

EXACT SOLUTIONS FOR THE EVOLUTION OF A BUBBLE IN AN IDEAL LIQUID IN A UNIFORM EXTERNAL ELECTRIC FIELD

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The dynamics of a bubble in a dielectric liquid under the influence of a uniform external electric field are considered. It is shown that in the situation where the boundary motion is determined only by electrostatic forces, the special regime of fluid motion can be realized for which the velocity and electric field potentials are linearly related. In the two-dimensional case, the corresponding equations are reduced to an equation similar in structure to the well-known Laplacian growth equation, which, in turn, can be reduced to a finite number of ordinary differential equations. This allows us to obtain exact solutions for asymmetric bubble deformations resulting in the formation of a finite-time singularity (cusp).

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1. INTRODUCTION

It is known that a liquid drop suspended in another liquid deforms when an external electric field is applied. In the case of two perfect dielectric fluids with no free charges at the interface, an initially spherical drop (or a gas bubble) is stretched by the electrostatic forces in the direction of the electric field [1–4]. For leaky dielectric fluids, the drop behavior becomes more complicated; its deformation also depends on the ratio of the conductivities of the fluids (see [5, 6] and the references therein).

Considerable interest is focused on the behavior of a conducting drop surrounded by an insulating fluid in an electric field [7–9]). In this situation, the electric field also stretches the drop. If the drop moves through the ambient fluid, the dynamic pressure of the flow should be taken into account. For irrotational flow, in the absence of an electric field, the drop is flattened along the direction of its motion (see, e. g., Ref. [10]). Considering a bubble instead of a drop corresponds to passing to the limit of zero density of the internal fluid. If the surface of the bubble is assumed to be perfectly conducting, then the electric field does not penetrate into the interior of the bubble as well as it does not penetrate into the conducting drop.

The problem of bubble motion, as well as any other problem concerning the dynamics of a free surface or interface, is extremely difficult to solve. Therefore, it is important to find ways to simplify the corresponding equations of motion. One known approach is to consider the Stokes flow of a viscous incompressible fluid, where the stream function satisfies the biharmonic equation (see, e. g., Refs. [8, 9, 11]). It is clear that the analysis essentially simplifies for a two-dimensional bubble [12]. The effect of an electric field on the motion of a two-dimensional bubble or drop surrounded by a viscous fluid was studied numerically in Refs. [13, 14]. In the case of two spatial dimensions, the conformal mapping technique can be effectively used for studying the bubble behavior. It allows one to reduce the original moving boundary problem to a fixed boundary problem (see the papers by Crowdy [11] and by Tanveer and Vasconcelos [15]).

In this paper, we show that if the boundary motion is determined only by electrostatic forces (capillary forces being ignored), it is possible to use a completely different method to simplify the equations of motion, which is applicable to studying the potential flow of an incompressible, inviscid fluid. The method is based on the consideration of a special regime of liquid mo-

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tion for which the velocity and electric field potentials are linearly dependent functions. Due to this dependence, the number of equations required for describing the motion of the bubble boundary can be reduced by half. The reduction can be carried out in the general three-dimensional case. In the particular case of a two-dimensional bubble, where it is possible to use the conformal mapping technique, the problem reduces to an equation similar to the Laplacian growth equation (LGE), whose time-dependent exact solutions can be found analytically. Its simplest (quasistationary) solution corresponds to an elliptical bubble moving with a constant velocity along the direction of the external electric field. Other (nontrivial) solutions describe the development of instability of the steady flow. Initially small deviations from the elliptical shape of the bubble grow rapidly; the bubble boundary is deformed asymmetrically, resulting in the formation of a singularity (a cusp) in a finite time.

We note that a similar approach was previously used in the analysis of the electrohydrodynamic instability of a charged free surface of liquid helium [16, 17] and also of an interface between two dielectric fluids [18]. A condition for instability of the plane boundary of liquid helium (the threshold value of the surface charge density) was found in Ref. [19]; it is a generalization of the instability criterion for the surface of a conducting liquid in an external electric field [20]. A functional relation between the electric and velocity potentials that underlies the analysis of boundary dynamics in Refs. [16–18] arises in the situation where electrostatic forces dominate over gravitational and capillary forces, i. e., if the system is far above the stability threshold [19].

2. INITIAL EQUATIONS

We consider the dynamics of the free boundary of a bubble in a perfect dielectric (nonconducting) fluid in the presence of an external uniform electric field. We assume that the electric field is directed along the x axis of the Cartesian coordinate system, and E is the external electric field strength. Let $D(t)$ be the region occupied by the fluid, $D_b(t)$ be the region corresponding to the bubble, and $\partial D(t)$ be the bubble boundary. We suppose that the surface of the bubble is conductive and the charge relaxation time is small, and hence the surface can be considered equipotential in the characteristic times of electrohydrodynamic phenomena. This situation can correspond to the bubble filled with a discharge plasma formed during electrical breakdown in a liquid dielectric.

We assume that the fluid is inviscid and incompressible and that the flow is irrotational (potential). The velocity and electric field potentials, ϕ and φ, satisfy the Laplace equations

$$\nabla^2\phi = 0, \quad \nabla^2\varphi = 0 \quad \text{in} \quad D(t). \tag{1}$$

The velocity potential ϕ obeys the dynamic boundary condition (the nonstationary Bernoulli equation on a free surface),

$$\rho\phi_t + \frac{\rho}{2}(\nabla\phi)^2 = \Delta p - \frac{\varepsilon_0\varepsilon}{2}(\nabla\varphi)^2 \quad \text{on} \quad \partial D(t). \tag{2}$$

Here, ε_0 is the vacuum permittivity, ε is the dielectric constant of the fluid, ρ is its density, and Δp is the difference between the fluid pressure at infinity and the pressure in the bubble, $\Delta p = p_\infty - p_b$ (the bubble is regarded as a constant-pressure region). We suppose that Δp does not vary with time and is defined by

$$\Delta p = \frac{\varepsilon_0\varepsilon}{2}E^2,$$

which corresponds to volume-preserving deformations of the bubble. The last term in the right-hand side of Eq. (2) is responsible for the electrostatic pressure at the bubble boundary resulting from the interaction between free surface charges and the external electric field. We note that the surface tension effects are not taken into account in (2); this corresponds to the formal limit of a strong external electric field.

Without loss of generality, the electric field potential can be assumed to be zero at the bubble boundary:

$$\varphi = 0 \quad \text{on} \quad \partial D(t). \tag{3}$$

Formally, the equation

$$\varphi(x, y, z, t) = 0$$

is the equation of a free surface. Then the condition that the velocity of the bubble surface coincides with the normal velocity of the ambient liquid (the kinematic boundary condition) can be written as

$$\varphi_t + \nabla\varphi\cdot\nabla\phi = 0 \quad \text{on} \quad \partial D(t). \tag{4}$$

The system is closed by the conditions

$$\phi \to 0, \quad \varphi \to -Ex, \quad |\mathbf{r}| \to \infty, \tag{5}$$

stating that the liquid is at rest and the electric field is uniform at an infinite distance from the bubble.

Multiplying kinematic boundary condition (4) by $\sqrt{\rho\varepsilon_0\varepsilon}$, then adding and subtracting dynamic (2) and kinematic (4) boundary conditions, we find

$$\rho\left(\phi \pm \varphi\sqrt{\frac{\varepsilon_0\varepsilon}{\rho}}\right)_t + \frac{\rho}{2}\left(\nabla\left(\phi \pm \varphi\sqrt{\frac{\varepsilon_0\varepsilon}{\rho}}\right)\right)^2 = \frac{\varepsilon_0\varepsilon}{2}E^2 \quad \text{on} \quad \partial D(t).$$

It follows from these expressions that it is convenient to introduce a pair of auxiliary potentials,

$$\psi^{(\pm)} \equiv \phi \pm \varphi\sqrt{\varepsilon_0\varepsilon/\rho}.$$

Then the initial equations (1)–(5) take the symmetric form

$$\nabla^2\psi^{(\pm)} = 0 \quad \text{in} \quad D(t), \tag{6}$$

$$\psi_t^{(\pm)} + \frac{1}{2}\left(\nabla\psi^{(\pm)}\right)^2 = \frac{\varepsilon_0\varepsilon}{2\rho}E^2 \quad \text{on} \quad \partial D(t), \tag{7}$$

$$\psi^{(\pm)} \to \mp Ex\sqrt{\varepsilon_0\varepsilon/\rho}, \quad |\mathbf{r}| \to \infty. \tag{8}$$

Equipotentiality condition (3) is then rewritten as

$$\psi^{(+)} = \psi^{(-)} \quad \text{on} \quad \partial D(t). \tag{9}$$

This form of the equations of motion turns out to be very convenient for the analysis of the bubble dynamics.

3. REDUCED EQUATIONS OF MOTION

An important feature of the system of equations (6)–(9) is that they are compatible with the conditions

$$\psi^{(-)} = +Ex\sqrt{\frac{\varepsilon_0\varepsilon}{\rho}} \quad \text{or} \quad \psi^{(+)} = -Ex\sqrt{\frac{\varepsilon_0\varepsilon}{\rho}}.$$

This proves the possibility of realizing the special regime of fluid motion for which the potentials are related by the linear expressions

$$\phi = \pm\sqrt{\varepsilon_0\varepsilon/\rho}\,(\varphi + Ex). \tag{10}$$

As follows from them, there exists a moving coordinate system in which the liquid moves along the electric field lines. Relations (10) allow eliminating one of the potentials from the initial equations of motion, which significantly simplifies their form.

For convenience, we switch to dimensionless variables,

$$\varphi \to \varphi ER, \quad t \to t\sqrt{\rho R^2/\varepsilon_0\varepsilon E^2}, \quad \mathbf{r} \to \mathbf{r}R,$$

where R is the characteristic size of the bubble. The reduced equations of motion, written in terms of the electric field potential, have the form

$$\nabla^2\varphi = 0 \quad \text{in} \quad D(t), \tag{11}$$

$$\varphi_t \pm \varphi_x \pm (\nabla\varphi)^2 = 0 \quad \text{on} \quad \partial D(t), \tag{12}$$

$$\varphi = 0 \quad \text{on} \quad \partial D(t), \tag{13}$$

$$\varphi \to -x, \quad |\mathbf{r}| \to \infty. \tag{14}$$

These two systems differ only by the time direction (they are related by the replacement $t \to -t$). Without loss of generality, we can consider only the system with the upper signs in Eq. (12).

Thus, analyzing the initial equations (1)–(5), we have shown that a special flow regime can be realized for which the velocity and electric field potentials are linearly related functions. This regime is described by the much simpler system (11)–(14). We emphasize that this result was obtained in the general three-dimensional case.

Below, we analyze system (11)–(14) in the particular case of two dimensions, where $\varphi = \varphi(x, y, t)$ (i.e., there is no dependence on the third spatial variable). This means that we consider the evolution of a two-dimensional bubble, for example, as in Refs. [10, 11]. It is clear that such a consideration is more of an academic rather than practical interest. However, in this case, it is possible to find exact solutions of the equations of motion using the conformal mapping technique; these solutions give us an insight into the behavior of a bubble in the considered special flow regime. Probably, the basic regularities of the bubble behavior are common for two- and three-dimensional cases.

We now proceed with the two-dimensional case. We introduce the complex variable $z = x + iy$ and assume that the surface $\partial D(t)$ is defined parametrically, $x = X(l, t)$ and $y = Y(l, t)$, or in the complex form $z = Z(l, t)$, where $Z = X + iY$ and l is some parameter, to be specified later. The complex electric potential $W \equiv \varphi - i\psi$ is an analytic function of the complex variable z outside the bubble, i.e., $W = W(z)$. Here, the function ψ is a harmonic conjugate of the electric field potential φ (the condition $\psi = \text{const}$ defines the electric field lines). The complex potential satisfies the condition $W \to -z$ at infinity. On the bubble surface, the following relations hold:

$$\varphi_t = -\operatorname{Re}(W_z Z_t), \quad \varphi_x = \operatorname{Re}(W_z), \quad (\nabla\varphi)^2 = |W_z|^2.$$

As a consequence, condition (12), which defines the motion of the boundary, can be rewritten as

$$\mathrm{Re}\left[(Z_t-1)/\overline{W_z}\right]=1 \quad \text{on} \quad \partial D(t). \tag{15}$$

We perform a time-dependent conformal mapping of the region $D(t)$ onto the region outside the unit circle in the parametric ξ-plane. Then the bubble boundary maps onto the circle $|\xi|=1$. In terms of new variables, the complex electric potential is given by the expression

$$W(\xi,t)=-\lambda(t)(\xi-\xi^{-1}), \tag{16}$$

where $\lambda(t)$ is a function of time. The inverse mapping $z(\xi,t)$ is unknown and has to be found. The function $z(\xi,t)$ is analytic for $|\xi|\geq 1$; it satisfies the condition $z\to\lambda(t)\xi$ at infinity. The equation describing the time evolution of the mapping function $z(\xi,t)$ can be obtained from (15). It has the form

$$\mathrm{Re}\left[(Z_t-1)\overline{z_\xi}/\overline{W_\xi}\right]=1, \quad |\xi|=1.$$

Substituting expression (16) for W here and using the parameterization $\xi=e^{il}$ for the circle $|\xi|=1$, where l is a real parameter varying in the range $0\leq l\leq 2\pi$, we finally obtain the equation of the LGE type (see, e. g., Ref. [21]):

$$\mathrm{Im}\left[(Z_t-1)\overline{Z_l}\right]=2\lambda(t)\cos l. \tag{17}$$

We demonstrate that for the considered flow regime, the bubble area (the area of its cross section by the plane xy) does not change with time. Indeed, according to the Green's formula, the area of the region $D_b(t)$ is given by the contour integral

$$S=\int\limits_{D_b(t)} dx\,dy=-\frac{1}{2}\,\mathrm{Im}\int\limits_0^{2\pi} Z\overline{Z_l}\,dl. \tag{18}$$

Differentiating this expression with respect to time, we obtain

$$S_t=-\,\mathrm{Im}\int\limits_0^{2\pi} Z_t\overline{Z_l}\,dl=\int\limits_0^{2\pi}(Y_l-2\lambda\cos l)\,dl=0,$$

where the integrand has been transformed with the help of (17). Then the area S and therefore the bubble volume are constant. Otherwise, we would have to take the change of the gas pressure inside the bubble into account. We recall that in the initial problem statement, the difference between pressures inside the bubble and in the liquid at infinity was assumed to be constant.

4. EXACT SOLUTIONS

A remarkable feature of Eq. (17) is that it admits reduction to a system of a finite number of ordinary differential equations (ODEs). The substitution of the form

$$Z(l,t)=\lambda(t)e^{il}+t+\sum_{n=0}^{N}\alpha_n(t)e^{-inl} \tag{19}$$

yields $N+2$ ODEs for the amplitudes $\lambda(t)$ and $\alpha_n(t)$ ($n=0,1,\ldots,N$). The nonlinear interaction of harmonics does not lead to the appearance of new harmonics. We note that expression (19) corresponds to the mapping function

$$z(\xi,t)=\lambda(t)\xi+t+\sum_{n=0}^{N}\alpha_n(t)\xi^{-in},$$

which is analytic for $|\xi|\geq 1$ and satisfies the required condition $z\to\lambda\xi$ at infinity.

As can be seen from the structure of Eq. (17), the system of ODEs for the harmonic amplitudes corresponding to ansatz (19) is linear in derivatives and, consequently, it can always be resolved with respect to them. This enables its efficient numerical solution. In some special cases, the system can be solved analytically.

The case $N=1$ is trivial: the bubble surface is always elliptical. The simplest case, where the bubble geometry already differs from elliptical and, consequently, the dynamics of its surface is nontrivial, corresponds to $N=2$. With further increase in the number of harmonics, the surface dynamics becomes more complicated; however, the main regularities of bubble evolution already appear at $N=2$.

So, we construct the simplest solutions of (17) corresponding to ansatz (19) with $N=2$. We seek the function Z in the form

$$Z(l,t)=\lambda(t)e^{il}+\alpha(t)+t+\beta(t)e^{-il}+\gamma(t)e^{-2il}, \tag{20}$$

where $\lambda(t)$, $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are real amplitudes. Separating the real and imaginary parts in (20), we obtain the following parametric expressions for the bubble profile:

$$x=X(l,t)=\alpha(t)+t+[\lambda(t)+\beta(t)]\cos l+ \\ +\gamma(t)\cos 2l, \tag{21}$$

$$y=Y(l,t)=[\lambda(t)-\beta(t)]\sin l-\gamma(t)\sin 2l. \tag{22}$$

Substituting (20) in (17) and collecting the terms corresponding to the same harmonics, we arrive at the system of four first-order ODEs

$$-\lambda\lambda_t + \beta\beta_t + 2\gamma\gamma_t = 0, \tag{23}$$

$$\beta\gamma_t - \lambda\alpha_t + \beta\alpha_t + 2\gamma\beta_t = 2\lambda, \tag{24}$$

$$\beta\lambda_t + 2\gamma\alpha_t - \lambda\beta_t = 0, \tag{25}$$

$$-\lambda\gamma_t + 2\gamma\lambda_t = 0. \tag{26}$$

Integrating (26) yields

$$\gamma = k\lambda^2, \tag{27}$$

where k is a constant. We can see that system (23)–(26) admits two types of solutions. The first family of solutions corresponds to the case $k = 0$:

$$\begin{gathered} \lambda = \lambda_0, \quad \beta = \beta_0, \quad \gamma = 0, \\ \alpha = -\frac{2\lambda_0}{\lambda_0 - \beta_0}\, t + \alpha_0, \end{gathered} \tag{28}$$

where λ_0, β_0, and α_0 are constants. This quasistationary solution can be thought of as the unperturbed solution. According to (21) and (22), the bubble surface is an ellipse

$$\frac{[x - \alpha(t) - t]^2}{(\lambda_0 + \beta_0)^2} + \frac{y^2}{(\lambda_0 - \beta_0)^2} = 1.$$

The "elliptic" bubble moves with a constant translational velocity $v = -(\lambda_0 + \beta_0)/(\lambda_0 - \beta_0)$ along the x axis. In the special case where $\beta_0 = 0$ (and hence $v = -1$), the bubble boundary is a circle of radius λ_0.

The second family of solutions corresponds to $k \neq 0$. Taking Eq. (27) into account, we use (23) and (25) to obtain

$$\beta = \pm \left(\lambda^2 - 2k^2\lambda^4 - s\right)^{1/2}, \tag{29}$$

$$\alpha = \frac{\beta}{2k\lambda} + \alpha_1. \tag{30}$$

Thus, the amplitudes α, β, and γ can be expressed in terms of λ. Finally, solving Eq. (24), we find the implicit dependence of the amplitude λ on time t:

$$-\frac{s + 2k^2\lambda^4}{4k\lambda^2} \pm \frac{4k^2\lambda^2 - 1}{2k\lambda}\left(\lambda^2 - 2k^2\lambda^4 - s\right)^{1/2} = 2(t - t_1). \tag{31}$$

Thus, in addition to the trivial solution (28), for which the second harmonic amplitude γ is identically zero, system of four ODEs (23)–(26) admits nontrivial (perturbed) solution (27), (29)–(31), for which the amplitude γ differs from zero, and the bubble is not elliptical. This solution contains four constants (integrals of motion): k, s, α_1 and t_1. The constant α_1 specifies the position of the bubble on the x axis; t_1 is the initial time moment. To clarify the meaning of the constant s, we calculate the bubble area using formula (18). Substituting (20) in (18), we obtain

$$S = \pi(\lambda^2 - \beta^2 - 2\gamma^2) = \pi s,$$

and hence s is proportional to the area S. Finally, the constant k characterizes the contribution of the second harmonic to the bubble evolution; in fact, it defines how much the bubble shape deviates from an ellipse (i.e., from the unperturbed state) at the initial moment of time.

Solution (27), (29)–(31) describes the evolution of the bubble boundary up to the formation of a singularity at some finite time $t = t_c$. Figure 1 shows the bubble profile (21) and (22) at successive instants of time. It is seen that the bubble whose initial shape is close to the unit circle is deformed asymmetrically, resulting in the formation of a cusp at one end of the bubble. A typical dependence of the amplitude λ on time is presented in Fig. 2. The cusp develops as the marked point $t = t_c$ is approached in the counterclockwise direction. We can see from the figure that all solutions of the second family exist only for a finite time interval.

Note that the cusp in curve 5 of Fig. 1 assumes an infinite number of harmonics in the variables $\{x, y\}$. In the conformal variables $\{\mathrm{Re}\,\xi, \mathrm{Im}\,\xi\}$, as demonstrated, it is sufficient to take a finite number of harmonics for the description of the singularity formation.

5. CONCLUSION

The original (three-dimensional) problem of bubble dynamics under the influence of electrostatic forces can be reduced to the analysis of much simpler equations describing the special flow regime where the velocity and electric field potentials are linearly related. In the case of two spatial dimensions, by using the conformal mapping technique, these equations can be reduced to the equation of LGE type, for which it is possible to construct a set of exact particular solutions. The simplest (quasistationary) solution (28) describes the motion of an elliptical bubble with constant velocity. Small perturbations of the initially elliptical boundary

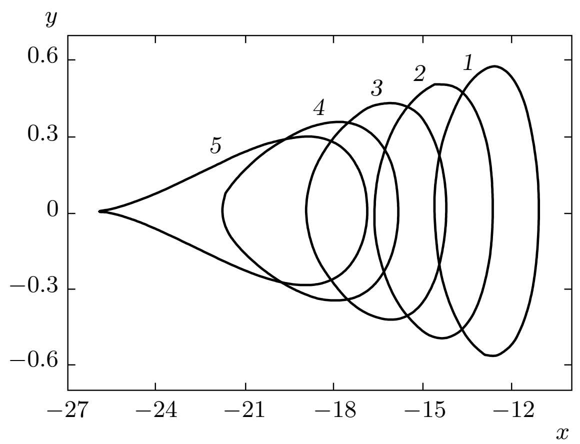

Fig. 1. Evolution of the bubble boundary corresponding to exact solutions (27), (29)–(31), where $s = 1$, $k = -0.02$, $t_1 = -11$, $\alpha_1 = 0$, and the upper sign "+" is taken. The boundary shape is shown for the successive instants $t = 0$, $t = 0.5$, $t = 0.9$, $t = 1.2$, and $t = t_c \approx 1.341$

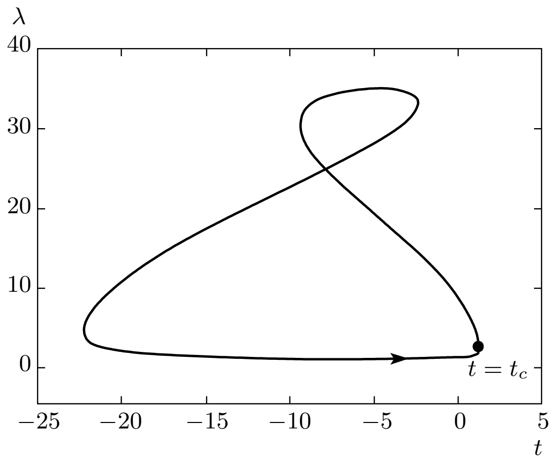

Fig. 2. Time dependence of the second harmonic amplitude λ for $s = 1$, $k = -0.02$, and $t_1 = -11$ described by Eq. (31). The instant of cusp formation corresponds to the marked point $t = t_c \approx 1.341$; the motion to this point occurs in the counterclockwise direction

increase with time, as is shown in Fig. 1. We can see that the cusp appears in a finite time at one end of the bubble (the solutions obtained do not have mirror symmetry about the vertical axis). The curvature of the boundary, its velocity, and the electric field strength become infinite at the singular point. It is clear that capillary forces, which are not taken into account in our study, can significantly change the dynamics of the system. But, we believe that the main result in this paper, namely, the demonstration of integrability of the corresponding free surface problem (even in the simplified version treated here), can be regarded as significant progress in theoretical studies of electrohydrodynamic phenomena.

As a rule, when analyzing the behavior of drops or bubbles in a uniform external electric field, researchers limit themselves to the case where the boundary possesses the fore-aft symmetry, which is determined by the symmetry of equilibrium configurations (see, for example, Refs. [8, 11]). In the present work, it has been shown that the fore-aft symmetry of the bubble can break, which leads to the formation of a cusp only at one side of the bubble. This result should be taken into account when studying the bubble dynamics, in particular with regard to the problem of electrical breakdown of dielectric liquids in the presence of gas bubbles.

We note that some results in this paper can be generalized to the case where a drop of incompressible dielectric liquid is considered instead of a bubble. Using the approach proposed in Refs. [18, 22], one can find that in the particular case where the ratio of the permittivities of the fluids is equal to the inverse ratio of their densities, the special regime of fluid motion can be realized for which the velocity and electric potentials are linearly dependent functions both inside and outside the drop.

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REFERENCES

1. Ch. T. O'Konski and F. E. Harris, J. Phys. Chem. **61**, 1172 (1957).

2. C. G. Garton and Z. Krasucki, Proc. Roy. Soc. London A **280**, 211 (1964).

3. J. D. Sherwood, J. Fluid Mech. **188**, 133 (1988).

4. J. D. Sherwood, J. Phys. A **24**, 4047 (1991).

5. S. J. Shaw and P. D. M. Spelt, Proc. Roy. Soc. London A **465**, 3127 (2009).

6. S. Mandal, K. Chaudhury, and S. Chakraborty, Phys. Rev. E **89**, 053020 (2014).

7. A. I. Grigorev, A. N. Zharov, and S. O. Shiryaeva, Zh. Tekhn. Fiz. **69**, 43 (1999).

8. N. Dubash and A. J. Mestel, J. Fluid Mech. **581**, 469 (2007).

9. J. G. Kim, D. J. Im, Y. M. Jung, and I. S. Kang, J. Coll. Int. Sci. **310**, 599 (2007).

10. E. B. McLeod Jr., Indiana Univ. Math. J. **4**, 557 (1955).

11. D. Crowdy, IMA J. Appl. Math. **73**, 740 (2008).

12. G. R. Baker and D. W. Moore, Phys. Fluids A **1**, 1451 (1989).

13. S. Mählmann and D. T. Papageorgiou, J. Fluid Mech. **626**, 367 (2009).

14. S. Mählmann and D. T. Papageorgiou, Theor. Comput. Fluid Dyn. **23**, 375 (2009).

15. S. Tanveer and G. L. Vasconcelos, J. Fluid Mech. **301**, 325 (1995).

16. N. M. Zubarev, Pis'ma v Zh. Eksp. Teor. Fiz. **71**, 534 (2000).

17. N. M. Zubarev, Zh. Eksp. Teor. Fiz. **121**, 624 (2002).

18. N. M. Zubarev, Phys. Fluids **18**, 028103 (2006).

19. L. P. Gor'kov and D. M. Chernikova, Pis'ma v Zh. Eksp. Teor. Fiz. **18**, 119 (1973).

20. Ya. I. Frenkel', Zh. Eksp. Teor. Fiz. **6**, 347 (1936).

21. M. Z. Bazant and D. Crowdy, in *Handbook of Materials Modeling*, ed. by S. Yip et al., Springer (2005), Vol. I, Art. 4.10.

22. E. A. Kochurin and N. M. Zubarev, Phys. Fluids **24**, 072101 (2012).