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NONLINEAR DYNAMICS OF MAGNETOHYDRODYNAMIC FLOWS OF HEAVY FLUID ON SLOPE IN SHALLOW WATER APPROXIMATION

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Magnetohydrodynamic equations for a heavy fluid over an arbitrary surface are studied in the shallow water approximation. While solutions to the shallow water equations for a neutral fluid are well known, shallow water magnetohydrodynamic (SMHD) equations over a nonflat boundary have an additional dependence on the magnetic field, and the number of equations in the magnetic case exceeds that in the neutral case. As a consequence, the number of Riemann invariants defining SMHD equations is also greater. The classical simple wave solutions do not exist for hyperbolic SMHD equations over an arbitrary surface due to the appearance of a source term. In this paper, we suggest a more general definition of simple wave solutions that reduce to the classical ones in the case of zero source term. We show that simple wave solutions exist only for underlying surfaces that are slopes of constant inclination. All self-similar discontinuous and continuous solutions are found. Exact explicit solutions of the initial discontinuity decay problem over a slope are found. It is shown that the initial discontinuity decay solution is represented by one of four possible wave configurations. For each configuration, the necessary and sufficient conditions for its realization are found. The change of dependent and independent variables transforming the initial equations over a slope to those over a flat plane is found.

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1. INTRODUCTION

Exact, explicit nonlinear solutions of magnetohydrodynamic (MHD) equations are rare. The shallow water magnetohydrodynamic (SMHD) equations are the alternative to solving the full set of magnetohydrodynamic equations for a heavy fluid with a free surface. These equations are derived from the MHD equations for an incompressible nonviscous fluid layer in the gravity field assuming that the pressure is hydrostatic, using the depth averaging, and taking the fluid layer depth to be much smaller than the characteristic size of the physical system. The derived system of equations [1, 2] is important in many applications of MHD to astrophysical and engineering problems. The SMHD approximation is widely used for the solar tachocline study [1, 3–5], for the description of spread of matter over a neutron-star surface during disc accretion [6, 7], for the study of the neutron-star atmosphere dynamics [8, 9], and for the study of exoplanets [10]. A similar approximation of the MHD equations for small Reynolds numbers is used to model the aluminum production processes [11, 12] and those in fusion technologies [13]. The nonlinear dynamics of the above flows is described by the full set of MHD equations for all scales. This system cannot be examined analytically and is still difficult to model numerically. Practically, the shallow water approximation has the same fundamental role in the plasma magnetohydrodynamics as a similar approximation has in the neutral fluid dynamics [14, 15]. The latter case is used widely to study the large-scale processes in Earth's atmosphere and oceans [16].

This paper is devoted to the study of nonlinear flows of a heavy fluid described by the SMHD equations over a nonflat surface and is an extension of the theory developed in [17] for magnetohydrodynamic shallow water equations over a flat plane. Indeed, results obtained

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in this paper are of particular interest for understanding astrophysical flows due to the lack of information on topographic futures of their lower boundaries. The theory developed in this paper can be a basis for justification of models of astrophysical objects. This new set of equations is also of general interests in solar fluid dynamics because the dynamical importance of a compositional stratified layer has been suggested in the tachocline [18]. In particular, the existence of the Sun's settling helium layer may lead to new nonlinear dynamics. Moreover, the Coriolis force and other external forces in the large-scale magnetohydrodynamic models can be naturally represented by using an effective nonflat surface, as is done in the study of large-scale atmospheric and oceanic flows (see, e. g., [19]). These equations serve as a basis for the development of multilayer stratified shallow water magnetohydrodynamic models, and for the development of finite-volume numerical methods for magnetohydrodynamic shallow water flows subjected to an external force (e. g., the Coriolis force or a hydraulic friction). In the appendix, we give a brief derivation of the SMHD equations on a nonhomogeneous boundary, describing the approximations made.

The SMHD equations are of hyperbolic type. However, while solutions of the shallow water equations for a neutral fluid are well known, magnetohydrodynamic shallow water equations over a nonflat boundary have an additional dependence on the magnetic field, and the number of equations in the magnetic case exceeds that in the neutral case. As a consequence, the number of Riemann invariants defining the magnetohydrodynamic shallow water equations increases. The hyperbolicity of magnetohydrodynamic shallow water equations (see [2] and J. A. Rossmanith's PhD dissertation, University of Washington (2002), Ch. 4) leads to the existence of discontinuous solutions, even if the initial conditions are differentiable, as well as to the existence of continuous ones. In the above papers, the properties of the SMHD equations as a hyperbolic system over a flat plane are studied, including linear wave modes, Riemann invariants, Rankine–Hugoniot conditions, and shock waves, and the numerical Roe-type Riemann solver is developed. In this paper, simple wave solutions of the SMHD equations over a nonflat surface are studied. The classical simple wave solutions do not exist for hyperbolic SMHD equations over an arbitrary surface due to the appearance of a source term. In this paper, we suggest a more general definition of simple wave solutions that reduce to the classical one in the case of zero source term. It is shown that these solutions exist only for the underlying surfaces that are slopes of constant inclination. Therefore, the main focus in this paper is on the study of magnetohydrodynamic shallow water flows on a sloping surface. New wave types appear in this case in contrast to solutions obtained in [4].

Magnetogravity rarefaction wave solutions, magnetogravity shock wave solutions and Alfvenic wave solutions for slopes are found. The characteristics of these waves are parabolas transforming to straight lines in the case of a flat plane. These particular waves are fundamental for studying nonlinear wave phenomena over a nonflat surface. The change of dependent and independent variables transforming the SMHD equations over a slope to those over a flat plane is found. The obtained change of variables is valid only for continuous solutions and fails for discontinuous ones. Hence, the full set of simple wave-type solutions on slopes cannot be found from those on a flat plane using this change of variables. For unified descriptions of fluid physics, we derive simple wave solutions, continuous and discontinuous ones, from the initial governing equations, although continuous ones can be obtained by transforming the solutions from [2]. The obtained continuous solutions allow finding trajectories of propagation of discontinuous solutions, and thus determine the domains of location of the solutions of the initial discontinuity decay problem as a combination of domains of continuous magnetohydrodynamic flows; for each of these, the suggested transformation of variables is applicable. It is used to find the exact solution of the initial discontinuity decay problem for the SMHD equation system over a slope. We find that the structure of the solution over a slope is the same as over a flat plane. The conditions for each wave configuration realization exactly match. It has been shown that the particular solutions in our case differ from those for incompressible shallow-water flows. Hence, the conditions of the realization of each configuration are different.

The initial discontinuity decay solution is represented by one of the following wave configurations: "two magnetogravity shock waves, two Alfvenic waves", "magnetogravity rarefaction wave, magnetogravity shock wave, two Alfvenic waves", "two magnetogravity rarefaction waves, two Alfvenic waves", "two hydrodynamic rarefaction waves and a vacuum region between them". Explicit expressions for continuous and discontinuous solutions obtained in Sec. 4 allow imposing a solution of the Riemann problem in explicit form.

In Sec. 2, the initial equations of shallow water magnetohydrodynamics over an arbitrary surface are presented. In Sec. 3, this set of equations is written in the Riemann invariant form and it is shown that sim-

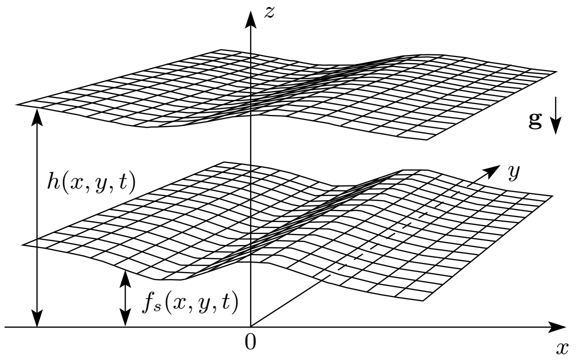

Fig. 1. Coordinate system and topography

ple wave solutions exist only for the underlying surfaces that are slopes of constant inclination. In Sec. 4, particular continuous and discontinuous simple wave solutions are found. In Sec. 5, the initial discontinuity decay problem solution for a slope is found. The main results are outlined in the conclusion.

2. SHALLOW WATER MAGNETOHYDRODYNAMIC EQUATIONS OVER AN ARBITRARY SURFACE

In this section, we consider a one-dimensional SMHD model to study the magnetic fluid flows with a free surface in the gravity field over an arbitrary boundary. The SMHD equations over an arbitrary boundary are obtained from the classical MHD equations [20] written for the fluid layer with a free surface in the gravity field over an arbitrary boundary $f_s(x)$ (Fig. 1). There, the z axis is parallel to the gravity force vector and is opposite in direction. Assuming that the magnetic fluid layer depth is small compared to the characteristic scale of the studied phenomena and the full pressure (the sum of magnetic and hydrodynamic pressures) is hydrostatic, the relevant system is averaged over the fluid layer depth and then admits a mean field description. We set $\tilde{B}_i = B_i/\sqrt{\rho}$ (ρ is a fluid density) to simplify the equations (the tilde sign is omitted in what follows) and write this system in the one-dimensional case:

$$\frac{\partial h}{\partial t} + \frac{\partial h u_1}{\partial x} = 0, \qquad (2.1)$$

$$\frac{\partial h u_1}{\partial t} + \frac{\partial (h u_1^2 - h B_1^2 + g h^2/2)}{\partial x} = -g h \frac{\partial f_s}{\partial x}, \qquad (2.2)$$

$$\frac{\partial h u_2}{\partial t} + \frac{\partial (h u_1 u_2 - h B_1 B_2)}{\partial x} = 0, \qquad (2.3)$$

$$\frac{\partial h B_1}{\partial t} = 0, \qquad (2.4)$$

$$\frac{\partial h B_2}{\partial t} + \frac{\partial (h B_2 u_1 - h B_1 u_2)}{\partial x} = 0, \qquad (2.5)$$

$$\frac{\partial h B_1}{\partial x} = 0. \qquad (2.6)$$

Here, x and t are the spatial and temporal coordinates, $h(x,t)$ is the fluid depth, $u_1(x,t)$ and $u_2(x,t)$ are the respective depth-averaged fluid velocities along x and y axes, $B_1(x,t)$ and $B_2(x,t)$ are the respective depth-averaged magnetic field components along x and y axes, and g is the gravitational constant.

System (2.1)–(2.5) is known as the SMHD system over an arbitrary surface (see J. A. Rossmanith's PhD dissertation, University of Washington (2002), Ch. 4). These equations are derived from initial nonviscous and nonresistive incompressible MHD equations by averaging over the fluid layer between a pair of material surfaces and using hydrostatic conditions for the sum of the magnetic and fluid pressure. Equations are derived in the mean-field approximation, neglecting the squares of velocity and magnetic field deviations from mean quantities. It is assumed that magnetic surfaces are, at the same time, material surfaces. For the details of the derivation of SMHD equations, see the Appendix. Equation (2.6) is a consequence of the magnetic field divergence-free equation in the initial MHD equations and is used to set the correct initial data. Equations (2.1)–(2.5) differ from those considered in [14] first and foremost by the number of independent quantities and consequently by the number of equations. As is shown below, this leads to an increase in the number of equations for Riemann invariants. The appearance of a magnetic field suggests nontrivial dependences of one-dimensional equations on both components of horizontal flows. Moreover, the magnetic field is included in the relations for the propagation speed of weak perturbations. In the next section, we find the simple wave solutions of this system.

3. RIEMANN WAVES FOR SMHD EQUATIONS OVER AN ARBITRARY SURFACE

In this section, we rewrite the initial equations (2.1)–(2.5) in the form of the Riemann invariants, which is more appropriate for further consideration. It immediately follows from Eqs. (2.4) and (2.6) that

$$h B_1 = \text{const}. \qquad (3.1)$$

We rewrite Eq. (2.1) in the form

$$\frac{\partial h}{\partial t} = -u_1 \frac{\partial h}{\partial x} - h \frac{\partial u_1}{\partial x}.$$

Thus the time derivatives in the initial equations are transformed to the form

$$\frac{\partial h\,\alpha}{\partial t} = h\frac{\partial \alpha}{\partial t} + \alpha\left(-h\frac{\partial u_1}{\partial x} - u_1\frac{\partial h}{\partial x}\right), \qquad (3.2)$$

where $\alpha = u_1, u_2, B_1, B_2$.

Using expressions (3.2) for the time derivatives in Eqs. (2.1)–(2.6), we obtain

$$\partial_t \begin{pmatrix} h \\ u_1 \\ u_2 \\ B_2 \end{pmatrix} + \begin{pmatrix} u_1 & h & 0 & 0 \\ c_g^2/h & u_1 & 0 & 0 \\ 0 & 0 & u_1 & -B_1 \\ 0 & 0 & -B_1 & u_1 \end{pmatrix} \times$$

$$\times\, \partial_x \begin{pmatrix} h \\ u_1 \\ u_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -g\partial f_s/\partial x \\ 0 \\ 0 \end{pmatrix}, \qquad (3.3)$$

$$hB_1 = \text{const}, \qquad (3.4)$$

where the propagation speed of weak perturbations is $c_g = \sqrt{B_1^2 + gh}$. Equations (3.3) reduce to those considered in [14] when $B_1 = B_2 \equiv 0$, and the expression for the propagation speed of weak perturbations then coincides with the classical one.

We derive the expressions for the Riemann invariants for Eqs. (3.3). For this, we find the eigenvectors of system (3.3). The left eigenvectors of (3.3) are $(c_g/h\,1\,0\,0)$, $(-c_g/h\,1\,0\,0)$, $(0\,0\,1\,1)$, and $(0\,0\,1\,{-1})$. Multiplying Eqs. (3.3) with the first eigenvector yields

$$\frac{\partial u_1}{\partial t} + \frac{c_g}{h}\frac{\partial h}{\partial t} + (u_1 + c_g)\left(\frac{\partial u_1}{\partial x} + \frac{c_g}{h}\frac{\partial h}{\partial x}\right) = = -g\frac{\partial f_s}{\partial x}. \qquad (3.5)$$

Introducing the function $\varphi(h) = \int (c_g/h)dh$, we rewrite Eq. (3.5) in the form

$$\frac{\partial r}{\partial t} + (u_1 + c_g)\frac{\partial r}{\partial x} = -g\frac{\partial f_s}{\partial x}, \qquad (3.6)$$

where $r = u_1 + \varphi(h)$. We note that the function $\varphi(h)$ cannot be expressed in elementary functions and is expressed in terms of elliptic integrals. However, it is a strictly increasing function. As a consequence, the inverse function φ^{-1} exists. Multiplying (3.3) with other eigenvectors yields

$$\frac{\partial s}{\partial t} + (u_1 - c_g)\frac{\partial s}{\partial x} = -g\frac{\partial f_s}{\partial x}, \quad s = u_1 - \varphi(h), \qquad (3.7)$$

$$\frac{\partial p}{\partial t} + (u_1 - B_1)\frac{\partial p}{\partial x} = 0, \quad p = u_2 + B_2, \qquad (3.8)$$

$$\frac{\partial q}{\partial t} + (u_1 + B_1)\frac{\partial q}{\partial x} = 0, \quad q = u_2 - B_2. \qquad (3.9)$$

The functions r, s, p, and q are called the Riemann invariants and system (3.4), (3.6)–(3.9) is called the shallow water magnetohydrodynamic equation system in the Riemann-invariant form.

The expressions for the velocities u_1 and u_2, the fluid depth h, and the magnetic field B_2 in terms of the Riemann invariants are as follows:

$$u_1 = \frac{r+s}{2}, \quad \varphi(h) = \frac{r-s}{2}, \qquad u_2 = p + q, \quad B_2 = p - q. \qquad (3.10)$$

According to the theory of hyperbolic equations, a Riemann wave is defined as a solution of Eqs. (3.4), (3.6)–(3.9) in which all but one Riemann invariants remain constant.

However, the classical Riemann wave solutions do not satisfy Eqs. (3.4), (3.6)–(3.9) due to the presence of the function $-g\partial f_s/\partial x$ in the right-hand side of the equations. We suggest a more general definition of simple wave solutions reducing to the classical one in the case of zero source term. We define the magnetogravity Riemann wave turned back as the solution satisfying Eqs. (3.6), (3.8), (3.9) identically, and the magnetogravity Riemann wave turned forward as the solution satisfying Eqs. (3.7)–(3.9) identically. Similarly, Alfvenic Riemann waves are defined as the solutions satisfying Eqs. (3.6), (3.7), (3.9) or (3.6)–(3.8) identically. The reasons for these definitions becomes clear below.

We assume that $p = p_0 = \text{const}$ and $q = q_0 = \text{const}$ in some area of the xt plane; then Eqs. (3.8) and (3.9) are identically satisfied in this area. We find the conditions for the expression for $r(x,t)$ satisfying Eq. (3.6) identically to exist in the above mentioned area. For this, we show that the expression $u_1 + c_g$ is dependent on s (and possibly on s and r) and $u_1 - c_g$ is dependent on r (and possibly on r and s). Definitely, c_g is a function of h, and hence a function of $\varphi(h)$. Therefore, if $u_1 \pm c_g = f(u_1 \mp \varphi)$, then $c_g = -\varphi + \text{const}$ and it is not the case. Consequently, $u_1 + c_g$ is dependent on s and $u_1 - c_g$ is dependent on r.

The functions $r(x,t)$ and $s(x,t)$ are linearly independent, and hence the factor at $u_1 + c_g$ has to be zero for Eq. (3.6) to be satisfied identically, whence $\partial r/\partial x \equiv \equiv 0$. However, if $\partial r/\partial t \equiv 0$, then Eq. (3.6) cannot be satisfied. Hence, $r(x,t)$ is a function of time only, $r = r(t)$, and therefore

$$\frac{\partial}{\partial x}\frac{\partial r}{\partial t} \equiv 0, \quad \frac{\partial}{\partial x}\left(-g\frac{\partial f_s}{\partial x}\right) \equiv 0.$$

We conclude that the solution satisfying Eq. (3.6) can exist only for the underlying surface $f_s(x)$ determined by $\partial^2 f_s/\partial x^2 \equiv 0$, i.e., $f_s = kx + b_0$. The magnetogravity wave turned back does not exist for other underlying surfaces. It can be similarly shown that the magnetogravity Riemann wave turned forward exists only for $\partial f_s/\partial x = k \equiv \text{const}$. Hence, the simple wave solutions only exist for underlying surfaces that are slopes of constant inclination, and we furthermore suppose that $\partial f_s/\partial x = k \equiv \text{const}$.

We consider system of equations (3.6)–(3.9). Taking into account that $\partial f_s/\partial x = k = \text{const}$, we rearrange it in the form

$$\partial_t \begin{pmatrix} h \\ u_1 \\ u_2 \\ B_2 \end{pmatrix} + \begin{pmatrix} u_1 & h & 0 & 0 \\ c_g^2/h & u_1 & 0 & 0 \\ 0 & 0 & u_1 & -B_1 \\ 0 & 0 & -B_1 & u_1 \end{pmatrix} \times \partial_x \begin{pmatrix} h \\ u_1 \\ u_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gk \\ 0 \\ 0 \end{pmatrix}. \quad (3.11)$$

We make the change of variables

$$\tilde{x} \to x + gkt^2/2, \quad \tilde{t} \to t. \quad (3.12)$$

Then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} + gk\tilde{t}\frac{\partial}{\partial \tilde{x}}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}. \quad (3.13)$$

This change of variables is nondegenerate, and after the change of u_1 given by

$$\tilde{u}_1 = u_1 + gkt \quad (3.14)$$

system (3.11) transforms into

$$\partial_{\tilde{t}} \begin{pmatrix} h \\ \tilde{u}_1 \\ u_2 \\ B_2 \end{pmatrix} + \begin{pmatrix} \tilde{u}_1 & h & 0 & 0 \\ c_g^2/h & \tilde{u}_1 & 0 & 0 \\ 0 & 0 & \tilde{u}_1 & -B_1 \\ 0 & 0 & -B_1 & \tilde{u}_1 \end{pmatrix} \times \partial_{\tilde{x}} \begin{pmatrix} h \\ \tilde{u}_1 \\ u_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.15)$$

After using transformation (3.12), (3.13), system (3.11) becomes the SMHD equation system on a flat surface, Eq. (3.15) ($k = 0$). This transformation is used below to solve the initial discontinuity decay problem over a slope, reducing it to the discontinuity decay problem over a flat plane solved in [18]. We note that the above change of variables is valid only for continuous solutions and fails for discontinuous ones. That is why we cannot exploit this change of variables to obtain the full set of simple wave-type solutions over slopes from those over a flat plane. For a unified description of fluid physics in this paper, we derived the above solutions, continuous and discontinuous ones, from the initial govering equations, although continuous ones can be obtained by transforming solutions from [2]. Indeed, the obtained transformation (3.15) is fruitful for understanding the Riemann problem solutions as soon as we know the discontinuity propagation trajectories, since this allows solving the initial discontinuity decay problem over a slope immediately using the solutions of the discontinuity decay problem over a flat plane.

In the next section, we find particular wave solutions for the shallow water magnetohydrodynamic equations over a slope. The obtained solutions are used in the following section to find exact explicit solutions of the initial discontinuity decay problem over a slope.

4. SIMPLE WAVE SOLUTIONS FOR SMHD EQUATIONS OVER A SLOPE

4.1. Continuous solutions, selfsimilarity, and degeneration of continuous Alfven waves

In this section, we study simple wave solutions for shallow water magnetohydrodynamic equations over a slope, which are a particular case of the initial equations (2.1)–(2.6) with $\partial f_s/\partial x = k \equiv \text{const}$. We consider a magnetogravity Riemann wave turned back. In this case, we have to satisfy Eqs. (3.6), (3.8), and (3.9) identically. It was shown in the preceding section that $p = p_0 = \text{const}$, $q = q_0 = \text{const}$ satisfy Eqs. (3.8) and (3.9), and it is easy to see that for $\partial f_s/\partial x = k \equiv \text{const}$, the expression

$$r = -gkt + r_0 \quad (4.1)$$

satisfies Eq. (3.6) identically. We now consider Eq. (3.7):

$$\frac{\partial s}{\partial t} + (u_1 - c_g)\frac{\partial s}{\partial x} = -gk. \quad (4.2)$$

Equation (4.2) transforms along the characteristics

$$\frac{dx}{dt} = u_1 - c_g \quad (4.3)$$

into the form

$$\frac{\partial s}{\partial t} + \frac{dx}{dt}\frac{\partial s}{\partial x} = -gk \Leftrightarrow \frac{ds}{dt} = -gk. \quad (4.4)$$

Integrating Eq. (4.4), we obtain

$$s\,(X(t),t) = \int_0^t \frac{ds}{dt}\,dt = -gkt + s\,(X(0),0)\,. \quad (4.5)$$

Substituting expression (4.5) in Eq. (4.3) yields

$$\frac{dx}{dt} = = -gkt + \frac{s\,(X(0),0) + r_0}{2} - c_g\Big|_{\substack{r=-gkt+r_0\\ s=-gkt+s(X(0),0)}}. \quad (4.6)$$

The variable c_g remains constant along characteristics (4.3). Indeed, $\varphi(h) = (r - s)/2$, whence $\varphi(h) = [r_0 - s(X(0),0)]/2 =$ const along characteristics (4.3). Because φ is a bijective function, it follows that $h =$ const and $c_g = c_g(h) =$ const along characteristics (4.6).

Integrating (4.6), we obtain the explicit expression for $X(t)$:

$$X(t) = \int_0^t \frac{dx}{dt}\,dt = -\frac{1}{2}gkt^2 + + \frac{s\,(X(0),0) + r_0}{2}\,t + c_g\,(h\,(X(0),0))\,t + X(0). \quad (4.7)$$

Characteristics (4.7) are parabolas in the xt plane. We note that in the case of a flat surface, the characteristics are straight lines,

$$X(t) = (u_1 + c_g)t + X(0), \quad (4.8)$$

and the change of variables

$$\tilde{x} = x + gkt/2, \quad \tilde{t} = t \quad (4.9)$$

transforms parabolas (4.7) into straight lines (4.8). This change of variables is used in what follows to reduce the discontinuity decay problem over a slope to the same problem over a flat surface. For the magnetogravity Riemann wave turned forward, the following relations hold:

$$p = p_0, \quad q = q_0, \quad s = -gkt + s_0, \quad (4.10)$$

$$r\,(X(t),t) = -gkt + r\,(X(0),0)\,, \quad (4.11)$$

$$X(t) = -\frac{1}{2}gkt^2 + \frac{r\,(X(0),0) + s_0}{2}\,t + + c_g\,(h\,(X(0),0))\,t + X(0). \quad (4.12)$$

If $\partial s/\partial x > 0$ in some domain of the xt plane for the Riemann magnetogravity wave turned back, then integral curves (4.3) are divergent. Taking expressions (4.1) and (4.5) into account, we obtain $u_1 = (r_0 + s)/2$, whence

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial s}{\partial x}, \quad \frac{\partial u}{\partial x} > 0.$$

Differentiating $r = u_1 + \varphi$ with respect to x yields

$$\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial h}\frac{\partial h}{\partial x} = 0,$$

whence $\partial h/\partial x < 0$ for $\partial\varphi/\partial h > 0$. Hence, we have a magnetogravity rarefaction wave. If $\partial s/\partial x < 0$ in some domain of the xt plane, then the integral curves are convergent, and we have a compression wave. In the domain of the xt plane in which $\partial s/\partial x = 0$, the characteristics are parallel lines, and we have a domain of a uniformly accelerated flow. It is well known from the theory of hyperbolic equations that converged characteristics in a dilatation wave intersect in a finite time, resulting in the magnetogravitational Riemann wave degeneration and thus the appearance of strong discontinuity.

The same results (except the sign) can be obtained for the Riemann magnetogravity wave turned forward ($s(x,t) = s_0 =$ const). We have a rarefaction wave for $\partial r/\partial x < 0$, a compression wave for $\partial r/\partial x > 0$, and a domain of uniformly accelerated flow for $\partial r/\partial x = 0$.

Using Eqs. (4.1), (4.5), (4.6), (4.10)–(4.12) and $p =$ const, $q =$ const, we obtain the relations for magnetogravity waves. For a magnetogravity Riemann wave turned back, the relations

$$\begin{aligned} &B_1(x,t)h(x,t) = B_1(x_0,0)h(x_0,0),\\ &B_2(x,t) = B_2(x_0,0),\\ &u_2(x,t) = u_2(x_0,0),\\ &u_1(x,t) + \varphi(x,t) + gkt = u_1(x_0,0) + \varphi(x_0,0) \end{aligned} \quad (4.13)$$

are satisfied in the domain of the wave. Moreover, along the lines

$$\frac{dx}{dt} = u_1(x_0,0) - c_g(x_0,0) - gkt, \quad (4.14)$$

the equation

$$u_1(x,t) - \varphi(x,t) + gkt = u_1(x_0,0) - \varphi(x_0,0) \quad (4.15)$$

is also satisfied. There, $\varphi(x_0,0) = \varphi(h(x_0,0))$.

For a magnetogravity Riemann wave turned forward, the relations

$$\begin{aligned} &B_1(x,t)h(x,t) = B_1(x_0,0)h(x_0,0), \\ &B_2(x,t) = B_2(x_0,0), \\ &u_2(x,t) = u_2(x_0,0), \\ &u_1(x,t) - \varphi(x,t) + gkt = u_1(x_0,0) - \varphi(x_0,0) \end{aligned} \tag{4.16}$$

are satisfied in the domain of the wave. Moreover, along the lines

$$\frac{dx}{dt} = u_1(x_0,0) + c_g(x_0,0) - gkt, \tag{4.17}$$

the equation

$$u_1(x,t) + \varphi(x,t) + gkt = u_1(x_0,0) + \varphi(x_0,0) \tag{4.18}$$

is also satisfied.

We consider an Alfvenic Riemann wave satisfying Eqs. (3.6)–(3.8). For this wave, we obtain that the relations

$$\begin{aligned} &u_1(x,t) + gkt = u_1(x_0,0), \\ &h(x,t) = h(x_0,0), \\ &B_1(x,t) = B_1(x_0,0), \\ &u_2(x,t) + B_2(x,t) = u_2(x_0,0) + B_2(x_0,0) \end{aligned} \tag{4.19}$$

are satisfied in the domain of the wave. Hence, along the characteristics

$$\frac{dx}{dt} = u_1 + B_1, \tag{4.20}$$

the equation

$$u_2(x,t) - B_2(x,t) = u_2(x_0,0) - B_2(x_0,0) \tag{4.21}$$

is also satisfied.

For the Alfvenic Riemann wave satisfying Eqs. (3.6), (3.7), and (3.9), the relations

$$\begin{aligned} &u_1(x,t) + gkt = u_1(x_0,0), \\ &h(x,t) = h(x_0,0), \\ &B_1(x,t) = B_1(x_0,0), \\ &u_2(x,t) - B_2(x,t) = u_2(x_0,0) - B_2(x_0,0) \end{aligned} \tag{4.22}$$

are satisfied in the domain of the wave. Hence, along the characteristics

$$\frac{dx}{dt} = u_1 - B_1, \tag{4.23}$$

the equation

$$u_2(x,t) + B_2(x,t) = u_2(x_0,0) + B_2(x_0,0) \tag{4.24}$$

is also satisfied.

We note that in the Alfvenic Riemann waves, all characteristics can be obtained from each other using parallel translation. They are also parabolic curves, the same as for characteristics of magnetogravity waves.

We next consider the practically important special case of the Riemann waves. A backward Riemann wave is called a centered wave if characteristics (4.3) form a group of curves that come out of one point (x_0, t_0). We let u' denote the parameter taking all values from the segment

$$\left[\lim_{x\to x_0-0}\left(u_1(x,t_0) - c_g(x,t_0)\right), \lim_{x\to x_0+0}\left(u_1(x,t_0) - c_g(x,t_0)\right)\right].$$

Then the solution is determined by the equations

$$\begin{aligned} &B_1(x,t)h(x,t) = B_1(x_0,0)h(x_0,0), \\ &B_2(x,t) = B_2(x_0,0), \\ &u_2(x,t) = u_2(x_0,0), \\ &u_1(x,t) - \varphi(x,t) + gkt = u_1(x_0,0) - \varphi(x_0,0) \end{aligned} \tag{4.25}$$

satisfied in the domain of the wave and the equation

$$u_1(x,t) + \varphi(x,t) + gkt = u_1(x_0,0) + \varphi(x_0,0) \tag{4.26}$$

satisfied along the lines

$$\frac{dx}{dt} = u' - gkt. \tag{4.27}$$

Because $\varphi - c_g$ is a monotonic function of h, it follows that $u_1(x_0,0)$ and $h(x_0,0)$ are uniquely determined on each characteristics by the relations $u_1(x_0,0) - \varphi(x_0,0) = \text{const}$ and $u' = u_1(x_0,0) - c_g(x_0,0)$. Equations (4.25)–(4.27) determine all the parameters in a centered magnetogravity wave turned back.

For a centered magnetogravity wave turned forward, we let u' denote the parameter taking all values from the segment

$$\left[\lim_{x\to x_0-0}\left(u_1(x,t_0) + c_g(x,t_0)\right), \lim_{x\to x_0+0}\left(u_1(x,t_0) + c_g(x,t_0)\right)\right].$$

Then the solution is determined by the equations

$$\begin{aligned} &B_1(x,t)h(x,t) = B_1(x_0,0)h(x_0,0), \\ &B_2(x,t) = B_2(x_0,0), \\ &u_2(x,t) = u_2(x_0,0), \\ &u_1(x,t) + \varphi(x,t) + gkt = u_1(x_0,0) + \varphi(x_0,0) \end{aligned} \tag{4.28}$$

satisfied in the domain of the wave and the equations

$$u_1(x,t) - \varphi(x,t) + gkt = u_1(x_0,0) - \varphi(x_0,0) \tag{4.29}$$

satisfied along the lines

$$\frac{dx}{dt} = u' + gkt. \tag{4.30}$$

Here, $u_1(x_0, 0)$ and $h(x_0, 0)$ can be found on each characteristic from $u_1(x_0, 0) + \varphi(x_0, 0) = \text{const}$ and u'. System (4.28)–(4.30) determines all the parameters in a centered magnetogravity wave turned forward.

We note that the obtained relations (4.19) and (4.22) for the Alfven wave are satisfied in a band that consists of parallel characteristics. Therefore, in the selfsimilar case, these relations can be fulfilled only on a single bundle line $x = \lambda t$, and hence the continuous Alfven wave degenerates.

4.2. Discontinuous solutions over a slope. The jump conditions

As shown above, any magnetogravitational dilatation wave leads to the appearance of high discontinuity in a finite time. In this section, the conditions that must be satisfied on the discontinuity lines are obtained. For this, we rewrite Eqs. (2.1)–(2.5) in the divergent form:

$$\begin{aligned}
&\frac{\partial h}{\partial t} + \frac{\partial h\, u_1}{\partial x} = 0,\\
&\frac{\partial h\, u_1}{\partial t} + \frac{\partial (hu_1^2 - hB_1^2 + gh^2/2)}{\partial x} = -g\frac{\partial b}{\partial x},\\
&\frac{\partial h\, u_2}{\partial t} + \frac{\partial (hu_1u_2 - hB_1B_2)}{\partial x} = 0,\\
&\frac{\partial h\, B_1}{\partial t} = 0,\\
&\frac{\partial h\, B_2}{\partial t} + \frac{\partial (hu_1B_2 - hB_1u_2)}{\partial x} = 0.
\end{aligned} \tag{4.31}$$

Integrating (4.31) on an arbitrary domain G homeomorphic to a square in the xt plane yields

$$\begin{aligned}
&\iint\limits_G \left(\frac{\partial h}{\partial t} + \frac{\partial h\, u_1}{\partial x}\right) dG = 0,\\
&\iint\limits_G \left(\frac{\partial h\, u_1}{\partial t} + \frac{\partial (hu_1^2 - hB_1^2 + gh^2/2)}{\partial x}\right) dG =\\
&\qquad\qquad = \iint\limits_G \left(-g\frac{\partial b}{\partial x}\right) dG,\\
&\iint\limits_G \left(\frac{\partial h\, u_2}{\partial t} + \frac{\partial (hu_1u_2 - hB_1B_2)}{\partial x}\right) dG = 0,\\
&\iint\limits_G \left(\frac{\partial h\, B_1}{\partial t}\right) dG = 0,\\
&\iint\limits_G \left(\frac{\partial h\, B_2}{\partial t} + \frac{\partial (hu_1B_2 - hB_1u_2)}{\partial x}\right) dG = 0.
\end{aligned} \tag{4.32}$$

Transforming volume integrals in (4.32) using the Green's formula we obtain

$$\begin{aligned}
&\oint\limits_{\partial G} h\, dx - (hu_1)\, dt = 0,\\
&\oint\limits_{\partial G} (hu_1)\, dx - \left(hu_1^2 - hB_1^2 + \frac{gh^2}{2}\right) dt =\\
&\qquad\qquad = \oint\limits_{\partial G} (gb)\, dt,\\
&\oint\limits_{\partial G} (hu_2)\, dx - (hu_1u_2 - hB_1B_2)\, dt = 0,\\
&\oint\limits_{\partial G} (hB_1)\, dx = 0,\\
&\oint\limits_{\partial G} (hB_2)\, dx - (hu_1B_2 - hB_1u_2)\, dt = 0.
\end{aligned} \tag{4.33}$$

Equations (4.33) represent the most general relations that are integral conservation laws and are valid for an arbitrary contour ∂G and, in particular, for the contour including the discontinuity lines of an appropriate solution.

Let $x = x(t)$ be the equation of a jump line; we suppose that it has a continuous tangent on the segment $[t_1, t_2]$. Assuming that the functions u_1, u_2, B_1, B_2, and h have a jump on the line $x = x(t)$ only and $b(x)$ has no jump, we set

$$\begin{aligned}
u_{1\,\mathrm{I}}(t) &= \lim_{x\to x(t)-0} u_1(x,t),\\
u_{1\,\mathrm{II}}(t) &= \lim_{x\to x(t)+0} u_1(x,t),\\
u_{2\,\mathrm{I}}(t) &= \lim_{x\to x(t)-0} u_2(x,t),\\
u_{2\,\mathrm{II}}(t) &= \lim_{x\to x(t)+0} u_2(x,t),\\
B_{1\,\mathrm{I}}(t) &= \lim_{x\to x(t)-0} B_1(x,t),\\
B_{1\,\mathrm{II}}(t) &= \lim_{x\to x(t)+0} B_1(x,t),\\
B_{2\,\mathrm{I}}(t) &= \lim_{x\to x(t)-0} B_2(x,t),\\
B_{2\,\mathrm{II}}(t) &= \lim_{x\to x(t)+0} B_2(x,t),\\
h_{\mathrm{I}}(t) &= \lim_{x\to x(t)-0} h(x,t),\\
h_{\mathrm{II}}(t) &= \lim_{x\to x(t)+0} h(x,t).
\end{aligned} \tag{4.34}$$

Let ∂G be the contour $ABCE$ with lines AB and CE located infinitely close to the line of the jump $x(t)$ on the left- and right-hand sides respectively (Fig. 2). Letting $D = D(t) = x'(t)$ denote the speed of the discontinuity, such that $dx = D(t)\, dt$, we obtain

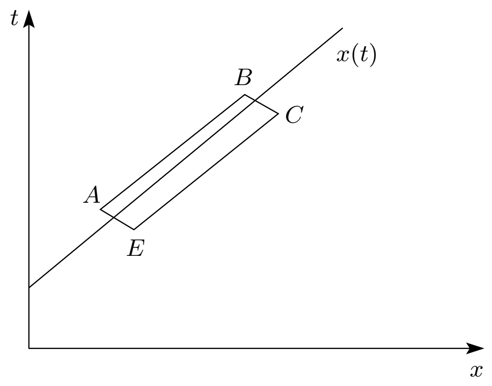

Fig. 2. Contour $ABCE$ with lines AB and CE located infinitely close to the line of the jump $x(t)$

$$\begin{aligned}
&\int\limits_{AB} (Dh - hu_1)\,dt \int\limits_{CE} (Dh - hu_1)\,dt = 0,\\
&\int\limits_{AB} \left(Dhu_1 - hu_1^2 + hB_1^2 - \frac{gh^2}{2}\right) dt -\\
&- \int\limits_{CE} \left(Dhu_1 - hu_1^2 + hB_1^2 - \frac{gh^2}{2}\right) dt = 0,\\
&\int\limits_{AB} (Dhu_2 - hu_1u_2 + hB_1B_2)\; dt -\\
&- \int\limits_{CE} (Dhu_2 - hu_1u_2 + hB_1B_2)\; dt = 0,\\
&\int\limits_{AB} (DhB_1)\,dt - \int\limits_{CE} (DhB_1)\,dt = 0,\\
&\int\limits_{AB} (DhB_2 - hu_1B_2 + hB_1u_2)\; dt -\\
&- \int\limits_{CE} (DhB_2 - hu_1B_2 + hB_1u_2)\; dt = 0.
\end{aligned} \tag{4.35}$$

The contour $ABCE$ is arbitrary, and hence Eqs. (4.35) are equivalent to the following conditions for the integrands:

$$\begin{aligned}
&Dh_{\rm I} - h_{\rm I}u_{1\,\rm I} = Dh_{\rm II} - h_{\rm II}u_{1\,\rm II},\\
&Dh_{\rm I}u_{1\,\rm I} - h_{\rm I}u_{1\,\rm I}^2 + h_{\rm I}B_{1\,\rm I}^2 - gh_{\rm I}^2/2 =\\
&= Dh_{\rm II}u_{1\,\rm II} - h_{\rm II}u_{1\,\rm II}^2 + h_{\rm II}B_{1\,\rm II}^2 - gh_{\rm II}^2/2,\\
&Dh_{\rm I}B_{1\,\rm I} = Dh_{\rm II}B_{1\,\rm II},\\
&Dh_{\rm I}u_{2\,\rm I} - h_{\rm I}u_{1\,\rm I}u_{2\,\rm I} + h_{\rm I}B_{1\,\rm I}B_{2\,\rm I} =\\
&= Dh_{\rm II}u_{2\,\rm II} - h_{\rm II}u_{1\,\rm II}u_{2\,\rm II} + h_{\rm II}u_{1\,\rm II}B_{2\,\rm II},\\
&Dh_{\rm I}B_{2\,\rm I} - h_{\rm I}u_{1\,\rm I}B_{2\,\rm I} + h_{\rm I}u_{2\,\rm I}B_{1\,\rm I} =\\
&= Dh_{\rm II}B_{2\,\rm II} - h_{\rm II}u_{1\,\rm II}B_{2\,\rm II} + h_{\rm II}u_{2\,\rm II}B_{1\,\rm II}.
\end{aligned} \tag{4.36}$$

We consider the case $h_{\rm I} \neq h_2$. Then the first three equations in (4.36) give

$$\begin{aligned}
&h_{\rm I}B_{\rm I} = h_{\rm II}B_{\rm II},\\
&D = \frac{h_{\rm I}u_{1\,\rm I} - h_{\rm II}u_{1\,\rm II}}{h_{\rm I} - h_{\rm II}},\\
&u_{1\,\rm I} - u_{1\,\rm II} = \pm(h_{\rm I} - h_{\rm II}) \times\\
&\times \sqrt{\frac{g(h_{\rm I} + h_{\rm II})/2 + (B_{1\,\rm I}h_{\rm I})^2/h_{\rm I}h_{\rm II}}{h_{\rm I}h_{\rm II}}}.
\end{aligned} \tag{4.37}$$

Substituting D from the second relation in (4.37) in the last two equations in (4.36) and rearranging the terms, we obtain

$$\begin{aligned}
&h_{\rm I}h_{\rm II}(u_{1\,\rm I} - u_{1\,\rm II})(u_{2\,\rm I} - u_{2\,\rm II}) =\\
&\quad = -(h_{\rm I} - h_{\rm II})(h_{\rm I}B_{1\,\rm I}B_{2\,\rm I} - h_{\rm II}B_{1\,\rm II}B_{2\,\rm II}),
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
&h_{\rm I}h_{\rm II}(u_{1\,\rm II} - u_{1\,\rm I})(B_{2\,\rm II} - B_{2\,\rm I}) =\\
&\quad = (h_{\rm I} - h_{\rm II})(h_{\rm II}u_{2\,\rm II}B_{1\,\rm II} - h_{\rm I}u_{2\,\rm I}B_{1\,\rm I}).
\end{aligned} \tag{4.39}$$

If $B_{2\,\rm I} = B_{2\,\rm II}$ and $u_{2\,\rm I} = u_{2\,\rm II}$, then Eqs. (4.38) and (4.39) are satisfied identically. Otherwise, we divide (4.38) by (4.39) and obtain $(u_{2\,\rm I} - u_{2\,\rm II})^2 = (B_{1\,\rm I} - B_{1\,\rm II})^2$, whence it follows that

$$u_{2\,\rm I} - u_{2\,\rm II} = \pm(B_{1\,\rm I} - B_{1\,\rm II}). \tag{4.40}$$

Substituting (4.40) in (4.38) and taking the third equation in (4.32) into account, we obtain

$$h_{\rm I}h_{\rm II}(u_{1\,\rm I} - u_{1\,\rm II}) = \pm(h_{\rm I} - h_{\rm II})h_{\rm I}B_{1\,\rm I}$$

and thus

$$u_{1\,\rm I} - u_{1\,\rm II} = \pm(h_{\rm I} - h_{\rm II})\frac{h_{\rm I}B_{1\,\rm I}}{h_{\rm I}h_{\rm II}}. \tag{4.41}$$

For the third equation in (4.37) and Eq. (4.41) to be satisfied simultaneously, the sum of the depths on both sides adjacent to the discontinuity must be zero, $h_{\rm I} + h_{\rm II} = 0$. This can be only in the case where each depth is equal to zero, i. e., the case of the fluid absence. Therefore, the assumption that B_2 and u_2 have a discontinuity is incorrect if Eqs. (4.37) have a nontrivial solution.

We consider the other case, where the free surface has no jump on the discontinuity, $h_{\rm I} = h_{\rm II}$. It follows from (4.37) that velocity and magnetic field components normal to the discontinuity have no jump as well, $B_{1\,\rm I} = B_{1\,\rm II}$ and $u_{1\,\rm I} = u_{1\,\rm II}$. Therefore, the first three equations in system (4.36) are satisfied identically. There are only two nontrivial relations at the discontinuity:

$$\begin{aligned}
&D = u_1 - B_1\frac{B_{2\,\rm I} - B_{2\,\rm II}}{u_{2\,\rm I} - u_{2\,\rm II}},\\
&(B_{2\,\rm I} - B_{2\,\rm II})^2 = (u_{2\,\rm I} - u_{2\,\rm II})^2.
\end{aligned} \tag{4.42}$$

Rearranging (4.42), we obtain

$$\begin{aligned} D &= u_1 \pm B_1, \\ B_{2\,\mathrm{I}} - B_{2\,\mathrm{II}} &= \mp(u_{2\,\mathrm{I}} - u_{2\,\mathrm{II}}). \end{aligned} \tag{4.43}$$

Relations (4.43) are identically those obtained for the Alfvenic Riemann waves without discontinuity, Eqs. (4.16)–(4.18), (4.19)–(4.21).

Thus, there are only two types of stable discontinuities with a nonzero mass flow through the discontinuity: discontinuity (4.37) with a free surface jump and transverse velocity and transverse magnetic field jumps, termed a magnetogravity shock wave, and discontinuity (4.43) with the tangential velocity jump and the tangential magnetic field jump, termed an Alfvenic wave. We note that the magnetogravity wave is an analog of a hydrodynamic jump for the classical shallow water equations, and the relations for this wave transform into those for the hydrodynamic jump as $B_1 \to 0$. The magnetogravity shock wave is supersonic in the medium before the wave and subsonic in the medium after the wave, as it is for the classical hydrodynamic jump [21] in the shallow water theory [22].

In general, system (4.36) admits the third type of stable discontinuities with the continuous tangential velocity component equal to the discontinuity velocity. It is termed the contact discontinuity. These discontinuities must be considered if the problem has different properties in the right and left half-spaces and these properties do no affect the discontinuity decay solution. An example of such a case is the fluid with different densities in the half-spaces separated by the discontinuity. Another example considered in this paper corresponds to the degeneration of an Alfvenic wave as $B_1 \to 0$. In this case, the mass flow through the discontinuity is equal to zero and the tangential magnetic field and velocity field components have the properties described above.

It is shown in [23] that the characteristic intersection envelope in quasilinear hyperbolic partial differential equations is itself a characteristic of the same system. Hence, the high discontinuity propagation trajectory $x = x(t)$ is also a parabola. In our case of SMHD flows over a slope, the magnetogravitational shock wave is due to the fall of a magnetogravitational dilatation wave. Hence, a strong discontinuity borders the magnetogravitational Riemann wave throughout the area of uniformly accelerated flow, and hence has a parabolic trajectory.

5. INITIAL DISCONTINUITY DECAY PROBLEM FOR SMHD EQUATIONS OVER A SLOPE

Here, we formulate the initial discontinuity decay problem for SMHD equations and list all possible wave configurations describing the nonlinear dynamics of the initial discontinuity decay. We find the realization conditions for each wave configuration. As it has been shown, the particular solutions in our case differ from those for incompressible shallow-water flows. Hence, the conditions for the realization of each configuration are different

5.1. Initial discontinuity decay problem statement

We consider Eqs. (2.1)–(2.5) with an arbitrary piecewise constant initial conditions for the left ($x < 0$) and right ($x > 0$) half-spaces:

$$\begin{aligned} &t = 0, \\ &h = h_{\mathrm{I}}, \quad u_1 = u_{1\,\mathrm{I}}, \quad u_2 = u_{2\,\mathrm{I}}, \\ &B_1 = B_{1\,\mathrm{I}}, \quad B_2 = B_{2\,\mathrm{I}} \quad \text{for } x < 0, \\ &h = h_{\mathrm{II}}, \quad u_1 = u_{1\,\mathrm{II}}, \quad u_2 = u_{2\,\mathrm{II}}, \\ &B_1 = B_{1\,\mathrm{II}}, \quad B_2 = B_{2\,\mathrm{II}} \quad \text{for } x > 0, \\ &B_{1\,\mathrm{I}} h_{\mathrm{I}} = B_{1\,\mathrm{II}} h_{\mathrm{II}}. \end{aligned} \tag{5.1}$$

The discontinuity for two semi-infinite magnetic fluids adjacent to the $x = 0$ plane at the initial instant and satisfying (5.1) is called the initial discontinuity [23]. The determination of the flow at $t > 0$ for these initial conditions is called the initial discontinuity decay problem solution for SMHD equations.

Without the loss of generality, it is assumed hereafter that the fluid depth in the right half-space is less than or equal to the fluid depth in the left half-space. It is shown below that in the absence of the fluid in the right half-space, the magnetic field component B_1 must be equal to zero in the left half-space, $B_{1\,\mathrm{I}} = 0$, and this leads to the absence of B_1 in the space-time domain of the solution. In this case, the solution is reduced to the classical dam break problem solution [21] with an additional convective transfer of the tangential velocity and magnetic field components. It is assumed that the right half-space magnetic fluid is at rest. The above two assumptions are easily satisfied when changing the coordinate system to the one with a proper axis direction and moving with a prescribed velocity.

We note that in nontrivial case $B_1 \neq 0$, the fluid depth is strictly positive in the space-time region of the solution because of the magnetic field divergence-free

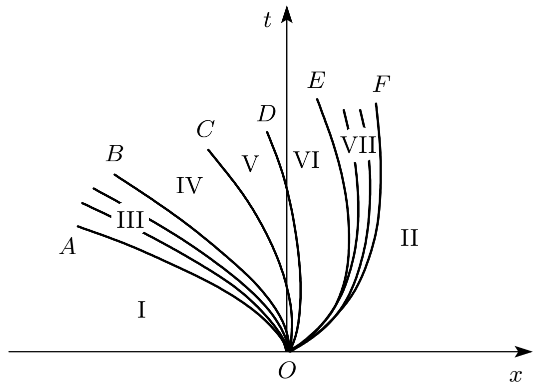

Fig. 3. Two magnetogravity rarefaction waves and two Alfvenic waves over a slope. I, II, IV, V, VI are regions of uniformly accelerated flow; III and VII are magnetogravity rarefaction waves; OC and OD are Alfvenic waves

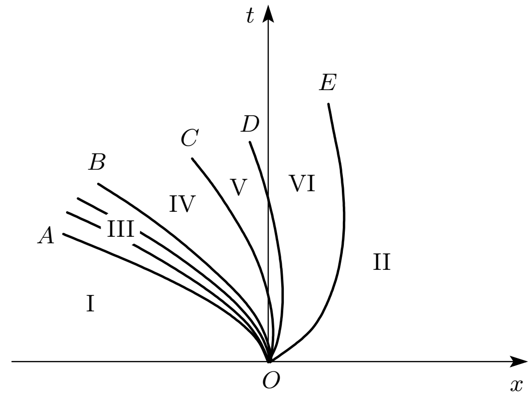

Fig. 4. Magnetogravity wave turned back, magnetogravity shock wave, and two Alfvenic waves over a slope. I, II, IV, V, VI are regions of uniformly accelerated flow; III is a magnetogravity rarefaction wave; OC and OD are Alfvenic waves; OE is a magnetogravity shock wave

condition. Hence, if there is no fluid in one half-space, then the normal magnetic field component in the other half-space degenerates. The case of the fluid absence in the left and right half-spaces leads to the entire solution degeneration (all physical values are constant and equal to zero) and is not considered here. If $B_{1\,\mathrm{I}} = B_{1\,\mathrm{II}} = 0$, then the problem reduces to the hydrodynamic initial discontinuity decay [21]. Indeed, in the case of a zero tangent magnetic field (the absence of the magnetic field), Eqs. (2.1)–(2.5) reduce to the classical shallow water system. In the case of a nonzero tangent magnetic field, the solution degenerates. It is shown below that Alfvenic waves merge and become a contact discontinuity, and the tangent velocity and magnetic field components are transferred convectively. This corresponds to the classical dam break problem [21]. This case is not specially considered below.

In what follows, we use the change of variables in (3.12), (3.13) to find the initial discontinuity decay problem solution over a slope. For this, we use the initial discontinuity decay problem solution on a flat plane [18] and perform the change of variables inverse to (3.12), (3.13):

$$\begin{aligned} x &= \tilde{x} - \frac{1}{2}gkt^2, \\ t &= \tilde{t}, \\ u &= \tilde{u} - gkt. \end{aligned} \tag{5.2}$$

It follows from (5.2) that the characteristics in the case of a slope are parabolas, whereas the characteristics in the case of a flat plane are straight lines. These characteristics are tangent at the initial point. Indeed,

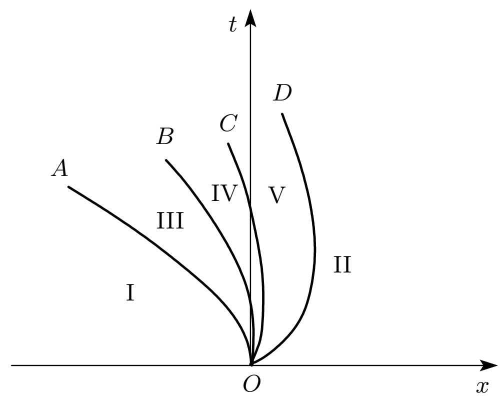

Fig. 5. Two magnetogravity shock waves and two Alfvenic waves over a slope. I, II, III, IV, and V are regions of uniformly accelerated flow; OA are OD are magnetogravity shock waves; OB and OC are Alfvenic waves

we note that the wave configurations of the Riemann problem solution over a slope are the same as over a flat plane [18]: "Two magnetogravity rarefaction waves, and two Alfvenic waves" (Fig. 3), "Magnetogravity rarefaction wave, magnetogravity shock wave, and two Alfvenic waves" (Fig. 4), "Two magnetogravity shock waves and two Alfvenic waves" (Fig. 5), and "Two hydrodynamic rarefaction waves and a vacuum region between them" (Fig. 6). The configurations realization conditions also match: when

$$u_{1\,\mathrm{I}} \geq (h_{\mathrm{I}} - h_{\mathrm{II}})\sqrt{\frac{g(h_{\mathrm{I}} + h_{\mathrm{II}})/2 + (B_{1\,\mathrm{I}} h_{\mathrm{I}})^2/h_{\mathrm{I}} h_{\mathrm{II}}}{h_{\mathrm{I}} h_{\mathrm{II}}}},$$

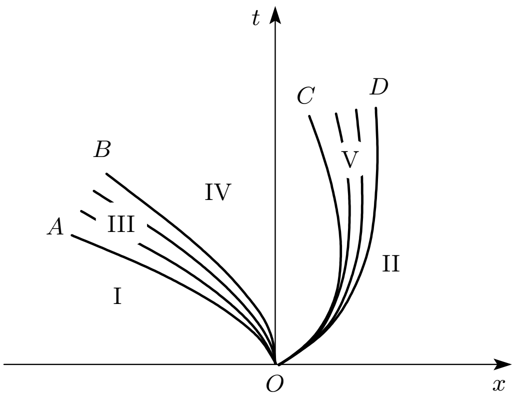

Fig. 6. Two hydrodynamic rarefaction waves and a vacuum region between them over a slope. I and II are regions of uniformly accelerated flow; III and V are magnetogravity rarefaction waves; IV is the vacuum region

the configuration "two magnetogravity shock waves, and two Alfvenic waves" is realized; when

$$u_{1\,\mathrm{I}} > \varphi(h_{\mathrm{II}}) - \varphi(h_{\mathrm{I}}),$$

$$u_{1\,\mathrm{I}} < (h_{\mathrm{I}} - h_{\mathrm{II}})\sqrt{\frac{g(h_{\mathrm{I}} + h_{\mathrm{II}})/2 + (B_{1\,\mathrm{I}}h_{\mathrm{I}})^2/h_{\mathrm{I}}h_{\mathrm{II}}}{h_{\mathrm{I}}h_{\mathrm{II}}}},$$

the configuration "magnetogravity rarefaction wave turned back, magnetogravity shock wave, and two Alfvenic waves" is realized; when

$$u_{1\,\mathrm{I}} \le \varphi(h_{\mathrm{II}}) - \varphi(h_{\mathrm{I}}),$$

the configuration "two magnetogravity rarefaction waves and two Alfvenic waves" is realized; and when

$$B_{1\,\mathrm{I}} = B_{1\,\mathrm{II}} = 0,$$

$$u_{1\,\mathrm{I}} < -2c_{g\,\mathrm{I}} - 2c_{g\,\mathrm{II}},$$

the configuration "two hydrodynamic rarefaction waves and a vacuum region between them" is realized.

It should be noted that the explicit form of the obtained solution of the initial discontinuity decay problem over a slope differs substantially from that over a flat plane, despite similar wave configurations and realization conditions. This is because the characteristics in the case of a slope are parabolas, whereas the characteristics in the case of a flat plane are straight lines. The Riemann problem solution found above forms a basis for the development of finite-volume numerical methods to compute continuous and discontinuous solutions without capturing the discontinuities [19, 24, 25].

6. CONCLUSION

In this paper, the nonlinear dynamics of the SMHD flows of heavy fluid is studied. It is shown that simple Riemann waves are not solutions of the SMHD equations in the case of an arbitrary nonhomogeneous boundary due to the source term $-g\,db/dx$ in the right-hand side of Eqs. (2.1)–(2.6). This is because the Riemann variables are not conserved along the characteristics, in contrast to the flat plane case, and classical simple wave solutions do not exist. Generalized simple waves [14] exist only for slopes $b = kx + c$, where $k, c = \mathrm{const}$. Generalized centered simple waves are obtained in this particular case. The obtained solutions can be interesting in and of themselves because they describe the nonlinear dynamics of a rotating magnetic fluid in the beta plane approximation. All continuous simple wave solutions over slopes are found: Alfvenic waves and magnetogravity waves. Discontinuous solutions are obtained, which are magnetogravitational and Alfven discontinuities. The change of variables transforming the SMHD equations over a slope to the equations over a flat plane is found. The exact explicit solution of the initial discontinuity decay problem over a slope is found. It is shown that these solutions are represented by one of the following configurations: "Two magnetogravity rarefaction waves and two Alfvenic waves", "Two magnetogravity shock waves and two Alfvenic waves", "Rarefaction magnetogravity wave, magnetogravity shock wave, and two Alfvenic waves", "Two hydrodynamic rarefaction waves and a vacuum region between them". Despite the same wave configurations in the case of a slope and a flat plane, these solutions are drastically different from each other. In the case of a flat plane, the characteristics of waves are straight lines and in the case of a slope, they are parabolas. The constant-flow regions in the flat plane solutions are transformed into the regions of constantly accelerated flow in the case of a slope. It follows from the obtained results that the solution of the initial discontinuity decay is a superposition of two solutions: the initial discontinuity decay solution for shallow water without a magnetic field (with the modified sound velocity $c_g = \sqrt{B_1^2 + gh}$) and two Alfvenic waves. When $B_1 \equiv 0$, the two Alfvenic waves merge and become the contact discontinuity. The "two hydrodynamic rarefaction waves and a vacuum region between them" configuration differs from the other configurations and can be realized only when the normal component of the magnetic field is equal to zero.

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APPENDIX

Derivation of shallow water magnetohydrodynamic equations

In this section, we derive shallow water equations for heavy magnetohydrodynamic flows in the field of gravity. We consider nonviscous and nonresistive magnetohydrodynamic equations for incompressible homogeneous (ρ = const) flows with the gravity force directed opposite the z axis:

$$\partial_t \begin{pmatrix} \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{pmatrix} + \partial_x \begin{pmatrix} \rho u_1^2 - \tilde{B}_1^2 + \tilde{p} \\ \rho u_1 u_2 - \tilde{B}_1 \tilde{B}_2 \\ \rho u_1 u_3 - \tilde{B}_1 \tilde{B}_3 \\ 0 \\ u_1 \tilde{B}_2 - u_2 \tilde{B}_1 \\ u_1 \tilde{B}_3 - u_3 \tilde{B}_1 \end{pmatrix} + \partial_y \begin{pmatrix} \rho u_1 u_2 - \tilde{B}_1 \tilde{B}_2 \\ \rho u_2^2 - \tilde{B}_2^2 + \tilde{p} \\ \rho u_2 u_3 - \tilde{B}_2 \tilde{B}_3 \\ u_2 \tilde{B}_1 - u_1 \tilde{B}_2 \\ 0 \\ u_2 \tilde{B}_3 - u_3 \tilde{B}_2 \end{pmatrix} + \partial_z \begin{pmatrix} \rho u_1 u_3 - \tilde{B}_1 \tilde{B}_3 \\ \rho u_2 u_3 - \tilde{B}_2 \tilde{B}_3 \\ \rho u_3^2 - \tilde{B}_3^2 + \tilde{p} \\ u_3 \tilde{B}_1 - u_1 \tilde{B}_3 \\ u_3 \tilde{B}_2 - u_2 \tilde{B}_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\rho g \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(A.1)}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0, \quad \text{(A.2)}$$

$$\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} = 0, \quad \text{(A.3)}$$

with the boundary conditions

$$\begin{aligned} u_3|_{z=f_s} &= u_1|_{z=f_s} \frac{\partial f_s}{\partial x} + u_2|_{z=f_s} \frac{\partial f_s}{\partial y}, \\ u_3|_{z=h} &= \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + \\ &\quad + u_1|_{z=h} \frac{\partial h}{\partial x} + u_2|_{z=h} \frac{\partial h}{\partial y}, \\ B_3|_{z=f_s} &= B_1|_{z=f_s} \partial_x f_s + B_2|_{z=f_s} \partial_y f_s, \\ B_3|_{z=h} &= B_1|_{z=h} \partial_x h + B_2|_{z=h} \partial_y h. \end{aligned} \quad \text{(A.4)}$$

Here, $h(x,y,t)$ is the fluid surface, $\mathbf{u}$ is the velocity vector, $\mathbf{B}$ is the magnetic field vector, ρ is the density, $\tilde{p} = p + \rho|B|^2/2$ is the magnetohydrostatic pressure, g is the acceleration of gravity, and $f_s = f_s(x,y)$ is the stream bed profile. Boundary conditions for the velocity are the nonslip condition on the bottom boundary, and the vertical velocity must be equal to the free surface velocity in magnitude. Boundary conditions for the magnetic field suggest that the magnetic fields on the bottom and on the free surface are parallel to those boundaries. We suppose that material surfaces are at the same time magnetic surfaces.

After substituting $\mathbf{B} = \tilde{\mathbf{B}}\rho^{-1/2}$, the third equation in system (2.1) can be written as

$$\partial_t u_3 + (\mathbf{u}\cdot\nabla)u_3 - (\mathbf{B}\cdot\nabla)B_3 + \rho^{-1}\partial_z\left(p + \frac{\rho}{2}|B|^2\right) = -g. \quad \text{(A.5)}$$

We consider the magnetohydrodynamic flows whose depth is smaller than the characteristic scale of fluid motions. Then the pressure can be considered magnetohydrostatic for such flows:

$$\partial_z\left(p + \frac{\rho}{2}|B|^2\right) = -\rho g. \quad \text{(A.6)}$$

Equations (A.1)–(A.3) are integrated over the vertical coordinate to obtain the magnetohydrodynamic equations in the shallow water approximation. Equations (A.1)–(A.3) with (A.6) taken into account are written as

$$\int_{f_s}^{h} \partial_t \begin{pmatrix} \rho u_1 \\ \rho u_2 \\ 0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} dz + \int_{f_s}^{h} \partial_x \begin{pmatrix} \rho u_1^2 - \rho B_1^2 + \tilde{p} \\ \rho u_1 u_2 - \rho B_1 B_2 \\ 0 \\ 0 \\ u_1 B_2 - u_2 B_1 \\ u_1 B_3 - u_3 B_1 \end{pmatrix} dz + \int_{f_s}^{h} \partial_y \begin{pmatrix} \rho u_1 u_2 - \rho B_1 B_2 \\ \rho u_2^2 - \rho B_2^2 + \tilde{p} \\ 0 \\ u_2 B_1 - u_1 B_2 \\ 0 \\ u_2 B_3 - u_3 B_2 \end{pmatrix} dz +$$

$$+\int\limits_{f_s}^{h}\partial_z\begin{pmatrix}\rho u_1u_3-\rho B_1B_3\\ \rho u_2u_3-\rho B_2B_3\\ \tilde{p}\\ u_3B_1-u_1B_3\\ u_3B_2-u_2B_3\\ 0\end{pmatrix}dz=$$

$$=\begin{pmatrix}0\\0\\-\rho gh\\0\\0\\0\end{pmatrix}, \quad (A.7)$$

$$\int\limits_{f_s}^{h}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}\right)dz=0, \quad (A.8)$$

$$\int\limits_{f_s}^{h}\left(\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z}\right)dz=0. \quad (A.9)$$

Using the Leibniz differentiation rules

$$\frac{\partial}{\partial x}\int\limits_{b(x)}^{a(x)}f(x,z)\,dz=\int\limits_{b(x)}^{a(x)}\frac{\partial}{\partial x}f(x,z)+f|_a\frac{\partial a}{\partial x}-f|_b\frac{\partial b}{\partial x},$$

we transform Eq. (2.8) to the form

$$\frac{\partial}{\partial x}\int\limits_{f_s}^{h}u_1dz-u_1|_{z=h}\frac{\partial h}{\partial x}+u_1|_{z=f_s}\frac{\partial f_s}{\partial x}+$$
$$+\frac{\partial}{\partial y}\int\limits_{f_s}^{h}u_2dz-u_2|_{z=h}\frac{\partial h}{\partial y}+$$
$$+u_2|_{z=f_s}\frac{\partial f_s}{\partial y}+u_3|_{z=h}-u_3|_{z=f_s}=0. \quad (A.10)$$

After inserting boundary condition (2.4), Eq. (2.10) becomes

$$\frac{\partial}{\partial x}\int\limits_{f_s}^{h}u_1dz-u_1|_{z=h}\frac{\partial h}{\partial x}+u_1|_{z=f_s}\frac{\partial f_s}{\partial x}+$$
$$+\frac{\partial}{\partial y}\int\limits_{f_s}^{h}u_2dz-u_2|_{z=h}\frac{\partial h}{\partial y}+$$
$$+u_2|_{z=f_s}\frac{\partial f_s}{\partial y}+\frac{\partial h}{\partial t}+u_1|_{z=h}\frac{\partial h}{\partial x}+u_2|_{z=h}\frac{\partial h}{\partial y}-$$
$$-u_1|_{z=f_s}\frac{\partial f_s}{\partial x}-u_2|_{z=f_s}\frac{\partial f_s}{\partial y}=0.$$

Summing similar terms, we obtain

$$\frac{\partial}{\partial x}\int\limits_{f_s}^{h}u_1dz+\frac{\partial}{\partial y}\int\limits_{f_s}^{h}u_2dz+\frac{\partial h}{\partial t}=0. \quad (A.11)$$

The remaining equations are transformed similarly. Assuming the pressure to be constant on the free surface ($p|_{z=h}=p_0$), we obtain from the third equation in system (2.7) that

$$\tilde{p}=p_0-\rho g(h-z). \quad (A.12)$$

The first equation in (2.7), taking (2.12) into consideration, is transformed as follows:

$$\rho\frac{\partial}{\partial t}\int\limits_{f_s}^{h}u_1dz-\rho u_1|_{z=h}\frac{\partial h}{\partial t}+\rho u_1|_{z=f_s}\frac{\partial f_s}{\partial t}+$$
$$+\rho\frac{\partial}{\partial x}\int\limits_{f_s}^{h}u_1^2dz-\rho u_1^2|_{z=h}\frac{\partial h}{\partial x}+\rho u_1^2|_{z=f_s}\frac{\partial f_s}{\partial x}-$$
$$-\rho\frac{\partial}{\partial x}\int\limits_{f_s}^{h}B_1^2dz+\rho B_1^2|_{z=h}\frac{\partial h}{\partial x}-\rho B_1^2|_{z=f_s}\frac{\partial f_s}{\partial x}+$$
$$+\rho g(h-f_s)\frac{\partial h}{\partial x}+\rho\frac{\partial}{\partial y}\int\limits_{f_s}^{h}u_1u_2dz-$$
$$-\rho u_1|_{z=h}u_2|_{z=h}\frac{\partial h}{\partial y}+\rho u_1|_{z=f_s}u_2|_{z=f_s}\frac{\partial f_s}{\partial y}-$$
$$-\rho\frac{\partial}{\partial y}\int\limits_{f_s}^{h}B_1B_2dz+\rho B_1|_{z=h}B_2|_{z=h}\frac{\partial h}{\partial y}-$$
$$-\rho B_1|_{z=f_s}B_2|_{z=f_s}\frac{\partial f_s}{\partial y}+\rho u_1|_{z=h}u_3|_{z=h}-$$
$$-\rho B_1|_{z=h}B_3|_{z=h}=0,$$

and, after inserting boundary condition (2.4) and summing similar terms:

$$\rho\frac{\partial}{\partial t}\int\limits_{f_s}^{h}u_1dz+\rho\frac{\partial}{\partial x}\int\limits_{f_s}^{h}u_1^2dz-$$
$$-\rho\frac{\partial}{\partial x}\int\limits_{f_s}^{h}B_1^2dz+\rho g(h-f_s)\frac{\partial h}{\partial x}+$$
$$+\rho\frac{\partial}{\partial y}\int\limits_{f_s}^{h}u_1u_2dz-\rho\frac{\partial}{\partial y}\int\limits_{f_s}^{h}B_1B_2dz=0. \quad (A.13)$$

The second equation in (2.7) is similarly transformed as

$$\rho\frac{\partial}{\partial t}\int\limits_{f_s}^{h} u_2 dz + \rho\frac{\partial}{\partial x}\int\limits_{f_s}^{h} u_1 u_2 dz - \\ - \rho\frac{\partial}{\partial x}\int\limits_{f_s}^{h} B_1 B_2 dz + \rho g(h-f_s)\frac{\partial h}{\partial y} + \rho\frac{\partial}{\partial y}\int\limits_{f_s}^{h} u_2^2 dz - \\ - \rho\frac{\partial}{\partial y}\int\limits_{f_s}^{h} B_2^2 dz = 0. \quad \text{(A.14)}$$

Analogously, the fourth and fifth equations in (2.7) take the form

$$\frac{\partial}{\partial t}\int\limits_{f_s}^{h} B_1 dz + \frac{\partial}{\partial y}\int\limits_{f_s}^{h} B_1 u_2 dz - \\ - \frac{\partial}{\partial y}\int\limits_{f_s}^{h} B_2 u_1 dz = 0, \quad \text{(A.15)}$$

$$\frac{\partial}{\partial t}\int\limits_{f_s}^{h} B_2 dz + \frac{\partial}{\partial x}\int\limits_{f_s}^{h} B_2 u_1 dz - \\ - \frac{\partial}{\partial x}\int\limits_{f_s}^{h} B_1 u_2 dz = 0. \quad \text{(A.16)}$$

The sixth equation in (2.7) becomes identical, and Eq. (2.8) is transformed as

$$\frac{\partial}{\partial x}\int\limits_{f_s}^{h} B_1 dz + \frac{\partial}{\partial y}\int\limits_{f_s}^{h} B_2 dz = 0. \quad \text{(A.17)}$$

We introduce the mean velocities and magnetic fields over the depth:

$$u_x = \frac{1}{h-f_s}\int\limits_{f_s}^{h} u_1(x,y,z,t)\,dz,$$

$$u_y = \frac{1}{h-f_s}\int\limits_{f_s}^{h} u_2(x,y,z,t)\,dz,$$

$$B_x = \frac{1}{h-f_s}\int\limits_{f_s}^{h} B_1(x,y,z,t)\,dz,$$

$$B_y = \frac{1}{h-f_s}\int\limits_{f_s}^{h} B_2(x,y,z,t)\,dz,$$

and write

$$u_1 = \frac{1}{h-f_s}\int\limits_{f_s}^{h} u_1(x,y,z,t)\,dz + u_1'(x,y,t),$$

where

$$\int\limits_{f_s}^{h} u_1'(x,y,z,t) = 0,$$

$$u_2 = \frac{1}{h-f_s}\int\limits_{f_s}^{h} u_2(x,y,z,t)\,dz + u_2'(x,y,t),$$

where

$$\int\limits_{f_s}^{h} u_2'(x,y,z,t) = 0,$$

$$B_1 = \frac{1}{h-f_s}\int\limits_{f_s}^{h} B_1(x,y,z,t)\,dz + B_1'(x,y,t),$$

where

$$\int\limits_{f_s}^{h} B_1'(x,y,z,t) = 0,$$

$$B_2 = \frac{1}{h-f_s}\int\limits_{f_s}^{h} B_2(x,y,z,t)\,dz + B_2'(x,y,t),$$

where

$$\int\limits_{f_s}^{h} B_2'(x,y,z,t) = 0.$$

It hence follows that

$$\int\limits_{f_s}^{h} u_1^2 dz = \int\limits_{f_s}^{h} u_x^2 dz + \int\limits_{f_s}^{h} 2u_1' u_x dz + \int\limits_{f_s}^{h} u_1'^2 dz.$$

Neglecting terms that are products of fluctuating terms, we obtain the following system from (A.11), (A.13)–(A.17):

$$\frac{\partial H}{\partial t} + \frac{\partial H u_x}{\partial x} + \frac{\partial H u_y}{\partial y} = 0, \quad \text{(A.18)}$$

$$\frac{\partial H u_x}{\partial t} + \frac{\partial (H u_x^2 - H B_x^2)}{\partial x} + \frac{\partial (H u_x u_y - H B_x B_y)}{\partial y} + \\ + gH\frac{\partial h}{\partial x} = 0, \quad \text{(A.19)}$$

$$\frac{\partial Hu_y}{\partial t}+\frac{\partial(Hu_xu_y-HB_xB_y)}{\partial x}+\frac{\partial(Hu_y^2-HB_y^2)}{\partial y}+ \\ +gH\frac{\partial h}{\partial y}=0, \quad (A.20)$$

$$\frac{\partial HB_x}{\partial t}+\frac{\partial(HB_xu_y-HB_yu_x)}{\partial y}=0, \quad (A.21)$$

$$\frac{\partial HB_y}{\partial t}+\frac{\partial(HB_yu_x-HB_xu_y)}{\partial x}=0, \quad (A.22)$$

$$\frac{\partial HB_x}{\partial x}+\frac{\partial HB_y}{\partial y}=0, \quad (A.23)$$

where $H = h - f_s$.

Equations (A.18)–(A.23) are magnetohydrodynamic equations in the shallow water approximation. In one-dimensional version (2.1)–(2.6), the indices x and y are respectively denoted as 1 and 2.

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