ЖЭТФ, 2013, том 144, вып. 4 (10), стр. 867–877

STRUCTURAL ELEMENTS OF COLLAPSES IN SHALLOW WATER FLOWS WITH HORIZONTALLY NONUNIFORM DENSITY

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Received March 2, 2013

The mechanisms and structural elements of instability whose evolution results in the occurrence of the collapse are studied in the scope of the rotating shallow water model with a horizontally nonuniform density. The diagram stability based on the integral collapse criterion is suggested to explain system behavior in the space of constants of motion. Analysis of the instability shows that two collapse scenarios are possible. One scenario implies anisotropic collapse during which the contact area of a collapsing drop-like fragment with the bottom contracts into a rotating segment. The other implies isotropic contraction of the area into a point.

DOI: 10.7868/S0044451013100180

1. INTRODUCTION

The shallow water approximation arises in many physical situations where the characteristic horizontal scale (perpendicular to the gravity acceleration) is much larger than the vertical dimension of the flow. In geophysical fluid dynamics, many oceanic and atmospheric large-scale gravity currents, flows in rivers, avalanches etc., can be investigated using layered models, in which the continuous vertical structure is approximated by a small stack of layers with varying thicknesses [1].

Besides geophysical fluid dynamics, the shallow water models can be useful for studying certain astrophysical phenomena. For example, a shallow water analogue was used to describe the shock instability taking place in the collapsing inner core prior to explosion of a protoneutron star [2]. The shallow water model can also describe the dynamics of the tachocline of a star, as was done in [3, 4] for the tachocline of the Sun.

In the simplest approximation, the fluid variables within each layer, such as density and the horizontal flow velocity, are assumed to be vertically uniform, depending only on horizontal coordinates and time. The simplest layer model is the shallow water model, describing equations for a single layer of an incompressible fluid with a free surface. Finer effects, for example, baroclinic effects due to unaligned density and pressure gradients in a continuously stratified fluid, can be modeled using two or more layers. Inasmuch as layer models with constant layer densities in general have difficulty representing thermodynamic phenomena such as heating or fresh medium forcing that can become important, Ripa [5] proposed to consider a family of layered models that permitted horizontal variations in fluid density within each layer. These density variations may be attributed, for example, to horizontal temperature gradients. In the ocean/atmosphere, gravity currents are driven by temperature and salinity inhomogeneities, or considered as turbidity currents whose density derives from suspended mud or silt [6].

One disadvantage of Ripa's models is that they cannot incorporate effects of the Rayleigh–Taylor instability because, by definition, buoyancy is supposed to be positive in each layer. To overcome these limitations, we have proposed a new one-layer model [7], whose dynamics is described by a relative buoyancy of alternating sign. As is shown in [7], the collapse (blow-up) is possible in such a model only under certain initial conditions when an integral criterion is fulfilled and the distribution of density (temperature) is such that the potential energy integral is nonpositive. This means

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that the mechanism responsible for initiating the collapse is the Rayleigh–Taylor instability.

Undoubtedly, the collapse sooner or later leads to small-scale processes ignored in the simplified model [7]. However, it is extremely unlikely that in a more complete dissipation-free model, the solutions change so dramatically that collapses are completely eliminated. In particular, as shown in [8, 9], accounting for the nonlinear dispersion due to nonhydrostatic pressure effects does not cancel collapses in shallow water models. Even if the collapsing solutions are eliminated or, what is more probable, the self-similarity is lost, such solutions are still of a certain value because they can be regarded as initial or intermediate asymptotes [10].

Paper [7] was limited to the study of only integral criteria and power laws of collapses. The obtained results therefore turn out to be incomplete because the problem of finding the space structure for self-similar solutions remained beyond the scope of that work. Here, we intend to fill this gap. By analogy with [8, 9, 11–13], it is natural to expect that development of a large-scale instability in the model discussed below also leads to disintegration of the strongly perturbed flow and to the occurrence of drop-like fluid fragments. It is these formations that play the role of structural elements from which it is possible to compile an overall picture of the instability up to the turbulence stage. Because drop-like fragments produce space–time singularities responsible for power-law tails in the short-wave range of the spectrum, the study of structural elements provides the key to understanding strong turbulence [14, 15].

This article is organized as follows. In Sec. 2, we construct the minimal model and formulate the governing equations in the shallow water approximation with horizontal density gradients. In Sec. 3, we discuss the rigorous integral criterion for isotropic collapse. We assume that this phenomenon arises at the final stage when the development of the instability has led to disintegration of strongly perturbed flows. After the formation of localized fluid fragments, a time comes when finite-time singularities form. The self-similar scenarios of collapses and their corresponding structural elements are considered in Sec. 4–6. We summarize our results in Sec. 7.

2. MINIMAL MODEL

We consider the simplest model that can be proved in the framework of the two-layer model (see the Appendix for more details). This model supposes that two incompressible fluids with densities $\varrho = \mathrm{const}$ and $\varrho + \varrho'(x_1, x_2, t)$ are separated by the surface $z = h(x_1, x_2, t)$ and contained between two rigid parallel planes $z = 0$ and $z = l$ under the action of gravity g. If the horizontally nonuniform density jump ϱ' between the fluids is small and the lower layer is sufficiently thin, such that the inequalities $\varrho'/\varrho \ll 1$ and $h/l \ll 1$ hold, then the shallow water approximation leads to the equations

$$\partial_t u_i + u_k \partial_k u_i - 2\Omega e_{ik} u_k = -\partial_i (h\tau) + \frac{1}{2} h \partial_i \tau, \quad (1)$$

$$\partial_t h + \partial_k (h u_k) = 0, \quad (2)$$

$$\partial_t \tau + u_k \partial_k \tau = 0. \quad (3)$$

These equations describe the depth-averaged flow in the lower layer, and our notation is as follows: $x_i = (x_1, x_2)$ are the Cartesian coordinates; $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$; e_{ik} is the unit antisymmetric tensor, $e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$; $u_i = (u_1, u_2)$ are horizontal components of the depth-averaged velocity in the layer; and h is its thickness. Because Ω is the constant angular velocity with which the layer is rotating about the vertical axis, the term $2\Omega e_{ik} u_k$ describes the Coriolis acceleration components. The field variable $\tau = g\varrho'/\varrho$ has the meaning of the relative buoyancy and can therefore take any sign.

In the cases where density variations are produced only by temperature ones, ΔT, and are linearly connected, the relative buoyancy can be computed as $\tau = -g\beta\Delta T$, where β is thermal expansion coefficient. This parameterization allows studying the heating and cooling effects in shallow water models [16, 17].

We note that in the case $\tau = 1$, Eqs. (1)–(3) reduce to the usual shallow water equations. The other limit case $\tau = -1$ leads to the so-called "inverted" shallow water model describing the layer of a heavy fluid bounded above by a solid slab. The equilibrium in the unperturbed state is provided by the pressure of a light fluid or a gas lying below. Examples of using the inverted shallow water model in various applications are presented in [18]. Understandably, such equilibrium is unstable and short-lived. The heavier fluid eventually falls down to the bottom. But initial and intermediate stages of the instability, when the system is far from the final state, are of the utmost importance. Their study provides a way for understanding the processes of vertical mixing in many physical applications, including atmospheric and ocean science.

There is one more useful interpretation of Eqs. (1)–(3) as equations of hydrodynamic type derived from first principles (conservation laws). As can be verified directly, if the variables h and τ are regarded as densities of mass and entropy, Eqs. (1)–(3) follow from the Hamiltonian formulation [19, 20] of two-dimensional motion of a nonbarotropic rotating gas with the Hamiltonian

$$H = \int d\mathbf{x} \left(h\frac{\mathbf{u}^2}{2} + \epsilon(h,\tau) \right),$$

where $\epsilon(h,\tau)$ is the internal energy density, which in our case is given by $\epsilon = h^2\tau/2$, and $d\mathbf{x} = dx_1 dx_2$.

In terms of the variables h, τ, and

$$\mathbf{m} = \frac{\delta H}{\delta \mathbf{u}} = h\mathbf{u}$$

(referred to as the hydrodynamic momentum density), nontrivial Poisson brackets defining the dynamics for the given family of models take the form

$$\{m_i, m'_k\} = \partial'_i(m'_k\delta) - \partial_k(m_i\delta) + 2\Omega h e_{ik}\delta, \qquad (4)$$

$$\{h, m'_k\} = -\partial_k(h\delta), \quad \{\tau, m'_k\} = -\delta\partial_k\tau. \qquad (5)$$

Primed field variables imply the dependence on the primed spatial coordinates, and $\delta = \delta(\mathbf{x} - \mathbf{x}')$ is the Dirac delta function.

Evolution (1)–(3) preserves the integrals of total mass Q and total energy H,

$$Q = \int d\mathbf{x}\, h, \quad H = \frac{1}{2}\int d\mathbf{x}\, \left(h\mathbf{u}^2 + h^2\tau \right). \qquad (6)$$

In addition, any system with Poisson brackets (4), (5) automatically preserves the integrals (Casimirs)

$$C = \int d\mathbf{x}\, \left(\partial_1 u_2 - \partial_2 u_1 + 2\Omega \right) F(\tau)$$

for any function $F(\tau)$. Among them, we note the conservation law

$$n = \int d\mathbf{x}\, \tau \left(\partial_1 u_2 - \partial_2 u_1 + 2\Omega \right). \qquad (7)$$

As we see in what follows, this quantity together with other constants of motion plays an important role in determining self-similar solutions considered in Secs. 4–6.

3. COLLAPSE CRITERION

If the fluid moves as a whole, then it is convenient to pass from old coordinates $\mathbf{x}$ to new ones $\mathbf{x}'$ connected with the center-of-mass reference frame. In this case, the primed and unprimed coordinates and velocities are related by the transformation

$$\mathbf{x} = \mathbf{X} + \mathbf{x}', \quad \mathbf{u} = Q^{-1}\mathbf{P} + \mathbf{u}', \qquad (8)$$

where coordinates of the center of mass $\mathbf{X}$ and components of the total momentum $\mathbf{P}$ are defined as

$$\mathbf{X} = Q^{-1}\int d\mathbf{x}\, h\mathbf{x}, \quad \mathbf{P} = \int d\mathbf{x}\, h\mathbf{u},$$

and, by virtue of Eqs. (1)–(3), are governed by the equations

$$\partial_t X_i = Q^{-1}P_i, \quad \partial_t P_i = 2\Omega e_{ik} P_k.$$

Because the transformation (8) leaves Eqs. (1)–(3) invariant, we do not change the notation and merely set $\mathbf{P} = 0$ and $\mathbf{X} = 0$ from the very beginning.

As shown in [7], the model in (1)–(3) admits a simple mechanical reduction in terms of the variables

$$V = \int d\mathbf{x}\, h\, x_i u_i, \quad M = \int d\mathbf{x}\, h \left(x_1 u_2 - x_2 u_1 \right), \quad I = \int d\mathbf{x}\, h\mathbf{x}^2. \qquad (9)$$

The integrals I, M, and V have the respective meaning of the moment of inertia, the kinetic moment, and the virial, and obey the closed system of equations

$$\partial_t I = 2V, \quad \partial_t V = 2H + 2\Omega M, \quad \partial_t M = -2\Omega V. \qquad (10)$$

Equations (10) give two more motion integrals

$$m = M + \Omega I, \qquad (11)$$

$$V_0^2 = \left(M + \Omega^{-1}H \right)^2 + V^2, \qquad (12)$$

and can be easily integrated to obtain

$$I = \Omega^{-2}\left(H + \Omega m \right) - \frac{V_0}{\Omega}\cos 2\Omega\left(t - t_0 \right), \qquad (13)$$

$$M = -\Omega^{-1}H + V_0 \cos 2\Omega\left(t - t_0 \right), \qquad (14)$$

$$V = V_0 \sin 2\Omega\left(t - t_0 \right), \qquad (15)$$

where t_0 is a constant of integration.

The integral I serves as an indicator of the isotropic collapse, in the course of which this positive-definite quantity undergoes specific temporal changes: I decreases with increasing t and reaches the value $I = 0$ at a finite point $t = t_0 > 0$. The condition for such behavior is the inequality

$$\left(H + \Omega m \right)^2 \le \Omega^2 V_0^2. \qquad (16)$$

We also note that this condition can be written in the equivalent form

$$2\left(H + \Omega m - \Omega^2 I \right) I \le V^2 = \frac{1}{4}(\partial_t I)^2, \qquad (17)$$

if we eliminate V_0 and M by using relations (11), (12).

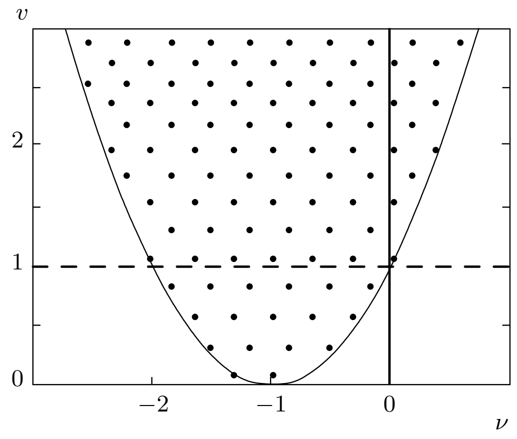

Fig. 1. Stability diagram. The collapse region is marked by dots

These inequalities are criterions for collapse in the rotating shallow water model with horizontally nonuniform density. Only under these conditions does the development of instability lead to the formation of a singularity in the point $\mathbf{x} = 0$.

According to (16), the stability of the system is determined by four constants of motion: H, m, V_0, and Ω. In place of them, it is more convenient to use two nondimensional parameters

$$\nu = \frac{H}{m\Omega}, \quad \upsilon = \left(\frac{V_0}{m}\right)^2$$

in terms of which the possible scenarios of stability and instability can be analyzed with the diagram shown in Fig. 1.

From the diagram of stability, we see that increasing the angular velocity $|\Omega|$, such that $\nu \to 0$ as $|\Omega| \to \infty$, allows the system to leave the collapse region only if $\upsilon \le 1$, i.e., $|V_0| \le |m|$. But in the opposite case $\upsilon > 1$ (and hence $|V_0| > |m|$), with the other parameters fixed, an analogous behavior of $|\Omega|$ does not lead to the same result.

In the case of isotropic collapsing, h behaves as a self-similar function, such that $h = \beta^{-2} f(\mathbf{x}/\beta)$, and we can therefore write the relation

$$I = \beta^2 C, \tag{18}$$

where $\beta(t)$ is a function of time and C is a positive constant depending on the shape factor f only.

On the other hand, expanding the function I in powers of $t_0 - t$ in the vicinity of the collapse time t_0, we approximately obtain

$$I \approx a_1 (t_0 - t) + a_2 (t_0 - t)^2 + \ldots, \tag{19}$$

where t_0, the coefficients a_1 and a_2, and the integrals of motion are related as

$$a_1 = 2\sqrt{V_0^2 - \Omega^{-2} (H + \Omega m)^2},$$
$$a_2 = 2(H + \Omega m),$$
$$H + \Omega m = V_0 \Omega \cos(2\Omega t_0).$$

The comparison of (18) with (19) allow us to make the following conclusions.

1. If $a_1 \neq 0$, i.e., inequality (16) is strict, then the isotropic collapse obeys the laws

$$\beta \sim (t_0 - t)^{1/2}, \quad h \sim \beta^{-2} \sim (t_0 - t)^{-1}. \tag{20}$$

2. If $a_1 = 0$, i.e., inequality (16) turns into an equality, then, instead of (20), we obtain the laws

$$\beta \sim (t_0 - t), \quad h \sim \beta^{-2} \sim (t_0 - t)^{-2}.$$

We note that system (10) can be viewed as a generalization of the virial theorem, which obtaining the Vlasov–Petrishchev–Talanov-type criterion for collapse in the nonlinear Schrödinger (NLS) equation. This criterion was first formulated for the two-dimensional NLS equation [21] and was later generalized to many other models. Among them is the NLS model in semiclassical limit [22], where the Zakharov equations transform into a hydrodynamic-type system. In particular, in the absence of rotation, system (10) reduces to the equation

$$\partial_t^2 I = 4H, \tag{21}$$

which coincides with that for the two-dimensional NLS equation, and after integration gives

$$I = 2Ht^2 + I_0' t + I_0.$$

Here, the initial data $I_0 = I|_{t=0}$ and $I_0' = (\partial_t I)|_{t=0} = 2V|_{t=0}$ are used as constants of integration. In this case, criterion (17) becomes

$$2HI \le V^2 = \frac{1}{4}(\partial_t I)^2 \tag{22}$$

and enables us to make the following conclusions.

1. If $H < 0$, the isotropic collapse always occurs. Because H can be represented as

$$H = K + \Pi, \quad K = \frac{1}{2}\int d\mathbf{x}\, h\mathbf{u}^2, \quad \Pi = \frac{1}{2}\int d\mathbf{x}\, h^2 \tau,$$

the inequality $H < 0$ implies that $\Pi|_{t=0} < -K|_{t=0}$. The only way to provide this condition is by appropriately choosing the initial distribution for the field τ, which, unlike h, can be sign-alternating.

2. If $H \geq 0$, the fulfilment of criterion (22) depends on I_0 and I_0', and hence at the initial time we have the condition

$$8HI_0 \leq I_0'^2, \tag{23}$$

where I_0' must be negative because I decreases with time.

On the other hand, based on the Cauchy inequality, we can write

$$(\partial_t I)^2 = 4V^2 \leq 8IK. \tag{24}$$

It is clear that inequalities (23) and (24) are consistent only if

$$H \leq \frac{I_0'^2}{8I_0} \leq K|_{t=0}.$$

As a consequence, we arrive at the condition $-K|_{t=0} \leq \leq \Pi|_{t=0} \leq 0$.

Therefore, irrespective of the sign of H, the collapse becomes possible if $\Pi|_{t=0} \leq 0$. The negative quantity $-K|_{t=0}$ plays the role of a critical level. For values $\Pi|_{t=0}$ below the critical level, no conditions are required, but above or at this level, the additional conditions must be satisfied.

4. STRUCTURAL ELEMENTS OF COLLAPSES

It is known [10] that self-similar solutions are intermediate asymptotic regimes of nondegenerate problems and are very useful in studying the final stages of strongly nonlinear processes, when the system forgets about the details related to the initial data and its behavior depends on the motion integrals. For any dynamical system, the existence of self-similar solutions reflects the existence of fundamental internal symmetries and allows deciding on the tendencies in the development of the instability at the final stage. This type of solutions is of particular importance for studying the phenomenon of collapse — formation of a singularity in a finite time [22–28].

As a self-similar substitution for h and τ, we consider the expressions

$$h = \frac{Q|G|}{\pi}(1+\gamma)f^{\gamma}, \quad \tau = \tau_0 f^{1-\gamma}, \tag{25}$$

$$f = 1 - (GU\mathbf{x})^2, \tag{26}$$

where $0 \leq \gamma \leq 1$, τ_0 is a magnitude-specified parameter, G and U are delation and rotation matrices

$$G = \begin{pmatrix} \beta_1^{-1} & 0 \\ 0 & \beta_2^{-1} \end{pmatrix}, \quad U = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

$\beta_1(t)$ and $\beta_2(t)$ are positive deformation parameters, and $\varphi(t)$ is the angle of rotation in the transformation $\mathbf{x}' = U\mathbf{x}$.

The ansatz in (25) and (26) describes a liquid drop concentrated on a compact carrier of elliptic shape

$$x_1'^2\beta_1^{-2} + x_2'^2\beta_2^{-2} = 1,$$

and rotated with the angular speed $\partial_t\varphi$.

Direct substitution in Eqs. (1)–(3) shows that expressions (25) and (26) are exact solutions if the velocity components $\mathbf{u}' = d\mathbf{x}'/dt$ in the rotating coordinate system $\mathbf{x}' = U\mathbf{x}$ obey the relations

$$u_1' = \frac{\alpha_1}{\beta_1}x_1' - \lambda\frac{\beta_1}{\beta_2}x_2', \quad u_2' = \frac{\alpha_2}{\beta_2}x_2' + \lambda\frac{\beta_2}{\beta_1}x_1'. \tag{27}$$

Equations (27) correspond to the uniform vorticity distribution inside the domain with an elliptical liquid contour boundary. The variables α_i, β_i, λ, and $\varphi' = = \varphi - \Omega t$ as functions of time satisfy the equations

$$\partial_t\alpha_i = -\beta_i\Omega^2 + (-1)^i 2(\beta_2-\beta_1)\lambda\partial_t\varphi' + \\ + \beta_i(\lambda - \partial_t\varphi')^2 + \frac{Q\tau_0(1+\gamma)^2}{\pi\beta_i\beta_1\beta_2}, \tag{28}$$

$$\partial_t\beta_i = \alpha_i, \tag{29}$$

$$m' = \lambda\beta_1\beta_2 - \frac{1}{2}\left(\beta_1^2+\beta_2^2\right)\partial_t\varphi', \tag{30}$$

$$n' = \frac{1}{2}\lambda\left(\beta_1^2+\beta_2^2\right) - \beta_1\beta_2\partial_t\varphi', \tag{31}$$

where m' and n' are parameters connected with motion invariants m and n by the relations

$$m' = (2+\gamma)\frac{m}{Q}, \quad n' = (2-\gamma)\frac{n}{2\pi\tau_0}.$$

We note that Eqs. (30) and (31) can be alternatively derived by substituting (25), (26), and (27) in the right-hand sides of (7), (11).

After eliminating λ and φ' from Eqs. (28)–(31), we obtain that variables α_i and β_i obey the canonical equations of motion

$$\partial_t\alpha_i = -\frac{\partial\mathcal{H}}{\partial\beta_i} = -\beta_i\Omega^2 + 2\frac{(n'+m')^2}{(\beta_1+\beta_2)^3} - \\ - (-1)^i 2\frac{(n'-m')^2}{(\beta_1-\beta_2)^3} + \frac{Q\tau_0(\gamma+1)^2}{\pi\beta_i\beta_1\beta_2}, \tag{32}$$

$$\partial_t\beta_i = \frac{\partial\mathcal{H}}{\partial\alpha_i} = \alpha_i, \tag{33}$$

which describe a system with two degrees of freedom and the Hamiltonian

$$\mathcal{H} = 2\frac{(\gamma+2)}{Q}(H+\Omega m) = \frac{1}{2}\left(\alpha_1^2+\alpha_2^2\right) + \left(\frac{n'+m'}{\beta_1+\beta_2}\right)^2 + \left(\frac{n'-m'}{\beta_1-\beta_2}\right)^2 + \frac{\Omega^2}{2}\left(\beta_1^2+\beta_2^2\right) + \frac{Q\tau_0(1+\gamma)^2}{\pi\beta_1\beta_2}. \quad (34)$$

Once we know the variables β_1 and β_2, we can find λ and $\partial_t\varphi$ as

$$\lambda = 2\frac{n'\left(\beta_1^2+\beta_2^2\right) - 2m'\beta_1\beta_2}{\left(\beta_1^2-\beta_2^2\right)^2},$$

$$\partial_t\varphi = \Omega + 2\frac{2n'\beta_1\beta_2 - m'\left(\beta_1^2+\beta_2^2\right)}{\left(\beta_1^2-\beta_2^2\right)^2}.$$

Hence, the ansatz in (25), (26), and (27) reduces the initial infinite-dimensional Hamiltonian system given by Eqs. (1)–(3) to a two-dimensional canonical system.

If motion invariants n and Ω are finite, it is convenient to convert the problem to dimensionless form by choosing the spatial scale L and characteristic time T such that

$$T = |\Omega|^{-1}, \quad L = |n'/\Omega|^{1/2}.$$

After nondimensionalizing, Hamiltonian (34) is rewritten as

$$\mathcal{H} = \frac{1}{2}\left(\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2\right) + \frac{\sigma-1}{\beta_1\beta_2} + \left(\frac{1+\mu}{\beta_1+\beta_2}\right)^2 + \left(\frac{1-\mu}{\beta_1-\beta_2}\right)^2,$$

where μ and σ are the nondimensional parameters

$$\mu = \frac{m'}{n'} = 2\pi\frac{(2+\gamma)}{(2-\gamma)}\frac{\tau_0 m}{Qn}, \qquad \sigma = 1 + 4\pi\left(\frac{1+\gamma}{2-\gamma}\right)^2\frac{Q\tau_0^3}{n^2}. \quad (35)$$

The corresponding equations of motion for α_i, β_i, and φ are given by

$$\partial_t\alpha_i = -\frac{\partial\mathcal{H}}{\partial\beta_i} = -\beta_i + \frac{\sigma-1}{\beta_i\beta_1\beta_2} + 2\frac{(1+\mu)^2}{(\beta_1+\beta_2)^3} - (-1)^i 2\frac{(1-\mu)^2}{(\beta_1-\beta_2)^3}, \quad (36)$$

$$\partial_t\beta_i = \frac{\partial\mathcal{H}}{\partial\alpha_i} = \alpha_i, \quad (37)$$

$$\partial_t\varphi = 1 + 2\,\mathrm{sign}(n')\frac{2\beta_1\beta_2 - \mu\left(\beta_1^2+\beta_2^2\right)}{\left(\beta_1^2-\beta_2^2\right)^2}. \quad (38)$$

5. ISOTROPIC SOLUTIONS

Below, we single out isotropic solutions of two types, rotating ($\Omega = 1$), and nonrotating ($\Omega = 0$).

We let $\Omega = 1$ and assume that solutions are radially symmetric. Then

$$\beta_1 = \beta_2 = \beta, \quad \alpha_1 = \alpha_2 = \alpha.$$

As analysis shows, such solutions are degenerate and are possible only if $\mu = 1$ or $m' = n'$. This is the reason why the rotational effect due to $\partial_t\varphi$ loses theoretical legitimacy and Eq. (38) is no longer valid. Instead, we can see directly from (30) and (31) that the functions φ and λ become linearly dependent,

$$\mu = 1 = \beta^2\left(1+\lambda-\partial_t\varphi\right).$$

Hence, isotropic solutions are rotationally invariant.

In this case, Eqs. (36) and (37) reduce to the form

$$\partial_t\alpha = -\frac{\partial H_1}{\partial\beta} = \frac{\sigma}{\beta^3} - \beta, \quad \partial_t\beta = \frac{\partial H_1}{\partial\alpha} = \alpha, \quad (39)$$

where

$$H_1 = \frac{1}{2}\left(\alpha^2+\beta^2+\frac{\sigma}{\beta^2}\right). \quad (40)$$

Analytic solutions of Eq. (39) can be written as

$$\beta = \sqrt{H_1 - \left(H_1^2-\sigma\right)^{1/2}\cos 2(t_0-t)}, \qquad \alpha = \frac{\left(H_1^2-\sigma\right)^{1/2}\sin 2t(t_0-t)}{\sqrt{H_1 - \left(H_1^2-\sigma\right)^{1/2}\cos 2(t_0-t)}}. \quad (41)$$

Relevant structures look like radially symmetric drops.

Without loss of generality, we can assume that σ equals either 0 or 1, or -1. With $\Omega = 1$, depending on the parameter σ, there are three different branches of solutions, hereafter referred to as the neutral ($\sigma = 0$), cold ($\sigma = 1$), and warm ($\sigma = -1$) rotating regimes. The parameter σ is chosen according to the rule

$$\sigma = \begin{cases} 0, & \tau_0 = \tau^*, \\ 1, & \tau_0 > \tau^*, \\ -1, & \tau_0 < \tau^*, \end{cases}$$

where the threshold value τ^* is determined from (35) with $\sigma = 0$, which yields

$$\tau^* = -\left(\frac{n^2}{4\pi Q}\right)^{1/3}\left(\frac{2-\gamma}{1+\gamma}\right)^{2/3}.$$

1. In the neutral regime, when $\sigma = 0$, the motion can occur only if $H_1 > 0$. As is shown in Fig. 2, the

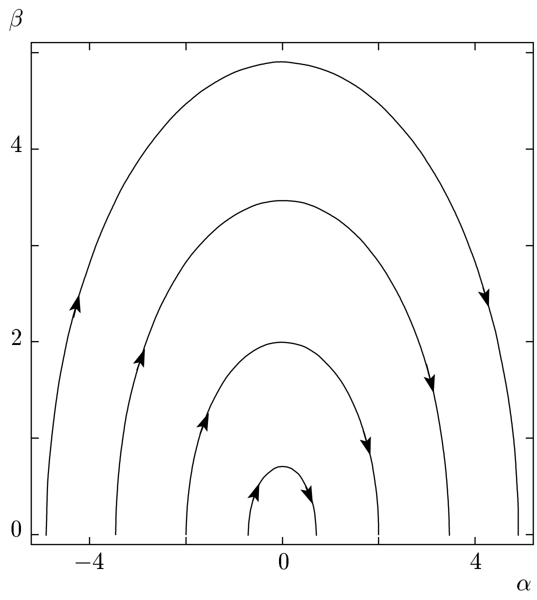

Fig. 2. Phase portrait of the rotating isotropic model in the neutral regime ($\sigma = 0$)

system moves along open trajectories in the form of semicircles. The arrows placed along the phase trajectories show the direction of motion in time. Relevant solutions for h look like drops collapsing according to the laws

$$\beta \sim t_0 - t, \quad \alpha \sim -\sqrt{2H_1}, \quad h \sim (t_0 - t)^{-2}.$$

2. In the cold regime, the motion can occur only if $H_1 \geq 1$. According to Fig. 3, the typical trajectories of the dynamical system are closed curves, which correspond to periodic solutions. Relevant solutions for h look like pulsating drops. The minimum $H_1 = 1$ is attained at the point $\alpha = 0$, $\beta = 1$ and corresponds to a stationary (nonpulsating) solution.

3. In the warm regime, the system moves along open trajectories shown in Fig. 4.

The collapse point is reached for both positive and negative values of H_1 as $\alpha \to \infty$, and $\beta \to 0$. In this regime, the variables β and α asymptotically (as $t \to t_0$) tend to zero and infinity, respectively, according to the laws

$$\beta \sim (t_0 - t)^{1/2}, \quad \alpha \sim (t_0 - t)^{-1/2}, \quad h \sim (t_0 - t)^{-1},$$

where the collapse time

$$t_0 = \frac{1}{2} \arccos \frac{H_1}{\sqrt{1 + H_1^2}}$$

is determined from (41) and the condition $\beta(t_0) = 0$.

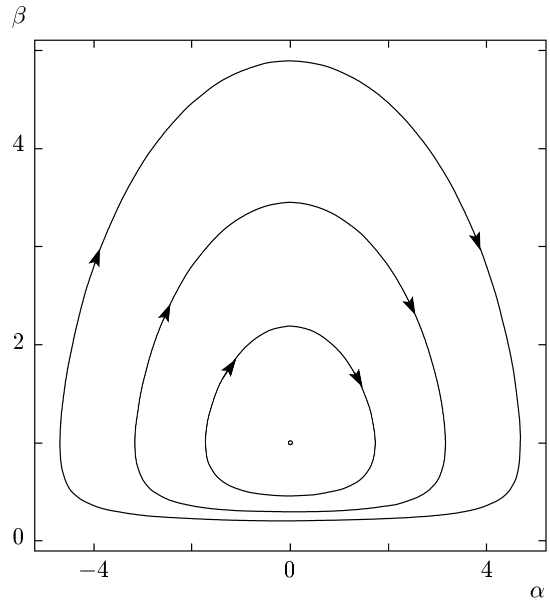

Fig. 3. Phase portrait of the rotating isotropic model in the cold regime ($\sigma = 1$)

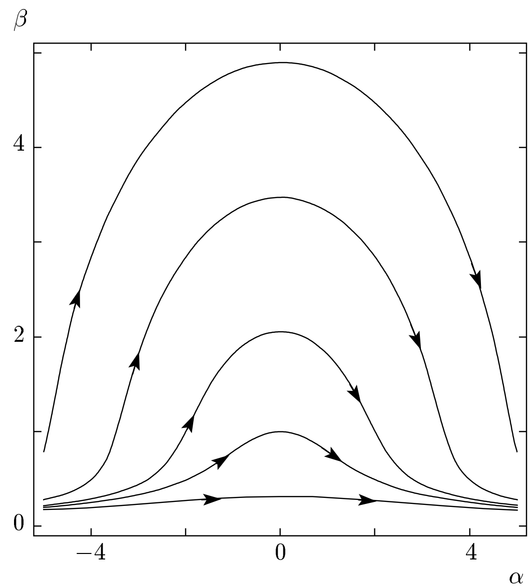

Fig. 4. Phase portrait of the rotating isotropic model in the warm regime ($\sigma = -1$)

We emphasize that in the case $\Omega = 1$, there are no spreading regimes for which $\beta \to \infty$ as $t \to \infty$ among solutions of Eqs. (39). These regimes occur only in nonrotating shallow water models with $\Omega = 0$. Because the proper analytic treatment implies dropping the term with β^2 from Hamiltonian (40), Eqs. (39) reduce to the form

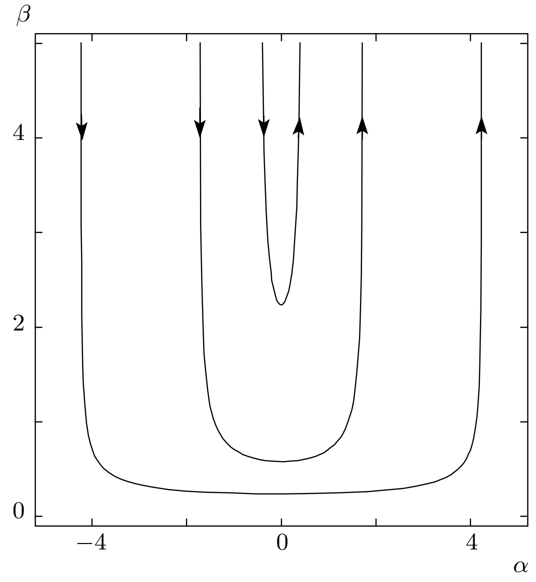

Fig. 5. Phase portrait of the nonrotating isotropic model in the cold regime ($\sigma = 1$)

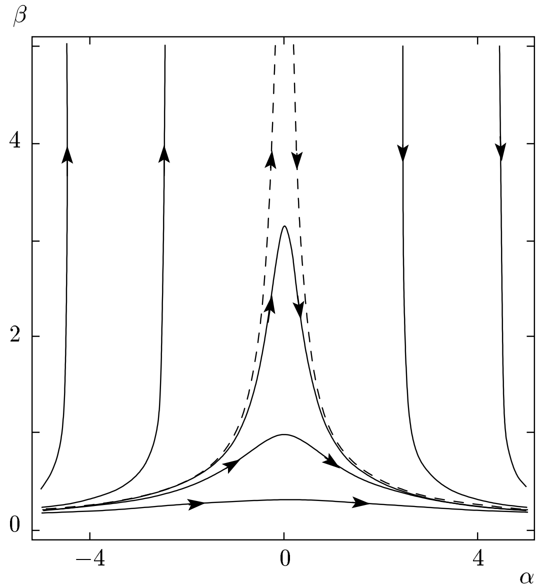

Fig. 6. Phase portrait of the nonrotating isotropic model in the warm regime ($\sigma = -1$)

$$\partial_t \alpha = -\frac{\partial H_0}{\partial \beta} = \frac{\sigma}{\beta^3}, \quad \partial_t \beta = \frac{\partial H_0}{\partial \alpha} = \alpha, \qquad (42)$$

where

$$H_0 = \frac{1}{2}\left(\alpha^2 + \frac{\sigma}{\beta^2}\right).$$

Phase trajectories of Eqs. (42) are presented in Figs. 5 and 6 for both cold and warm nonrotating regimes. The steady-state case $\sigma = 0$ is of no interest because of its triviality. Regardless of the sign of σ, spreading regimes are realized only on trajectories with $H_0 > 0$. In Fig. 6, the spreading ($H_0 > 0$) and collapsing ($H_0 < 0$) regimes are separated by a dashed line. Hence, if $\sigma = -1$ (i. e., a regime is warm) and $H_0 < 0$, the topology of phase trajectories is independent of whether the shallow water model is rotating.

6. ANISOTROPIC SOLUTIONS

Equations (36) and (37) can have solutions that violate the radial symmetry. In such situations, using positive-definite integral (9) to test the anisotropic collapse is not a good idea, since this quantity reaches zero only if β_1 and β_2 vanish simultaneously.

We first consider the anisotropic collapse scenario according to which the cross-sectional area $s = \pi\beta_1\beta_2$ tends to zero due to the unilateral compression along one of the semiaxes (e. g., β_1), whereas the other semiaxis β_2 remains finite (Fig. 7). As a result, the elliptic contact area of a collapsing liquid fragment contracts into a line segment rather than into a point.

Analysis of the anisotropic collapse solutions in the vicinity of the point $t = t_0$ results in the following

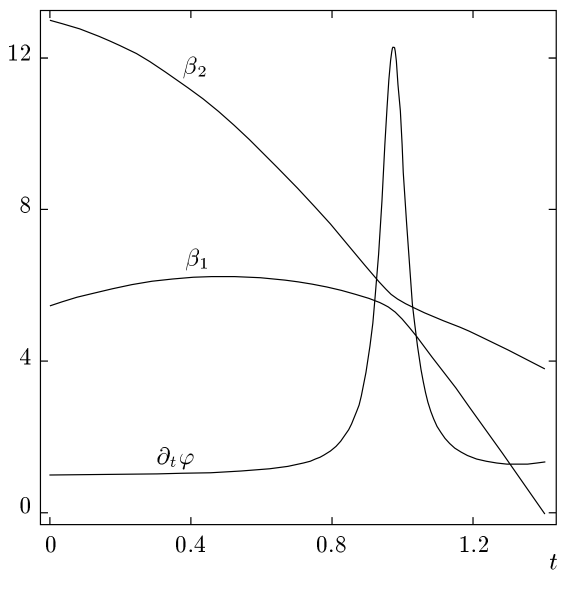

Fig. 7. Anisotropic collapse. The calculation was performed for the parameters $\sigma = -4$ and $\mu = 2.8$, and the initial values $\beta_1(0) = 5.5$, $\beta_2(0) = 13$, $\alpha_1(0) = 3$, and $\alpha_2(0) = -2$

asymptotic forms of β_1 and β_2:

$$\beta_1 \approx b\,(t_0-t)^{2/3} + a\,(t_0-t)^{4/3}, \qquad (43)$$

$$\beta_2 \approx -\frac{9}{2} b^{-3}(\sigma-1) + \frac{b^5}{9(\sigma-1)}\,(t_0-t)^{4/3}. \qquad (44)$$

Here, b and a are constants dependent on the initial conditions and closely connected with other constants of motion by

$$H = \frac{3^4}{2^3}\,\frac{(\sigma-1)^2}{b^6} + b^6\,\frac{2^3}{3^4}\,\frac{1+\mu^2}{(\sigma-1)^2} + \frac{10}{9} ba.$$

Asymptotic forms (43) and (44) signify two things. First, the anisotropic collapse is possible only if $\sigma < 1$ and, correspondingly, requires a negative value of τ_0. Second, because $h \sim (\beta_1\beta_2)^{-1}$, such a collapse obeys the law

$$h \sim (t_0-t)^{-2/3}.$$

By contrast, the isotropic collapses follow the comparatively faster laws $h \sim (t_0-t)^{-1}$ or $h \sim (t_0-t)^{-2}$.

We note that as $t \to t_0$, the contact area s between the liquid and the bottom shrinks into a line segment that rotates with the constant angular velocity

$$\partial_t\varphi = 1 - 2\mu\beta_2^{-2} = 1 - \frac{2^3}{3^4}\,\frac{\mu b^6}{(\sigma-1)^2}.$$

Collapsing solutions in the flat model have the same character as in a two-dimensional model, with the only difference that the contact area s shrinks not into a line segment but into an infinite axis perpendicular to the flow plane.

The flat model for shallow water follows from Eqs. (32) and (33) if one of the semiaxes (e.g., β_2) and, correspondingly, the total mass Q tend to infinity such that $Q/\beta_2 \to \text{const}$, $\alpha_2 \to 0$, and $\Omega = n = m = 0$. Collapses in the flat model therefore represent an idealization that ignores effects of rotation.

In this case, the nondimensional equations of motion are written as

$$\partial_t\alpha_1 = -\frac{\partial H'}{\partial\beta_1} = \frac{\sigma}{\beta_1^2}, \quad \partial_t\beta_1 = \frac{\partial H'}{\partial\alpha_1} = \alpha_1, \qquad (45)$$

$$H' = \frac{1}{2}\alpha_1^2 + \frac{\sigma}{\beta_1},$$

where $\sigma = \operatorname{sign}\tau_0$ is the only nondimensional parameter.

Phase portraits of nonlinear system (45) have no qualitative differences from the ones presented in Figs. 5, and 6. As analysis shows, depending on the parameter σ, there exist two kinds of collapsing regimes.

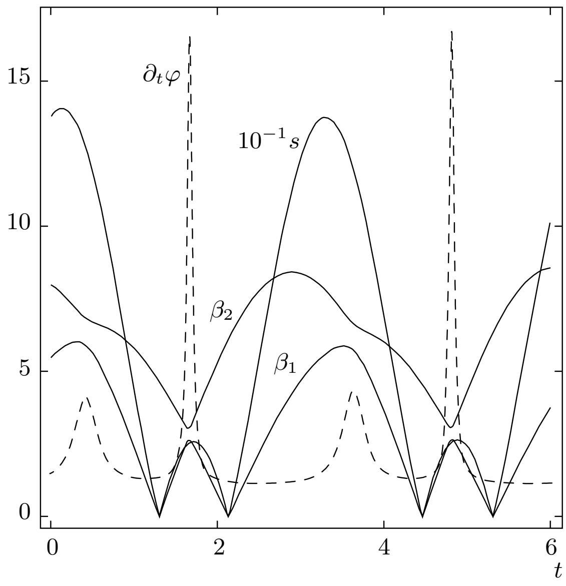

Fig. 8. Oscillation regime. The calculation was performed for the parameters $\sigma = 2$ and $\mu = 3.8$, and the initial values $\beta_1(0) = 5.5$, $\beta_2(0) = 8$, $\alpha_1(0) = 3$, $\alpha_2(0) = -2$

If $\sigma < 0$, the variables β and h asymptotically (as $t \to t_0$) tend to zero and infinity, respectively, according to the laws

$$\beta_1 \sim (t_0-t)^{2/3}, \quad h \sim (t_0-t)^{-2/3}.$$

But if $\sigma = 0$, these variables obey the laws

$$\beta_1 \sim (t_0-t), \quad h \sim (t_0-t)^{-1}.$$

In the absence of collapses, system (36)–(38) describes nonlinear oscillations. Such behavior of the system agrees completely with laws (13)–(15), according to which the oscillatory period π/Ω is indispensable for the rotating shallow water model. Because this period defines the maximum time scale in the system, characteristic times of all other feasible effects, including collapse, should be smaller.

A typical example of time behavior for basic functions is shown in Fig. 8.

Although collapse is physically impossible ($\sigma > 1$), the minimal and maximal values of the area s (height h) can be very small and very large. Specifically, the numerical experiment in Fig. 8 gives $s_{max} = 137.76$ and $s_{min} = 0.095$. Another notable fact is the periodic bursts of the angular velocity $\partial_t\varphi$ (marked by a dashed line) at the instant when the semiaxes β_1 and β_2 come close to each other.

7. CONCLUSIONS

We summarize the main results in the work. The main goal of this paper was to study structural elements of collapses in the shallow water model with horizontally nonuniform density. The diagram of stability based on the rigorous integral criterion for isotropic collapse allows to making some qualitative conclusions about the system behavior in the space of constants of motion. In particular, depending on the ratio between two integrals of motion V_0 and m, an amplification of rotation, i. e., an increase in the angular velocity Ω leads both to stabilization of the flow, if $|V_0| < |m|$, and to destabilization, if $|V_0| \geq |m|$.

In our opinion, the collapse phenomenon arises at the final stage when the development of instability has led to disintegration of the strongly perturbed flows. Once the fluid forms localized (drop-like) fragments, the collapse eventually occurs and leads to the formation of finite-time singularities.

Analysis of the instability shows that two collapse scenarios are possible depending on whether the contact area between the drop and the bottom is contractible into a segment or into a point. In the course of anisotropic collapsing, the contact area contracts to a spinning segment and the drop height h obeys the law $h \sim (t_0 - t)^{-2/3}$. By contrast, the isotropic scenario implies that the contact area contracts to a point. Because of this, the height h follows relatively faster laws $h \sim (t_0 - t)^{-1}$ and $h \sim (t_0 - t)^{-2}$ in warm and neutral regimes, respectively. In the absence of collapses, a drop-like fragment undergoes nonlinear oscillations with the period π/Ω. This period is the largest time scale of the rotating shallow water system.

This work was supported by the RFBR (grant № 12-05-00168), by the Presidium of the Russian Academy of Sciences (Program Fundamental Problems of Nonlinear Dynamics), and by the President of the Russian Federation (Program NSh-4166.2006.5).

APPENDIX

Formulation of model

We consider two layers of inviscid incompressible fluids that move under the action of gravity g in the Cartesian frame x, y, z rotating around the vertical axis z with a constant angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)$. We suppose the layers are separated by a surface $z = h(x, y, t)$ and are contained between two rigid parallel planes $z = 0$ and $z = l$ as shown in Fig. 9.

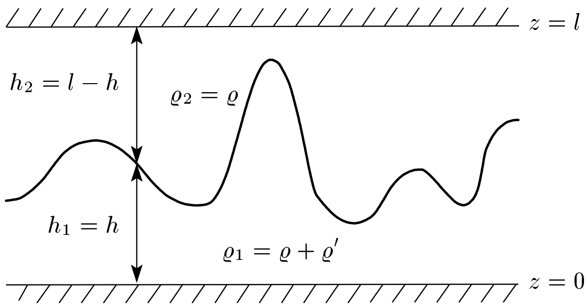

Fig. 9. Vertical structure of the two-layer model

Our purpose is to derive description for this flow in the shallow water approximation taking into account that the density jump between the layers is small and horizontally nonuniform. The corresponding motion equations can be obtained from the results in [5] if the free boundary condition for the uppermost layer in Ripa's model is replaced with the rigid-lid approximation.

This modification does not change Ripa's equations which, as before, are given by

$$(\partial_t + \mathbf{u}\cdot\nabla)\,\mathbf{u} - 2\Omega\times\mathbf{u} + \nabla\tilde{p} = \tilde{h}\nabla\theta, \qquad (46)$$

$$\partial_t h + \nabla\cdot(h\mathbf{u}) = 0, \quad (\partial_t + \mathbf{u}\cdot\nabla)\,\theta = 0 \qquad (47)$$

in each layer, where $h(x, y, t)$ is the thickness, $\mathbf{u}(x, y, t)$ is the velocity, and $\theta(x, y, t)$ is the buoyancy. We note that the variables $\mathbf{u}(x, y, t)$ and $\theta(x, y, t)$ must be interpreted as vertically layer-averaged quantities. The other two variables $\tilde{h}(x, y, t)$ and $\tilde{p}(x, y, t)$ are treated as the height of the center of mass of the layer and the effective pressure in the absence of inhomogeneities.

In the rigid-lid approximation, Ripa's definition of $\tilde{h}$ for the ith layer remains unchanged,

$$\tilde{h}_i = \sum_{k=1}^{i} h_k - \frac{1}{2}h_i, \qquad (48)$$

but the definition for $\tilde{p}$ must be modified as

$$\tilde{p}_i = \theta_i \sum_{k=1}^{i} h_k + \sum_{k=i+1}^{n} \theta_k h_k + p', \qquad (49)$$

where $p'(x, y, t) \neq \text{const}$ is the rigid-lid pressure, and the layer thicknesses satisfy the condition

$$\sum_{k=1}^{n} h_k = l, \qquad (50)$$

where the total depth l is a constant.

In the case of two layers, assuming that $\varrho_2 = \varrho = \text{const}$ and $\varrho_1 = \varrho + \varrho'(x, y, t)$, we obtain in accordance with (48)–(50) that

$$\tilde{h}_1 = \frac{1}{2}h_1, \quad \tilde{h}_2 = h_1 + \frac{1}{2}h_2 = l - \frac{1}{2}h_2, \tag{51}$$

$$\tilde{p}_1 = \theta_1 h_1 + \theta_2 h_2 + p', \quad \tilde{p}_2 = l\theta_2 + p', \tag{52}$$

$$\theta_1 = g\left(1 + \frac{\varrho'}{\varrho}\right), \quad \theta_2 = g. \tag{53}$$

After substitution of relations (51)–(53) to Eqs. (46) and (47), these equations take the form

$$(\partial_t + \mathbf{u}_1 \cdot \nabla)\,\mathbf{u}_1 - 2\mathbf{\Omega} \times \mathbf{u}_1 + \nabla p' = -\frac{1}{2h_1}\nabla(h_1^2\tau), \tag{54}$$

$$(\partial_t + \mathbf{u}_2 \cdot \nabla)\,\mathbf{u}_2 - 2\mathbf{\Omega} \times \mathbf{u}_2 + \nabla p' = 0, \tag{55}$$

$$\partial_t h_1 + \nabla \cdot (h_1 \mathbf{u}_1) = 0, \tag{56}$$

$$\partial_t h_2 + \nabla \cdot (h_2 \mathbf{u}_2) = 0, \tag{57}$$

$$\partial_t \tau + \mathbf{u}_1 \cdot \nabla \tau = 0, \tag{58}$$

where subscripts "1, 2" show layer numbers. We note that the reduced gravity $\tau = \theta_1 - \theta_2 = g\varrho'/\varrho$ can take any sign depending on the difference between positive-definite quantities θ_1 and θ_2.

Because of the condition $h_1 + h_2 = l$, it follows from Eqs. (57) that

$$\nabla \cdot (h_1 \mathbf{u}_1 + (l - h_1)\mathbf{u}_2) = 0. \tag{59}$$

Using (59) and combining (54) and (55), it is easy to find the expression for the pressure gradient

$$\Delta\left(lp' + \frac{1}{2}h_1^2\tau\right) = \nabla \cdot \nabla \cdot (h_1 \mathbf{u}_1 \mathbf{u}_1 + (l - h_1)\mathbf{u}_2 \mathbf{u}_2). \tag{60}$$

Let U be the scale of the velocity $\mathbf{u_1}$, L be the horizontal length scale, and $h_1 \ll l$, with $\varepsilon = h_1/l$ being a small parameter. Then, if $O(h_1\tau/U^2) = 1$, Eqs. (59) and (60) imply the estimations

$$\mathbf{u}_2 = O(\varepsilon U), \quad p' = O\left(\varepsilon U^2\right).$$

This result allows eliminating the pressure gradient $\nabla p'$ from Eq. (54). Thus, using the thin-layer approximation and omitting the layer subscript, we obtain, in the leading order, the closed system of equations (1)–(3).

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