimensional CP($N-1$) model with a varying degree of supersymmetry (or no supersymmetry at all). The Lagrangian of the model is

$$\mathcal{L} = \sum_{i,\bar{j}=1}^{N-1} G_{i\bar{j}}\, \partial_\mu \phi^{\dagger \bar{j}}\, \partial^\mu \phi^i + \text{fermions}, \tag{23}$$

where

$$G_{i\bar{j}} = \frac{2}{g^2}\left(\frac{\delta_{i\bar{j}}}{\chi} - \frac{\phi^{\dagger\, i}\phi^{\bar{j}}}{\chi^2}\right), \qquad \chi = 1 + \sum_{m}^{N-1} \phi^{\dagger\, m}\phi^m, \tag{24}$$

and $g^2$ is the asymptotically free coupling constant. In the large-$N$ limit, this model is exactly solvable [12–14]. To the leading order in $1/N$, the solution is known, but cannot be expressed in terms of (1) because instantons are irrelevant at strong coupling. Since the solution is known, we can still represent it in the form of a generic trans-series. In the first subleading $1/N$ correction, we return to a generic contrived situation, to be discussed below, similar to that in QCD.

A common feature of both theories above as well as many others from this class is the fact that the coupling constant is not a *bona fide* constant; it runs. In more detail,

$$g^2(Q) = \frac{B_{inst}}{\beta_0 \log(Q/\Lambda)}, \tag{25}$$

where

$$B_{inst} = \begin{cases} 8\pi^2, & \text{QCD}, \\ 4\pi, & \text{CP}(N-1), \end{cases} \qquad \beta_0 = \begin{cases} \dfrac{11}{3}N, & \text{YM}, \\ N, & \text{CP}(N-1), \end{cases} \tag{26}$$

and $Q$ is an appropriate momentum scale (assuming $Q \gg \Lambda$). Here, $\beta_0$ is the first coefficient of the $\beta$ function. In the upper line on the right, it is given for a pure Yang–Mills theory. When characteristic values of $Q$ become close to $\Lambda$, the running constant is undefined and all calculations in terms of gluons and quarks become meaningless.

As wee see, the genuine parameter of QCD is not the dimensionless $g^2$, but rather the dynamical QCD scale $\Lambda$ invisible in the classical Lagrangian. That is the phenomenon of dimensional transmutation inherent to all strongly coupled asymptotically free field theories. The series in $g^2$ becomes the series in $1/\log(Q)/\Lambda$, exponential terms $\exp(-c/g^2(Q))$ reduce to powers

$$\left(\frac{\Lambda}{Q}\right)^{c\beta_0/8\pi^2},$$

while terms exponential in $Q$, $\sim \exp(-cQ/\Lambda)$, which also appear in QCD and similar theories in the $g^2$ perturbation theory, have to emerge from

$$\exp(-c\exp(\tilde{c}/g^2(Q))).$$

Complete failure of quark–gluon calculations at $Q \sim \Lambda$ blocks the program of "analytic" resurgence in terms of trans-series in QCD. However, some kind of resurgence is possible, known as the operator product expansion. Now we proceed to a more systematic (albeit brief) discussion of OPE.

## 5. OPE VERSUS TRANS-SERIES

Instead of a general introduction to Wilson's operator product expansion (which would require a lot of time[6)]), we briefly discuss OPE from a somewhat nonstandard standpoint: following the logic of Sec. 3. devoted to resurgence in quantum mechanics.

[6)] For a review of OPE in QCD, see [15].

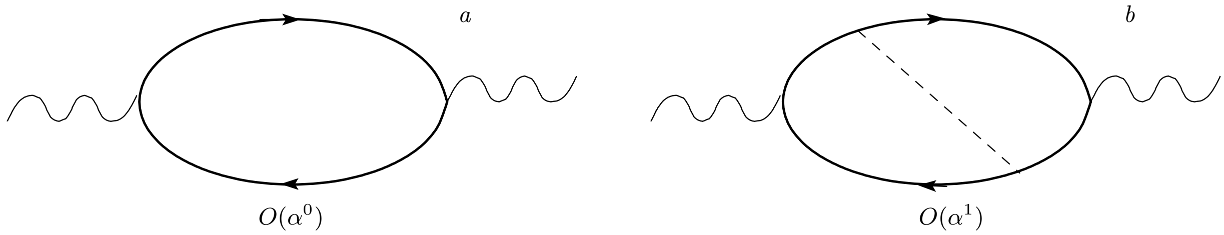


**Fig. 6.** The leading and the next-to-leading terms in the expansion of the Adler function. The external current $j_\mu$ injecting a quark–antiquark pair in the vacuum (and then annihilating it) is denoted by wavy lines

We start our discussion from the two-point function

$$\Pi_{\mu\nu}(q) = i \int d^4x\, e^{-iqx} \left\langle T\, [j_\mu(x) j_\nu(0)] \right\rangle = \\ = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right) \Pi(Q^2), \quad (27)$$

where $j_\mu = \bar\psi \gamma_\mu \psi$ is the quark current, and we set

$$Q^2 = -q^2, \quad (28)$$

such that $Q^2$ is positive in the Euclidean domain. We limit ourselves to large values of the Euclidean momentum, $Q \gg \Lambda$, such that the perturbation theory can be used. In fact, instead of $\Pi(Q^2)$, for technical reasons it is convenient analyze the so-called Adler function defined as

$$D(Q^2) = -4\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2}. \quad (29)$$

The first two terms in the expansion of the Adler function are defined by the diagrams in Fig. 6, with the coupling constant

$$\alpha \equiv \frac{g^2}{4\pi}. \quad (30)$$

Given the external momentum $Q$ flowing through the wavy line, it is easy to see that it is the running coupling $\alpha(Q)$ that enters Fig. 6*b*. Indeed, the momentum flowing through the gluon line in Fig. 6*b* is $k \sim Q$.

Moving to higher orders in $\alpha$, we find more and more complicated multiloop graphs. Among them, a special role belongs to the bubble-chain diagrams, depicted in Fig. 7. These graphs (referred to as renormalons) were extensively studied in the late 1970s [16] (for reviews, see [17, 18]).

When we say bubble chains, we should be careful. Generally speaking, the very definition of a bubble chain in the form of Fig. 7 is not quite accurate. The appropriate renormalon graphs cannot be isolated in the form of a bubble chain because in this form they are not even gauge invariant. An honest-to-god renormalon calculation is quite contrived.

There is a useful trick, however. We add $N_f$ flavors to the theory, where $N_f$ is treated as a free parameter. Then, instead of the full calculation of the genuine "bubble chain", with gluon degrees of freedom in the bubbles, we calculate only the matter bubbles (which are gauge invariant in the chain of Fig. 7), and then replace

$$\beta_0^f \equiv -\frac{2}{3} N_f \to \beta_0, \quad (31)$$

where $\beta_0$ is the first coefficient in the $\beta$ function that includes everything: gluons (plus ghosts in the covariant gauges) and matter fields[7)].

It is easy to see that the renormalon contribution to the $D$ function is sign-nonalternating and factorially divergent in higher orders, $\Delta D_{renorm} \sim n!\alpha^n$ $(n \gg 1)$. If $n$ is large, the estimate $k \sim Q$ is no longer valid. Both observations — the absence of sign alteration and factorial divergence — become obvious after a closer look at Fig. 6*b* before integrating over $k$. The exact result for a fixed $k^2$ was found by Neubert [19]. For illustrative purposes, it is sufficient to use a simplified interpolating expression [20] collecting all bubble insertions in the gluon propagator: no bubbles, one bubble, two bubbles, and so on,

$$D = C \times Q^2 \int dk^2 \frac{k^2 \alpha_s(k^2)}{(k^2+Q^2)^3}, \quad (32)$$

which coincides with the exact expression [19] in the limits $k^2 \ll Q^2$ and $k^2 \gg Q^2$, up to minor irrelevant details. The coefficient $C$ in Eq. (32) is a numerical constant and $\alpha(k^2)$ is the running gauge coupling, which can be represented as

$$\alpha(k^2) = \frac{\alpha(Q^2)}{1 - \dfrac{\beta_0 \alpha(Q^2)}{4\pi} \ln(Q^2/k^2)}. \quad (33)$$

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7) We note that $\beta_0^f$ and $\beta_0$ have opposite signs.

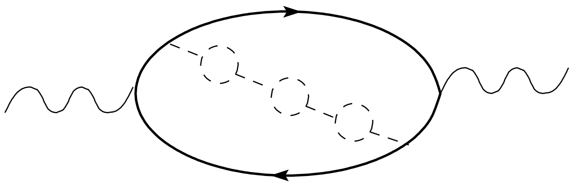

**Fig. 7.** The bubble-chain diagrams representing renormalons. Solid lines denote quark propagators, while dashed lines are for gluons

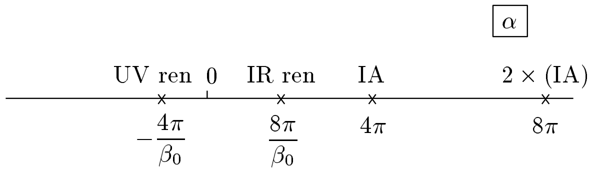


**Fig. 8.** The Borel plane for the Adler function in QCD. The singularity to the left of the origin is due to an ultraviolet renormalon, which does not concern us here. The nearest singularity to the right of the origin is due to the IR renormalon shown in Fig. 7. The IA singularities lie much farther to the right

We focus on the infrared domain. Omitting the overall constant $C$, we obtain

$$D(Q^2) = \frac{1}{Q^4}\alpha \sum_{n=0}^{\infty}\left(\frac{\beta_0\alpha}{4\pi}\right)^n \times \\ \times \int dk^2 k^2 \left(\ln\frac{Q^2}{k^2}\right)^n, \quad \alpha \equiv \alpha(Q^2), \tag{34}$$

which can be rewritten as

$$D(Q^2) = \frac{\alpha}{2}\sum_{n=0}^{\infty}\left(\frac{\beta_0\alpha}{8\pi}\right)^n \int dy\, y^n e^{-y}, \\ y = 2\ln\frac{Q^2}{k^2}. \tag{35}$$

The $y$ integration in Eq. (35) represents all bubble-chain diagrams after integration over the loop momentum $k$. The $y$ integral from zero to infinity is $n!$. A characteristic value of $k^2$ saturating the integral is

$$y \sim n \text{ or } k^2 \sim Q^2 \exp\left(-\frac{n}{2}\right). \tag{36}$$

Thus, we observe a factorial divergence of the coefficients. The corresponding singularity in the Borel plane is depicted in Fig. 8.

If $Q^2$ is fixed and $n$ is sufficiently large, $n > n_*$, where

$$n_* = 2\ln\frac{Q^2}{\Lambda^2}, \tag{37}$$

then the factorial divergence of the coefficients in (35) is purely formal and cannot be trusted. At small $k^2 \lesssim \Lambda^2$, the diagrams in Figs. 6*b* and 7 (in fact, any Feynman diagrams) cease to properly represent non-Abelian dynamics due to the strong coupling in the IR. Equation (36) shows that if $n > n_*$, the characteristic values of $k^2$ saturating the integral do fall off below $\Lambda^2$. The point $n = n_*$ represents the optimal truncation point: at this point, the terms of the asymptotic series are minimal. Formally, if we discard the domain $k^2 < \Lambda^2$, at $n > n_*$ the factorial growth is suppressed (see Fig. 10), and the series must be truncated:

$$D(Q^2) \to \frac{\alpha_s}{2}\sum_{n=0}^{n_*}\left(\frac{\beta_0\alpha_s}{8\pi}\right)^n n! \tag{38}$$

At this point, the road we have to take in QCD and similar strongly coupled theories diverges from that in quantum mechanics. In the latter, the validity of the semiclassical approximation combined with the clear-cut picture of the vacuum structure allows achieving full resurgence. In field theories, the vacuum structure is determined by infrared dynamics, the theory of which is still lacking, and semiclassical approximations are bound to fail. What can we do under the circumstances?

## 6. OPERATOR PRODUCT EXPANSION

I remember that after the first seminar on the SVZ sum rules [11] in 1978, Eugene Bogomol'nyi used to ask me each time we met: "Look, how can you speak of power corrections in the two-point functions at large $Q^2$ when even the perturbative expansion (i. e., the expansion in $1/\ln(Q^2/\Lambda^2)$) is not well defined? Isn't it inconsistent"?

Now, with the discussion of Sec. 5 in mind, I am able to answer the above Bogomol'nyi question in a positive way, namely:

Consistent use of Wilson's OPE makes everything well-defined at the conceptual level. Technical implementation may not always be straightforward, however. Moreover, the resulting OPE formula contains unknown vacuum condensates in the form of power corrections. In turn, their summation presents an unsolved problem.

The operator product expansion in asymptotically free theories is a book-keeping device separating short-distance (weak-coupling) contributions from those coming from large distances (strong-coupling domain). To this end, we introduce an auxiliary parameter $\mu$, a separation scale between large and short distances. The latter contribution resides in the OPE coefficient functions $C_i(Q,\mu)$, while the former contribution is encoded in the matrix elements of the corresponding operators $O_i(\mu,\Lambda)$,

$$D(Q,\Lambda) = \sum_{i=0}^{\infty} C_i(Q,\mu)\ \langle O_i(\mu,\Lambda)\rangle\,. \tag{39}$$

Generally speaking, OPE is applicable whenever we deal with problems that can be formulated in Euclidean space–time and in which we can regulate typical Euclidean distances by a varying large external momentum $Q$. Factorization (39) is technically meaningful (i.e., allows carrying out constructive calculations of $C_i(Q,\mu)$) if we can choose

$$\mu \ll Q, \quad \text{but } \mu \gg \Lambda. \tag{40}$$

Then the coefficients $C_i(Q,\mu)$ can be found semiclassically, even though they by no means reduce to the PT. The matrix elements $\langle O_i(\mu,\Lambda)\rangle$ cannot be determined semiclassically. As a book-keeping device, OPE cannot fail [11], as long as no arithmetic mistake is made en route.

A remarkable observation was made in the 1990s. Perturbative analysis (e.g., that of renormalons) prompts us that certain nonperturbative condensates must be present in QCD. Moreover, we can even determine their dimension from the the position of singularities in the Borel plane. Unfortunately, by far not all condensates are visible in the analysis of PT high orders. For instance, all condensates related to the spontaneous breaking of chiral symmetry leave no trace in any order of the perturbation theory, nor in its divergence.

The values of condensates that are visible in the PT divergence are not determined by the PT analysis either[8)].

## 7. OPE AND RENORMALONS IN QCD

After this brief digression, we return to Adler function (27) at large Euclidean $q^2$, where OPE can be consistently built through separation of large- and short-distance contributions.

For simplicity, taking into account that my purpose today is illustrative, I ignore the second inequality in (40) and set the separation scale at $\mu = \Lambda$ rather than at $\mu \gg \Lambda$. This would be inappropriate in quantitative analyses; however, my task is to explain a qualitative situation. Being auxiliary, the parameter $\mu$ eventually cancels from the master formula (42) anyway (see below).

We take a closer look at Eqs. (32) and (33). The unlimited factorial divergence in (35) is a direct consequence of the integration over $k^2$ in (34) all the way down to $k^2 = 0$. Not only is this nonsensical because of the pole in (33) at $k^2 = \Lambda^2$; this is not what we should do in calculating coefficient functions in OPE. The coefficients must include $k^2 > \Lambda^2$ by construction. The domain of small $k^2$ (below $\Lambda^2$) must be excluded from $c_0$ and referred to the vacuum matrix element of the gluon operator $G^2_{\mu\nu}$. Indeed, in the sum in Eq. (35), all terms with $n > n_*$ can be written as (see Figs. 9 and 10)

$$\Delta D(Q^2) = \frac{\alpha}{2}\sum_{n>n_*}\left(\frac{\beta_0\alpha}{8\pi}\right)^n n_*^n e^{-n_*} = \frac{\alpha}{2}\sum_{n>n_*}\frac{\Lambda^4}{Q^4}, \tag{41}$$

where we used the fact that

$$\frac{\beta_0\alpha(Q^2)}{8\pi} = \frac{1}{2\ln(Q^2/\Lambda^2)} = \frac{1}{n_*}.$$

Of course, we cannot calculate the gluon condensate from the above expression for the tail of the series (35) representing the large-distance contribution, for a number of reasons. In particular, the value of the coefficient in front of $\Lambda^4/Q^4$ remains uncertain in (41) because Eq. (33) is no longer valid at such momenta.

We do not expect the gluon Green's functions used in calculation in Fig. 7 and in Eq. (33) to retain any meaning in the strong-coupling nonperturbative domain. A qualitative feature — the power dependence $(\Lambda/Q)^4$ in (41) — is correct, however.

We note with satisfaction that the fourth power of the parameter $\Lambda/Q$, which we find from this tail, exactly matches the OPE contribution of the operator $\langle G^2_{\mu\nu}\rangle$ (see [11]).

Summarizing, we see that the consistent use of OPE cures the problem of the renormalon-related factorial divergence of coefficients in the $\alpha$ series, absorbing the IR tail of the series in the vacuum expectation value of the gluon operator $G^2_{\mu\nu}$ and similar higher-order operators. Although the value of $\langle G^2_{\mu\nu}\rangle$ cannot be calculated from renormalons, the very fact of its existence can be established.

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[8)] Prevalent in the 1970s and early 1980s was a misconception that the OPE coefficients are determined exclusively by perturbation theory, while the matrix elements of the operators involved are purely nonperturbative. Attempts to separate perturbation theory from "purely nonperturbative" condensates gave rise to inconsistencies (see, e.g., [21]).

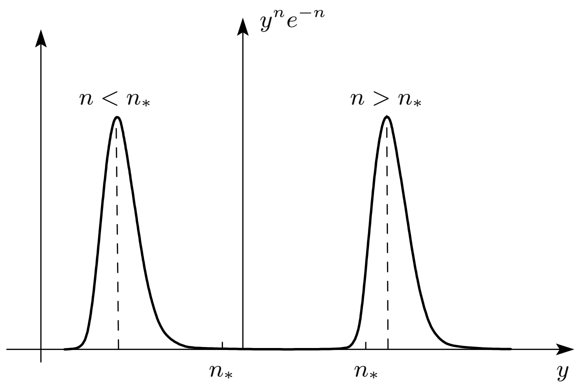


**Fig. 9.** The plot of the integrand in Eq. (34) for two values of $n$, "small" and "large". A sharp peak at $y \sim n$ saturates the integral. In the left plot, $n < n_* = 2\ln(Q^2/\Lambda^2)$ and the forbidden domain $k^2 \sim \Lambda^2$ does not contribute to the factorial factor. In the right plot, $n > n_*$. The $y$ integration has to be cut off at $y = n_*$, which kills the factorial growth

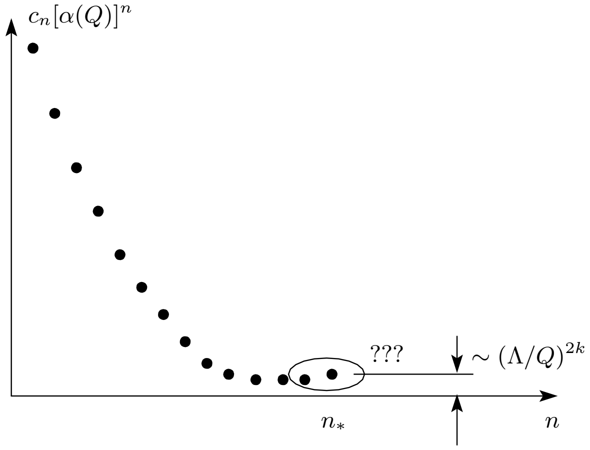


**Fig. 10.** The PT expansion for the Adler function is asymptotic. We can trust it only up to a point of optimal truncation. A "tail" beyond this point tells us of the existence of an operator of dimension $2k$ representing this tail not accessible by PT calculation. (In the case at hand, $k = 2$)

## 8. SOURCES OF FACTORIALS AND THE MASTER FORMULA

From quantum mechanics, we learned that the factorial divergence can arise from instantons. In QCD, the instantons are ill-defined in the IR and, strictly speaking, nobody knows what to do with them[9]. If we consider QCD in the 't Hooft limit of a large number of colors, instantons decouple. The corresponding singularity in the Borel plane (see Fig. 8) moves to the right infinity. At the same time, none of the essential features of QCD disappears in the 't Hooft limit. Therefore, in our simplified consideration, we can forget about instantons. Perhaps, they will be needed later.

If so, we can write a single (simplified) "master" formula for QCD and similar theories. At large Euclidean momenta, the correlation functions of type (27) can be represented as

$$D(Q^2) = \sum_{n=0}^{n_*^0} c_{0,n} \left(\frac{1}{\ln(Q^2/\Lambda^2)}\right)^n + \sum_{n=0}^{n_*^1} c_{1,n} \left(\frac{1}{\ln(Q^2/\Lambda^2)}\right)^n \left(\frac{\Lambda}{Q}\right)^{d_1} + \sum_{n=0}^{n_*^2} c_{2,n} \left(\frac{1}{\ln(Q^2/\Lambda^2)}\right)^n \left(\frac{\Lambda}{Q}\right)^{d_2} + \ldots + \text{“exponential terms”}. \quad (42)$$

Equation (42) is simplified in a number of ways. First, it is assumed that the currents in the left-hand side have no anomalous dimensions, and so do the operators appearing in the right-hand side. They are assumed to have only normal dimensions given by $d_i$ for the $i$th operator. Second, we ignore the second and all higher coefficients in the $\beta$ function, and hence the running coupling is represented by a pure logarithm. All these assumptions are not realistic in QCD[10]. We stick to them to make the master formula more concise. Inclusion of higher orders in the $\beta$ function and anomalous dimensions in both the left- and right-hand sides would give rise to rather contrived additional terms and factors containing $\log\log$'s, $\log\log\log$'s $(\log\log/\log)$'s, etc.[11]. This is a purely technical, rather than conceptual, complication, however.

So far, we discussed the divergence/convergence of the perturbative series explaining that the regulating parameter $\mu$ in OPE allows making the PT meaningful[12]. Expansion (42) runs not only in powers of $1/\ln Q^2$ but also in powers of $\Lambda/Q$. This is a double expansion, and the power series in $\Lambda/Q$ is also infinite in its turn. Does it have a finite radius of convergence?

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[9] This statement is a slight exaggeration. We refer to [22] for an alternative point of view on instantons in the QCD vacuum.

[10] They could be made somewhat more realistic in $\mathcal{N} = 2$ super-Yang–Mills theories.

[11] Multiple logarithms are elements of the trans-series analysis too, see [1].

[12] Factorial divergence of PT series due to a factorially large number of Feynman graphs with many loops is suppressed in the 't Hooft limit.

Needless to say, this is an important question. The answer to it is negative. Twenty years ago I argued in [23] (see also [15]) that the power series in (42) are factorially divergent in high orders. This is a rather straightforward observation following from the analytic structure of $D(Q^2)$. In a nutshell, because the $Q^2$ singularities in $D(Q^2)$ run all the way to infinity along the positive real semi-axis of $q^2$, the $1/Q^2$ expansion cannot be convergent. The last line in Eq. (42) symbolically represents a divergent tail of the power series.

## 9. SUPERSYMMETRIC YANG–MILLS THEORY

Factorial divergence of the perturbative series in supersymmetric theories was only scarcely discussed in the past [24–26]. Meanwhile, this is an interesting question because renormalons in supersymmetric theories have peculiarities related to peculiarities of the operator product expansion in supersymmetric Yang–Mills theory.

As we already know, the renormalons are in a one-to-one correspondence with particular gluon operators in OPE. There is a one-to-one correspondence between the given bubble-chain graph and an appropriate operator in OPE (see, e. g., [18]).

The SYM Lagrangian is

$$\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{i}{g^2} \bar{\lambda}^a \bar{\sigma}^\mu \, \mathcal{D}_\mu \lambda^a. \qquad (43)$$

The only difference with the QCD Lagrangian in (22) is in the fermion sector: the fundamental quarks are replaced by a Majorana spinor in the adjoint representation of the gauge group.

Supersymmetry of the model implies that an infinite class of gluonic operators cannot have nonvanishing vacuum expectation values (VEVs). This fact tells us that the conventional renormalon analysis must be modified. Below, we discuss a modification needed, but first see why gluonic operator VEVs must vanish in the SYM theory, in contradistinction to QCD.

### 9.1. Why gluon operators have vanishing VEVs in SYM?

The operator $G^a_{\mu\nu} G^a_{\mu\nu} + i \bar{\lambda}^a \bar{\sigma}^\mu \, \mathcal{D}_\mu \lambda^a$ is the highest component of $\mathrm{Tr} W^2$, where

$$W_\alpha = i \left( \lambda_\alpha + i\theta_\alpha \, D - \theta^\beta G_{\alpha\beta} - i\theta^2 \, \mathcal{D}_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \right). \qquad (44)$$

Supersymmetry allows only the lowest components of super fields to develop a nonvanishing VEV. In a pure SYM theory, without matter, $D = 0$, and therefore

$$G_{\alpha\beta} \sim D_{\{\beta} W_{\alpha\}} + \ldots, \qquad (45)$$

where the braces denote symmetrization, $D_\beta$ is the spinorial derivative, and the ellipses stand for higher components irrelevant for our purposes.

Gluonic operators in the pure SYM theory must contain at least two $G$ factors; in other words, their generic form is

$$O_g \propto G \ldots G \propto D_{\{\beta} W_{\alpha\}} \ldots D_{\{\tilde{\beta}} W_{\tilde{\alpha}\}}. \qquad (46)$$

The ellipses above represent any number of covariant derivatives and extra $W$ factors under the condition that the overall number of the $W$ factors be even. Taking Tr (which singles out color-singlet parts) is implied but not explicitly indicated.

The right-hand side in (46) can be identically rewritten as

$$D_{\{\beta} W_{\alpha\}} \ldots D_{\{\tilde{\beta}} W_{\tilde{\alpha}\}} = D_{\{\beta} \left( W_{\alpha\}} \ldots D_{\{\tilde{\beta}} W_{\tilde{\alpha}\}} \right) + \\ + W_\alpha \left( D_\beta \ldots D_{\{\tilde{\beta}} W_{\tilde{\alpha}\}} \right). \qquad (47)$$

The first term is a full superderivative and, as such, can have no nontrivial VEV. The lowest component of the second term, at the very least, contains $\lambda$ and $\mathcal{D}_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}$ (the last factor vanishes due to the equation of motion). Thus, the lowest-dimensional operator that could in principle appear in OPE is a two-$\lambda$ operator. However, this cannot appear too because if we calculate the OPE coefficients perturbatively (and renormalons are perturbative objects), then two-$\lambda$ operators have a wrong $R$ parity, while $\lambda\bar{\lambda}$ operators can have the Lorentz spin zero only in the combination

$$\mathrm{Tr} \, \lambda_\alpha \mathcal{D}_{\alpha\dot{\gamma}} G^{\dot{\gamma}\dot{\beta}} \bar{\lambda}^{\dot{\beta}}, \qquad (48)$$

which reduces to a four-fermion operator by virtue of the equation of motion. Thus, in a pure SYM, the OPE in fact starts from four-gluino operators of dimension 6 and the four-lambda operators with possible additional insertions of covariant derivatives and $G$ or $\lambda\lambda$ or $\lambda\bar{\lambda}$ factors, which have dimensions higher than 6. No purely gluonic operator can have a nonvanishing VEV in SYM.

The above argument based on $R$ parity is applicable to the two-point functions of the type

$$i \int d^4x \, e^{iqx} \left\langle O(x), \, O^\dagger(0) \right\rangle, \\ O = \mathrm{Tr} \, \bar{\lambda}_{\dot{\alpha}} \lambda_\alpha \ \text{ or } \ \mathrm{Tr} \, \lambda^\alpha \lambda_\alpha. \qquad (49)$$

Since the IR renormalons are in a one-to-one correspondence with OPE, we conclude that the bubble chain in Fig. 7, which normally is responsible for the non-Borel-summable divergence of higher orders (closest to the origin in the Borel plane) must be canceled by something else.

We note, however, that an easy way of identifying the "bubble chains" in QCD was through matter loops with the subsequent extraction of the $N_f$ factor plus substitution (31). This trick does not work in SYM (see (43)). We are forced to introduce matter fields.

### 9.2. Matter loops in SYM

To identify the renormalon bubble chain through matter loops, we should expand supersymmetric gluodynamics (43) to include $N_f$ matter fields in the fundamental representation of SU($N$),

$$\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{i}{g^2} \bar{\lambda}^a \bar{\sigma}^\mu \mathcal{D}_\mu \lambda^a + \\ + \sum_f \left( \mathcal{D}^\mu \overline{q_f} \mathcal{D}_\mu q_f + i \overline{\psi_f} \bar{\sigma}^\mu \mathcal{D}_\mu \psi_f \right) + \\ + \left[ -\frac{m}{2} \psi^f_\alpha \psi^\alpha_f + i\sqrt{2} \left( \psi_f \lambda^a T^a \right) \overline{q_f} + \text{H.c.} \right] - V(q_f), \quad (50)$$

where

$$V(q_f) = \frac{g^2}{2} \left( \sum_f \overline{q_f} T^a q_f \right)^2 + \sum_f |m|^2 |q_f|^2. \quad (51)$$

Here, $q$ and $\psi$ are the respective squark and quark fields. The mass terms in (50) and (51) are irrelevant and can be safely omitted[13].

In addition to the bubbles depicted in Fig. 7, bubble chains now develop elsewhere, as in Fig. 11.

Unlike the familiar QCD example (Fig. 7), the matter bubbles in the SYM theory appear even in the diagrams without gluon insertions, such as the graph depicted in Fig. 12. Each bubble produces $N_f g^2 \log p^2$, where $p$ is the momentum flowing through the gluino line. However, this particular diagram would correspond to the operator

$$\bar{\lambda}_{\dot{\alpha}} i \mathcal{D}^{\dot{\alpha}\alpha} \lambda_\alpha, \quad (52)$$

which reduces via the equation of motion to another dimension-4 operator,

$$\sum_f \phi^j \left( \bar{\lambda}^i_j \bar{\psi}_i \right),$$

[13] We should remember that each flavor is represented by two squarks and two Weyl quarks, one in the fundamental and another in the antifundamental representation of SU($N$).

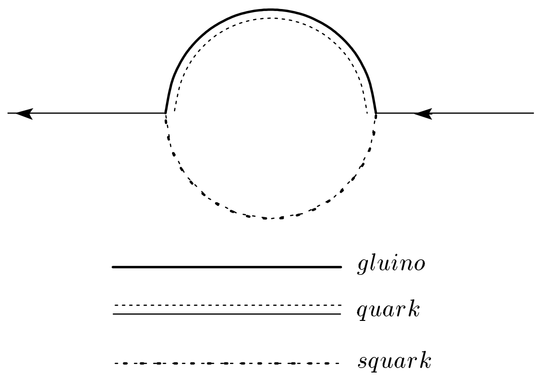


**Fig. 11.** Elementary bubble insertion in the gluino line

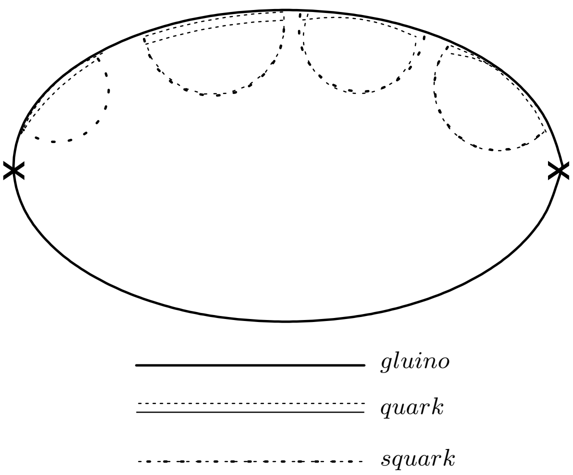


**Fig. 12.** Additional bubble diagrams in SYM, with matter insertions in the gluino line. $N_f$ matter bubbles produce the $N_f$ factor

which has no analog in QCD. We note that in SYM with matter, chiral symmetry is broken *a priori*, and is replaced by $R$ symmetry of the U(1) type. In addition, there is an anomalous part in the equation of motion to be used in Eq. (52), which produces the operator ${G_{\mu\nu}}^2$.

Calculating the elementary bubble insertion in the gluino line is straightforward. It is determined by the graph in Fig. 11. In fact, there is no need for an explicit calculation of this diagram. It represents the $Z$ factor of the gluino field. However, supersymmetry guarantees that the renormalization of the gluon and gluino fields are identical.

If we start from Lagrangian (50) normalized at a momentum scale $Q$ (then the corresponding coupling is $g^2(Q)$) and evolve it down to $p$, then the operator $(i/g^2)\bar{\lambda}^a \bar{\sigma}^\mu \mathcal{D}_\mu \lambda^a$ in the Lagrangian evolves as

$$\frac{i}{g^2(Q)}\bar{\lambda}^a\bar{\sigma}^\mu\,\mathcal{D}_\mu\lambda^a \to \frac{i}{g^2(p)}\bar{\lambda}^a\bar{\sigma}^\mu\,\mathcal{D}_\mu\lambda^a. \tag{53}$$

The corresponding $Z$ factor can be easily read off, for instance, by proceeding to the canonically normalized gluino kinetic term. In this way, we find

$$Z^{-1} = \frac{g^2(Q)}{g^2(p)} = 1 - \frac{g^2}{4\pi}(3N - N_f)\log\frac{Q^2}{p^2}, \tag{54}$$

where the matter bubble produces only the $N_f$ part, of course. In other words, the (truncated) diagram in Fig. 11 produces

$$(p_\mu\gamma^\mu)\,\frac{N_f g^2}{4\pi}\log\frac{Q^2}{p^2}. \tag{55}$$

In summary, we see that the standard method of the renormalon analysis, which works well in QCD, is not so straightforward in supersymmetric gluodynamics (i. e., gluons plus gluinos) because the introduction of matter dramatically changes the OPE operator basis. In pure YM and in QCD with massless quarks, it is one and the same dimension-4 operator that acquires a VEV and is responsible for the leading renormalon singularity. This is in sharp contradistinction with what happens in SYM.

### 9.3. Renormalons, OPE, and diagrams in SYM

We elucidate the last statement. The role of the IR renormalon bubble chain in a given diagram is to make the line to which bubbles are attached soft [17, 18]. At a critical value of $n$, the integration momentum flowing through the bubble line becomes of the order of $\Lambda$. Hence, in the framework of OPE, this line must be "cut" and becomes a part of the operator with a VEV, which then represents the tail of the renormalon. For instance, if we consider the graph in Fig. 7, the solid lines carry a large momentum, while the dashed one is soft. Correspondingly, this bubble chain is in a one-to-one correspondence with the operators $G^2$, $G\mathcal{D}^2G \to G^3$, and so on. In Fig. 12, the upper part of the graph is soft, while the lower part is hard. One of the operators corresponding to this bubble chain is $\mathrm{Tr}\,\lambda_\alpha\mathcal{D}_{\alpha\dot\beta}\bar{\lambda}^{\dot\beta}$. Four-gluino operators are obtained from the chains depicted in Fig. 13. In this graph, we should "cut" the gluino lines with bubble insertions. A large external momentum is passed through the graph, via lines without bubbles.

The problem of interpretation arises only with the bubble chains attached to the gluon lines, because the

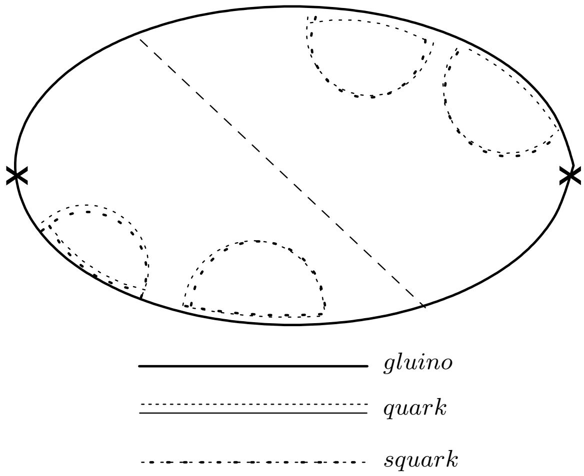


**Fig. 13.** Lines with bubbles are soft. Those without bubbles are hard

corresponding operators that should conspire with the tail of such renormalons can have no VEVs. Based on arguments that are not discussed here, I am inclined to conjecture that the renormalon depicted in Fig. 7 is canceled by the renormalon depicted in Fig. 12. If the numerical coefficient $c$ is right, the operator that these two graphs (combined together) give in OPE is

$$-\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \frac{ic}{g^2}\,\bar{\lambda}^a\bar{\sigma}^\mu\,\mathcal{D}_\mu\,\lambda^a \to 0. \tag{56}$$

This question should be explored further, however, see [27].

## 10. CONCLUSIONS

1. Resurgence in the sense it is carried out in quantum mechanics, encounters conceptual difficulties in strongly coupled Yang–Mills theories.

2. The best we can do is to use Wilson's operator product expansion adapted to QCD, which has conceptual similarities with the resurgence program.

3. In SYM theories, there are additional technical problems with renormalons, not addressed in the past, which are not yet fully solved.

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