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ELECTRIC FIELD ENHANCEMENT BETWEEN TWO PARALLEL CYLINDERS DUE TO PLASMONIC RESONANCE

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We theoretically examine the electric field enhancement in the narrow gap between two parallel cylinders due to the plasmonic resonance. The resonance condition and the field enhancement factor are found explicitly. It is shown that the resonance occurs at the frequencies lower than the plasma frequency. This effect results from the special geometry: the gap width between parallel cylinders is much smaller than their radii. It is also shown that the enhancement coefficient is much larger than the one for a single cylinder and is determined together with the resonance frequency by the system geometry.

1. INTRODUCTION

Electrodynamic properties of materials consisting of metal granules immersed into a dielectric medium attract great experimental interest. There are numerous experiments where electromagnetic waves propagate through such systems (we refer to monograph [1] for an introduction to the subject). In the case where the geometry of metal grains is such that there are narrow gaps between separate grains, the field enhancement effect is observed, which is as follows. The field value in the narrow gaps is much larger than the incident wave field and exhibits peak values at particular frequencies of the incident wave. In a disordered metal–dielectric composite, a number of sharp peaks are observed in the spatial distribution of the electric field when the system is exposed to an external electromagnetic wave [2–4].

There is an extensive literature on the problem of two metallic or dielectric spheres in an electric field. The problem of two metallic spheres in a dielectric medium was considered in Ref. [5]. The same problem for two dielectric spheres was studied in [6]. The problem of two cylinders in homogeneous field can be found in [7]. All those works, however, do not consider the effect of plasmonic resonance. It was considered for two remote metallic spheres in [8] using the perturbation theory, which is not appropriate for close spheres or cylinders. The problem of two close metallic spheres exposed to an electromagnetic wave was investigated in [9–11], where a system of recurrent relations was solved numerically.

In this paper, we theoretically examine the effect of plasmonic resonance for the field between two parallel cylinders. We find explicit expressions for the field, the resonance conditions, and the field enhancement coefficients. We also propose the general method of investigating such effects in more complicated 2D-geometry systems: cylinders of arbitrary cross sections. We supply our analytic solution with general physical considerations providing a qualitative explanation of the problem.

2. PROBLEM FORMULATION AND GENERAL CONSIDERATIONS

We consider the system of two parallel infinitely long metallic cylinders in a dielectric medium. We investigate the electric field distribution between and around these cylinders when a linearly polarized wave is incident on the system, with its electric field vector directed perpendicular to the cylinder axes and parallel to the line connecting the centers of their cross sections (Fig. 1). In this case, the problem is effectively two-dimensional: the field is the same in any plane perpendicular to the cylinders. We use the Cartesian coordinates with the z axis directed along the cylinders axes, the y axis directed along the line connecting the cylin-

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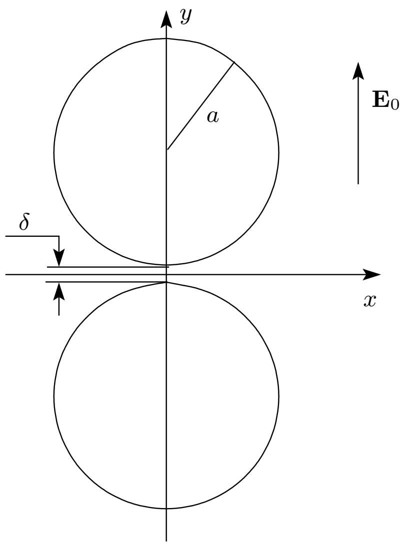

Fig. 1. Narrow gap between metallic cylinders

der cross-section centers, and the x line passes between the cylinders perpendicular to their axes (Fig. 1).

We assume the wavelength λ in the dielectric medium to be larger than the cylinder radius a, $\lambda \gg a$. We first consider the cylinders of the same radius, and let δ be the gap width. The wave field can be written as $\mathrm{Re}(\mathbf{E}_0 e^{-i\omega t})$. Due to the smallness of the cylinder radii compared to the wavelength, the electric field can be described in the quasistatic approximation; we therefore consider the problem of metallic cylinders in a homogeneous electric field. The electric field can be considered to be potential and can be described, disregarding the magnetic field, in terms of a scalar potential: $\mathbf{E} = -\nabla\phi$. Then the equation for the potential is $\nabla^2\phi = 0$.

To formulate the boundary conditions, let ε_d be the permittivity of the dielectric medium, assumed to be of the order of unity, and ε_m be the permittivity of the metal. We assume that the imaginary part of ε_d can be neglected and that the imaginary part of ε_m is small compared to its real part. The permittivities of both metal and dielectric are functions of frequency. We let $\varepsilon = \varepsilon_m/\varepsilon_d$ be the permittivity contrast. Then the boundary conditions are as follows: the potential must be continuous on the metal surface, which is equivalent to the condition that the tangential electric field be continuous and the normal derivatives of the potential differ by the factor ε:

$$\phi_{out} = \phi_{in}, \qquad (1)$$

$$\frac{\partial \phi_{out}}{\partial n} = \varepsilon \frac{\partial \phi_{in}}{\partial n}, \qquad (2)$$

where ϕ_{out} and ϕ_{in} are the respective potentials in the metal and the dielectric. Both conditions are to be imposed on the surface of cylinders. We therefore have the problem of finding the harmonic potential that satisfies conditions (1) and (2) on the cylinder surfaces.

To rigorously demonstrate the physical effect of the resonance, we first consider the field in the system with only one cylinder. The potential is then given by well-known formulas [12]

$$\phi_{out} = -\mathbf{E}_0 \cdot \boldsymbol{\rho} + \frac{\varepsilon - 1}{\varepsilon + 1} a^2 \frac{\mathbf{E}_0 \cdot \boldsymbol{\rho}}{\rho^2}, \qquad (3)$$

$$\phi_{in} = -\frac{2}{1+\varepsilon} \mathbf{E}_0 \cdot \boldsymbol{\rho}, \qquad (4)$$

where $\boldsymbol{\rho}$ is the two-dimensional radius vector. It follows from these formulas that the resonance value is $\varepsilon = -1$.

In the optical spectral region, the permittivity of a good metal can be approximated by the Drude–Lorentz formula

$$\varepsilon_m \sim -\left(\frac{\omega_p}{\omega}\right)^2 \left(1 - \frac{i}{\omega\tau}\right), \qquad (5)$$

where ω_p is the plasma frequency and τ is the electron relaxation time (we assume that $\omega\tau \gg 1$). Therefore, the resonance occurs at a frequency close to the plasma one [13, 14]. We also note that as can be seen from (3), the field is localized inside the cylinder and around it, gradually decaying with the distance. The enhancement coefficient is of the order of $1/\varepsilon''$ and is independent of the cylinder radius.

We now investigate the field in the system with two cylinders separated by a narrow gap, seeking resonances that occur at large negative values of the permittivity. We expect the enhanced field to be confined in the gap, which can be accounted for as follows. In seeking the resonance due to the gap, we have to find the standing waves that can exist in this gap. The gap approximately retains the constant width δ at the distances of the order of $\sqrt{a\delta}$ from its center. Hence, we are to find the standing waves in the flat gap of the width δ and length $\sqrt{a\delta}$ between two metals. It is known that the propagation constant β of an electromagnetic wave along a narrow gap is related to the metal permittivity as $\varepsilon = -\operatorname{cth}(\beta\delta/2)$ [15]. The standing wave condition (taking into account that $|\varepsilon| \gg 1$) can be written as $\beta\sqrt{a\delta} \approx \pi n$, where n is an integer. We thus arrive at the following estimate of the resonance permittivity:

$$\varepsilon_{res} \sim -\frac{1}{n}\sqrt{\frac{a}{\delta}}, \qquad (6)$$

which is large due to the condition $a \gg \delta$. We note that the resonance permittivity is determined by the gap geometry.

We now estimate the order of magnitude of the field enhancement coefficient. We assume that we are close to the resonance corresponding to $n = 1$ in (6), i.e., $\varepsilon \approx -\sqrt{a/\delta}$. We let E_c denote the field strength in the center of the gap and E_0 the external field. The field on the x axis is $E(x)$. The field inside the gap, i.e., at $x \ll \sqrt{a\delta}$, is approximately constant, equal to E_c. In the region $a \gg x \gg \sqrt{a\delta}$, the distance between the cylinders is of the order of x^2/a, and hence the potential difference between the cylinders at these distances is $(\Delta\phi)_{out} \sim E(x)x^2/a$. On the other hand, the potential change inside the metal can be estimated as $(\Delta\phi)_{in} \sim E(x)x/\varepsilon$. It follows that $(\Delta\phi)_{in}/(\Delta\phi)_{out} \sim \sqrt{a\delta}/x \ll 1$. We conclude that the potential difference between the cylinders is constant in the region $x \gg \sqrt{a\delta}$, i.e., $(\Delta\phi)_{out} = \text{const}$. We then explicitly write the field dependence on x, $E(x) \sim (\Delta\phi)_{out}a/x^2$. This potential difference between the cylinders gives rise to a dipole moment (per unit length) of the system, which can be estimated as

$$d \sim \int\limits_{\sqrt{a\delta}}^{a} dx\, E(x)\frac{x^2}{a} \sim (\Delta\phi)_{out}a. \tag{7}$$

The field far from the cylinders, $x \gg a$, is determined by this dipole and is $E(x) \sim (\Delta\phi)_{out}a/x^2$, which is the same as the field in the region $a \gg x \gg \sqrt{a\delta}$. Thus, there are only two asymptotic regions, $x \ll \sqrt{a\delta}$ and $x \gg \sqrt{a\delta}$, with the field values given by

$$E \sim E_c, \quad x \ll \sqrt{a\delta}, \tag{8}$$

$$E \sim (\Delta\phi)_{out}a/x^2, \quad x \gg \sqrt{a\delta}. \tag{9}$$

Relating these asymptotic formulas, we can estimate $(\Delta\phi)_{out} \sim E_c\delta$.

In order to estimate the relation between E_c and E_0, we have to calculate the energy dissipation in the system. Most dissipation occurs in the metal near the gap, where field penetrates the metal at a depth of the order of $\sqrt{a\delta}$. We write the dissipation rate per unit length according to the standard formula [16]

$$Q \sim \omega\varepsilon'' E_c^2(1/\varepsilon')^2 a\delta. \tag{10}$$

At the resonance permittivity $\varepsilon' \sim \sqrt{a/\delta}$, we find

$$Q \sim \omega\varepsilon'' E_c^2\delta^2. \tag{11}$$

This dissipation should be balanced by the work produced by the external field E_0 on the system. The power of this work is $P \sim \omega\, dE_0$. Comparing this with the dissipation rate, we find

$$\frac{E_c}{E_0} \sim \frac{1}{\varepsilon''}\frac{a}{\delta}. \tag{12}$$

Formulas (6) and (12) are the results of our estimations. They show that the resonance in our system occurs at larger negative values of permittivity, i.e., at lower frequencies than in the system of one cylinder. The enhancement coefficient is larger than in the case of a single cylinder and depends on the geometry of the system (cylinder radii and the gap width). It can be shown that in the case of two close cylinders of greatly different radii, the effect of the resonance is determined by the smallest radius (which determines the geometry of the gap).

We thus estimated the resonance conditions and the enhancement coefficient from the general physical reasoning. We emphasize that the resonance permittivity value is mostly determined by the narrow gap geometry (its width and length), while the enhancement coefficient is determined by the geometry of the whole system.

It is worth estimating the losses due to radiation. The radiation intensity per unit length can be written as [17]

$$I \sim \frac{\omega^3}{c^2}d^2 \sim \frac{\omega^3}{c^2}E_c^2\delta^2a^2, \tag{13}$$

where d is the dipole moment (per unit length). In our estimation of the field enhancement, we assumed these radiation losses to be small compared to Ohmic ones, given by formula (11). This leads to the condition

$$\varepsilon'' \gg (a/\lambda)^2. \tag{14}$$

3. ANALYTIC SOLUTION

We now turn to a rigorous solution of the problem of two cylinders in an external electric field. We let Φ_0 denote the potential of the external electric field of the strength E_0. Let Φ^{in} be the potential inside the cylinders and Φ^{out}_{ind} the induced potential due to the presence of the cylinders. The full potential outside the cylinders is $\Phi_0 + \Phi^{out}_{ind}$.

Because we consider the external wave polarized in the xy plane (see Fig. 1) perpendicular to the cylinders axis, the problem is essentially two-dimensional. To

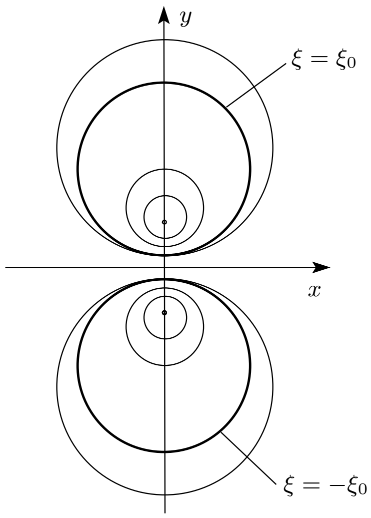

Fig. 2. Lines $\xi = \text{const}$ in bipolar coordinates

solve the Laplace equation, we use the so-called bipolar coordinate system in the xy plane [7]. We define dimensionless bipolar coordinates ξ and η as

$$x = \frac{C \sin\eta}{\operatorname{ch}\xi - \cos\eta}, \quad y = \frac{C \operatorname{sh}\xi}{\operatorname{ch}\xi - \cos\eta}, \tag{15}$$

$$-\infty < \xi < \infty, \quad 0 < \eta < 2\pi, \tag{16}$$

where C is the transformation constant. Coordinate lines $\xi = \text{const}$ are circles (Fig. 2):

$$x^2 + (y - C \operatorname{cth}\xi)^2 = (C/\operatorname{sh}\xi)^2. \tag{17}$$

Let ξ_0 and $-\xi_0$ correspond to the lines of cross sections of the metallic cylinders. We then find $C = a \operatorname{sh}(\xi_0/2)$. It can be verified that the condition $a \gg \delta$ corresponds to $\xi_0 \ll 1$, $\xi_0 \approx \sqrt{\delta/a}$.

The Laplace operator in this coordinate system is given by

$$\nabla^2 \equiv \frac{1}{h^2(\xi,\eta)} \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} \right), \tag{18}$$

where $h = C/(\operatorname{ch}\xi - \cos\eta)$ is the scaling function. We now can separate the variables in the Laplace equation. The eigenfunctions of operator (18) are $\exp(\pm n\xi)\cos n\eta$ or $\exp(\pm n\xi)\sin n\eta$. Using the symmetry of the external potential (symmetric with respect to the y axis and antisymmetric with respect to the x axis), we find the following expressions for potentials:

$$\Phi^{in} = E_0 a \sum A_n e^{-n\xi} \cos n\eta, \quad \xi > 0, \tag{19}$$

$$\Phi^{out}_{ind} = E_0 a \sum B_n \operatorname{sh} n\xi \cos n\eta, \tag{20}$$

where the unknown coefficients A_n and B_n are to be found from the boundary conditions.

We need to expand the external field potential $\Phi_0 \equiv -E_0 y$ into a series of Laplace eigenfunctions. Simple integration yields

$$\Phi_0 = -\operatorname{sign}\xi\, E_0 a \times \\ \times \operatorname{sh}\xi_0 \left(\sum_1^\infty 2e^{-n|\xi|} \cos n\eta + 1 \right). \tag{21}$$

The boundary conditions are

$$\Phi^{in}(\xi_0) = \Phi_0(\xi_0) + \Phi^{out}_{ind}(\xi_0), \tag{22}$$

$$\varepsilon \left.\frac{\partial\Phi^{in}}{\partial\xi}\right|_{\xi_0} = \left.\frac{\partial\Phi_0}{\partial\xi}\right|_{\xi_0} + \left.\frac{\partial\Phi^{out}_{ind}}{\partial\xi}\right|_{\xi_0}. \tag{23}$$

We can now find the coefficients A_n and B_n:

$$B_n = -\frac{2(1-\varepsilon)\exp(-n\xi_0)\operatorname{sh}\xi_0}{(\varepsilon + \operatorname{cth} n\xi_0)\operatorname{sh} n\xi_0}, \tag{24}$$

$$A_n = -\frac{2\exp(n\xi_0)\operatorname{sh}\xi_0}{(\varepsilon + \operatorname{cth} n\xi_0)\operatorname{sh} n\xi_0}. \tag{25}$$

Hence, the resonance conditions are

$$\varepsilon = -\operatorname{cth} n\xi_0. \tag{26}$$

This expression is formally exact because we did not use the smallness of the gap between the cylinders. For small gaps $\xi_0 \ll 1$, we find

$$\varepsilon \approx -1/n\xi_0 \tag{27}$$

for not very large n. Because $\xi_0 \approx \sqrt{\delta/a}$, we can write

$$\varepsilon_{res} \approx -\frac{1}{n}\sqrt{\frac{a}{\delta}}, \tag{28}$$

which is consistent with formula (6) found from general physical considerations.

We next investigate the structure of the electric field with the potential given by formulas (20) and (24). We assume that we are close to the resonance with $n = 1$, i.e., $\varepsilon' \approx -\sqrt{a/\delta}$. Then we can disregard all the terms in sum (20) except the first one. We also consider only the induced field. According to (20) and (24), the potential can be written as

$$\Phi^{out}_{ind} = -\frac{2E_0 a(1-\varepsilon)\exp(-\xi_0)}{\varepsilon + \operatorname{cth}\xi_0} \operatorname{sh}\xi \cos\eta. \tag{29}$$

Taking into account that $\xi_0 \ll 1$ and $|\varepsilon'| \approx \sqrt{a/\delta} \gg 1$, we can rewrite this expression as

$$\Phi_{ind}^{out} = -\frac{2E_0 a^{3/2}}{i\varepsilon'' \delta^{1/2}} \operatorname{sh}\xi \cos\eta. \tag{30}$$

To find the field, we now need to differentiate this potential with respect to the coordinates. We are mostly interested with the field on the line perpendicular to the cylinders axes and passing between them (the x axis in Cartesian coordinates, see Fig. 1). It is clear from the symmetry considerations (and also apparent from calculations) that there is only the y component of the field on this line. Therefore, we obtain

$$E_y = -\frac{\partial \Phi_{ind}^{out}}{\partial \xi}\frac{\partial \xi}{\partial y} - \frac{\partial \Phi_{ind}^{out}}{\partial \eta}\frac{\partial \eta}{\partial y}. \tag{31}$$

Calculating the derivatives, we find

$$E_y = \frac{2E_0}{i\varepsilon''}\frac{a}{\delta}\cos\eta(1-\cos\eta). \tag{32}$$

At the origin of Cartesian coordinates, i. e., in the center of the gap, $-\eta = \pi$. Hence,

$$E_c = -\frac{4E_0}{i\varepsilon''}\frac{a}{\delta}, \tag{33}$$

which agrees with estimate (12). If follows from (15) that $\eta \ll 1$ corresponds to $x \gg \sqrt{a\delta}$, and we can therefore write the field in this region as

$$E(x) = \frac{4E_0 a^2}{i\varepsilon''}\frac{1}{x^2}, \tag{34}$$

which corresponds to the field $4E_0 a^2/i\varepsilon''$ of the two-dimensional dipole and is also consistent with the qualitative estimaties.

Thus, the explicit analytic solution confirmed our qualitative estimaties based on the general physical principles. Here, we have investigated the field between and around the cylinders of the same radii with a narrow gap between them. However, using the above-discussed bipolar coordinate system also allows solving the problem with two cylinders of arbitrary radii and gap between them. These solutions are more cumbersome than the ones for cylinders of the same radii, and we do not present them here.

Finally, we remark on the purely static problem of two metallic cylinders (i.e., with $\varepsilon \to \infty$) in an external field. In this case, there is no field inside the cylinders, and their potentials are constant. It follows that the potential difference between the cylinders is $\Delta\Phi = 2E_0 a \operatorname{sh}\xi_0$. Because the gap width is δ, the order of magnitude of the field inside the gap is

$$E_c \sim E_0\sqrt{a/\delta}. \tag{35}$$

This increase in the field strength is due to a geometrical factor only and is not related to the plasmonic resonance. If the cylinder permittivity has a finite value, then this geometric effect is still present (although is weaker); it vanishes at $\varepsilon = 1$. Thus, we conclude that the enhancement due to plasmonic resonance (expression (33)) is larger than any possible geometric enhancement.

4. CONFORMAL TRANSFORMATIONS

The bipolar coordinates (ξ, η) can be obtained from the Cartesian coordinates (x, y) by the conformal transformation

$$\xi + i\eta = \ln\frac{x+iy-iC}{x+iy+iC}. \tag{36}$$

This is why the scaling functions of bipolar coordinates are equal and hence the Laplace operator has the simple form given by (18). The Laplace operator eigenfunctions contain $\cos\eta$ or $\sin\eta$ because the coordinate η has the period 2π due to the logarithm in (36). This simple form of the Laplace equation and consequently the simple eigenfunctions remain the same for any coordinates obtained from the Cartesian system by means of a conformal map if the η coordinate has the period 2π. This implies that the formal resonance condition (26) remains the same, although the constant ξ_0 is related to the characteristics of the systems differently. One can think of a transformation that produces cylinders of noncircular cross section (for example, of a prolate form) or a chain of cylinders.

5. CONCLUSIONS

We have investigated the plasmonic resonance in a narrow gap between two metallic cylinders embedded into a dielectric medium. We have estimated the results using the general physical reasoning, thus revealing the nature of the effects leading to field enhancement in such systems. We then solved the problem rigorously by using the so-called bipolar coordinates. The analytic solution yielded the resonance permittivity given by (28) and the field enhancement ratio given by (33), which confirmed our qualitative estimates based on general principles (formulas (6) and (12)). The main result of the study is that contrarily to the case of one cylinder, where the resonance position and the enhancement ratio are determined solely by the permittivity value and not by the cylinder radius, in the case of two cylinders, these characteristics of the resonance are dependent on the system geometry, i. e., the cylinder radii and the gap between the cylinders. We have shown that in the case of a narrow gap (with the gap width much smaller than the cylinders radius), it is possible to obtain the resonance at the large negative values of the permittivities (in contrast to $\varepsilon = -1$ for a single cylinder), which corresponds according to Drude representation (5) to frequencies smaller than the plasma one, mostly in the optical region. We also demonstrated that significant field enhancement can be obtained by adjusting the system geometry, i. e., by narrowing the gap between the cylinders.

Finally, we have proposed that our method can potentially be used to investigate more complex systems, i. e., cylinders of a noncircular cross section or a chain of cylinders. The chains of metal nanostructures have recently attracted considerable interest particularly because of their plasmonic modes [18].

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