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THE VELIKHOV AND ANTI-VELIKHOV EFFECTS IN THE THEORY OF MAGNETOROTATIONAL INSTABILITY

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A theory of magnetorotational instability (MRI) allowing an equilibrium plasma pressure gradient and nonaxisymmetry of perturbations is developed. This approach reveals that in addition to the Velikhov effect driving the MRI due to negative rotation frequency profile, $d\Omega^2/dr < 0$, there is an opposite effect (the anti-Velikhov effect) weakening this driving (here, Ω is the rotation frequency and r is the radial coordinate). It is shown that in addition to the Velikhov mechanism, two new mechanisms of MRI driving are possible, one of which is due to the pressure gradient squared and the other is due to the product of the pressure and density gradients. The analysis includes both the one-fluid magnetohydrodynamic plasma model and the kinetics allowing collisionless effects. In addition to the pure plasma containing ions and electrons, the dust plasma is considered. The charged dust effect on stability is analyzed using the approximation of immobile dust. In the presence of dust, a term with the electric field appears in the one-fluid equation of plasma motion. This electric field affects the equilibrium plasma rotation and also gives rise to a family of instabilities of the rotating plasma, called the dust-induced rotational instabilities.

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1. INTRODUCTION

Application of the magnetorotational instability (MRI) concept [1, 2] to the problem of accretion disks [3] became an important event in physics. It helped to resolve the long-standing puzzle of anomalous viscosity in the disks [4].

Paper [3] has stimulated numerous astrophysical investigations. The original studies in this astrophysical trend in the MRI theory are cited in [5]. The current status of the research and future perspectives in this field are summarized in review [6], where it was emphasized that one of the main topics here is rapid spontaneous spin-up of plasma with no apparent momentum input. In addition to the traditional areas of astrophysical applications such as star formation processes, mass transfer between binary stars, and active galactic nuclei, the solar dynamo [7, 8] was also mentioned [6] as a phenomenon where the angular momentum transport is an issue.

It is generally accepted that the MRI drive in a perfectly conducting magnetized medium [1–3] is the effect of differential rotation, i. e., the Velikhov effect. Following the approach of the local dispersion relation proposed in [3] and used in [5], one obtains the instability criterion of the axisymmetric perturbations in the form

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$$\frac{d\Omega^2}{d\ln r} + k^2 v_A^2 < 0. \tag{1.1}$$

It is assumed here that the equilibrium configuration is axisymmetric, i. e., independent of the azimuthal angle θ in the cylindrical coordinates (r, θ, z). The equilibrium magnetic field is along the axis z, and the medium is involved in azimuthal rotation with the angular frequency $\Omega = \Omega(r)$. Here, v_A is the Alfvén velocity and $k^2 = k_r^2 + k_z^2$, where k_r and k_z are the radial and longitudinal wave numbers of the perturbations. The term $d\Omega^2/d\ln r$ describes the mentioned Velikhov effect. Instability criterion (1.1) can be called the Velikhov–Balbus–Hawley (VHB) instability criterion. It can be satisfied only for a decreasing rotation frequency profile, $d\Omega^2/d\ln r < 0$.

One of the motivations of this paper was to elucidate whether the Velikhov effect is the only driving mechanism of MRI. As a tool, we use the magnetohydrodynamic (MHD) approach going back to the paper by Frieman and Rotenberg (FR) [9], which has been effectively applied and developed in [10–12].

Local instability criterion (1.1) was derived in [3] within the so-called local approach, assuming that the radial dependence of a perturbation $\widetilde{F}(r)$ can be expressed in the form $\bar{F}(r)\exp(ik_r r)$, where $\bar{F}(r)$ is the amplitude with a negligibly weak radial dependence.

The FR approach deals with two variables, one of which is the perturbed radial magnetic field $\widetilde{B}_r$ and the other is the sum of the perturbed pressures of the medium and the magnetic field, denoted as p_* and called the FR variable. The MHD approach leads to a pair of first-order differential equations for $\widetilde{B}_r$ and p_* [10–12], the Hameiri–Bondeson–Iacono–Bhattacharjee (HBIB) type equations. The equation for $\widetilde{B}_r$ contains the radial derivative of p_*, and therefore, when this variable is eliminated in order to obtain the mode equation, two contributions appear from the equation for p_*. One inolves the radial derivative of $\widetilde{B}_r$, which is canceled by a similar term in the HBIB equation for $\widetilde{B}_r$. The second is expressed in terms of the radial derivative of the medium equilibrium parameters. This part, in general, should be taken into account in the local dispersion relation. As a result, it combines a local part and a differential contribution. Such a dispersion relation can be called the canonical dispersion relation. The crucial question is whether the local dispersion relation obtained by means of the MHD approach coincides with that derived by the local approach used, in particular, in [3, 5]. The answer is that such a coincidence occurs only in the absence of the differential part in the FR local dispersion relation!

One can say that papers [1, 2] were aimed at applications to a laboratory medium. On the contrary, paper [3] with an analysis of accretion disks is clearly astrophysical. Then, based on Eq. (1.1), one can suggest that response of the laboratory and astrophysical media to the MRI is identical. But the physics of the equilibrium rotation in these cases is different in general. As a rule, it is assumed that the astrophysical rotation is caused by the gravitation force [3]. In contrast, this force plays no role in the laboratory plasma. If there is no equilibrium azimuthal magnetic field, the main reason for the plasma rotation is the positive plasma pressure gradient, $p_0' > 0$, where p_0 is the pressure and the prime is the radial derivative. Then we can refer to either the simplest astrophysical situation, where the rotation is stipulated solely due to the gravitation force, or the simplest laboratory case, where the only reason for plasma rotation is the positive plasma pressure gradient. For the "astrophysical plasma", we use the term "gravitation-dominated plasma". Also, in addition to "laboratory plasma", the term "gravitation-free plasma" is used in what follows. This implies that the case $g = 0$ is relevant to not only the laboratory devices but also some space configurations.

In Refs. [1–3], the perturbations were assumed to be incompressible, i. e., those with

$$\nabla \cdot \widetilde{\mathbf{V}} = 0, \tag{1.2}$$

where $\widetilde{\mathbf{V}}$ is the perturbed medium velocity. In the simplest astrophysical situation, the incompressibility assumption leads to the vanishing of the perturbed pressure $\widetilde{p}$,

$$\widetilde{p} = 0. \tag{1.3}$$

In contrast to this, in the laboratory case, the same assumption leads to a nonvanishing $\widetilde{p}$ defined, for axisymmetric perturbations [1, 2], by the equation

$$\frac{\partial \widetilde{p}}{\partial t} + \widetilde{V}_r p_0' = 0, \tag{1.4}$$

where $\widetilde{V}_r$ is the perturbed radial velocity.

Dealing with $p_0' \neq 0$, we must allow for the perturbed mass density $\widetilde{\rho}$, which, in the incompressible approximation, is determined by the continuity equation

$$\frac{\partial \widetilde{\rho}}{\partial t} + \widetilde{V}_r \rho_0' = 0, \tag{1.5}$$

where ρ_0 is the equilibrium mass density. Thus, for the simplest laboratory plasma, in contrast to the simplest astrophysical scenario, we should deal with effects of p_0' and, in general, the effects of ρ_0'.

Then a question arises: are the MRI developments in these two situations the same or different? The answer to this question is one of the goals of this paper. With the above remarks, we should expect that the scenarios must be different. We show this in formulas.

The MRI in the simplest astrophysical situation has already been studied in detail, including the analysis of the MRI dependence on β, the ratio of the plasma pressure to the magnetic field pressure (see, e. g., Refs. [5, 13, 14]). The recent theory of MRI in this case has also considered the kinetic effects [5, 15–19] including the effects of plasma pressure anisotropy [15, 18, 19]. Here, we restrict the study by the one-fluid plasma model only, assuming the plasma to be ideal.

Thus, there is a rather wide open area in the MRI theory: a branch allowing for the pressure gradient effects. The present paper is a step in its development.

Another insufficiently studied area of the MRI theory is that with nonaxisymmetric perturbations. Correct description of such perturbations within the MHD approach necessitates taking the differential part of the mode equation into account. Closing this gap in the MRI theory is another goal of this paper.

According to the plasma equilibrium condition, the rotation frequency in the simplest laboratory situation is determined solely by the pressure gradient, and hence $\Omega^2 = p_0'/r\rho_0$. In this case, the Velikhov effect is related to the pressure gradient, $d\Omega^2/d\ln r = d(p_0'/r\rho_0)/d\ln r$. In the MHD approach, the Velikhov effect is contained in the local part of the mode equation, but the term with $(p_0'/r\rho_0)'$, i. e., with $d\Omega^2/d\ln r$, also enters the differential part of this equation, although with the opposite sign. In contrast to the Velikhov effect, this term can be considered responsible for "the anti-Velikhov effect". This leads to an "annihilation" of the Velikhov and anti-Velikhov effects.

One of the most intriguing questions of our investigation is whether the Velikhov effect, predicted as the only driving mechanism of MRI for the astrophysical plasma, remains in force in the laboratory plasma. In other words, whether the MRI can be driven when the Velikhov and anti-Velikhov effects are completely "annihilated" such that $d\Omega^2/d\ln r = 0$. The answer is that there are two additional driving mechanisms in this case, one due to the squared pressure gradient effect and the other is the cross effect of the pressure and density gradients.

It turns out that the differential part of the mode equation is nonvanishing not only at $p_0' \neq 0$ but also at $p_0' = 0$ if the perturbations are nonaxisymmetric, $m \neq 0$, where m is the azimuthal mode number. Analysis of a locally nonaxisymmetric MRI based on the properly derived dispersion relation is also given here.

We have mentioned a variety of physically different configurations where a rotating plasma can be subjected to a rather wide family of specific MRIs. Therefore, it seems reasonable to develop a unified theory of MRI similar to the unified theory of instabilities of nonrotating plasma dealing with the beforehand-calculated plasma permittivity tensor and the standard general dispersion relation written in terms of this tensor, as described in our papers [20–24]. The parameters D, C_1, C_2, and C_3 in HBIB-type equations play the role similar to that of the permittivity tensor components. Therefore, we call them the canonical parameters of the MHD theory. Because an additional canonical parameter Λ appears in our theory, they can also be called the primary canonical parameters, and Λ the secondary canonical parameter. It was already mentioned that Λ contains the local and differential parts. These values denoted by a and b can be called the local and differential secondary canonical parameters. Using the canonical mode equation, we derive a local dispersion relation in terms of D, C_2, and Λ, which we call the canonical local dispersion relation.

Investigation of linear and nonlinear collective phenomena in a dusty plasma is a wide area of recent studies in space and laboratory plasma physics [25–35]. According to Refs. [36–38], this applies to fusion-oriented systems and, first of all, the tokamaks, among the laboratory devices dealing with the dusty plasma.

On the other hand, dusty plasmas are of long-lasting interest for astrophysics, in particular, for the physics of accretion and protoplanetary disks [39–43]. According to these studies, collective phenomena are important for such disks in relation to plasma turbulence in them. As one of the possible candidates for generating such a turbulence, the MRI was considered in Refs. [39–43]. The importance of MRI was demonstrated in other astrophysical problems as well as in various problems of applied physics. Recent plasma-physical investigations of MRI open new areas where this instability can be important.

A first step in the analysis of collective phenomena in a rotating dusty plasma was made in brief communication [44], where the linear problem of instability in such a plasma was studied assuming the dust grains heavy enough to be immobile. One of the goals of the present paper is to give a more detailed theory of this instability.

Section 2 contains a derivation of the canonical mode equation and the canonical local dispersion relation for the one-fluid plasma model. In Sec. 3, we use this dispersion relation to analyze the axisymmetric perturbations in this plasma model. Section 4 addresses the collisionless plasma, and Sec. 5 deals with the dusty plasma. Discussions are given in Sec. 6.

2. MRI THEORY IN THE STANDARD MHD PLASMA MODEL

2.1. Preliminaries

2.1.1. Basic equations

We start with the standard MHD plasma equation of motion

$$\rho\,\frac{d\mathbf{V}}{dt} = -\nabla p + \rho\mathbf{g} - \frac{1}{4\pi}\left\{\nabla\,\frac{\mathbf{B}^2}{2} - (\mathbf{B}\cdot\nabla)\mathbf{B}\right\}, \tag{2.1}$$

where $\mathbf{V}$ is the plasma velocity, $\mathbf{B}$ is the magnetic field, p is the plasma pressure, ρ is the plasma mass density, $\mathbf{g}$ is the gravity force, and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla. \tag{2.2}$$

We use the Ohm law in the form

$$\mathbf{E} + [\mathbf{V}\times\mathbf{B}]/c = 0, \tag{2.3}$$

where $\mathbf{E}$ is the electric field and c is the speed of light. Equation (2.3) leads to the standard freezing-in condition

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla\times[\mathbf{V}\times\mathbf{B}] = 0. \tag{2.4}$$

In addition, we use the Maxwell equation

$$\nabla\cdot\mathbf{B} = 0, \tag{2.5}$$

the plasma continuity equation

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{V} = 0, \tag{2.6}$$

and the adiabaticity condition

$$\frac{d}{dt}\left(\frac{p}{\rho^{\Gamma}}\right) = 0, \tag{2.7}$$

where Γ is the adiabatic exponent.

2.1.2. Equilibrium

We consider a cylindrical plasma rotating in the azimuthal direction θ with the angular frequency $\Omega = \Omega(r)$, where r is the radial coordinate. The equilibrium magnetic field $\mathbf{B}_0$ is assumed to be uniform and directed along the cylinder axis z, $\mathbf{B}_0 = (0, 0, B_0)$, and we assume the gravitational force $\mathbf{g}$ to have only the radial component, $\mathbf{g} = (g, 0, 0)$. In accordance with Ohm law (2.3), there is an equilibrium electric field $\mathbf{E}_0 = (E_0, 0, 0)$ related to the rotation frequency

$$\Omega = V_0/r \tag{2.8}$$

by

$$E_0 = -r\Omega B_0/c, \tag{2.9}$$

where $V_0 = V_0(r)$ is the azimuthal equilibrium plasma velocity.

With (2.8), it follows from the equilibrium part of the radial component of the plasma equation of motion (2.1) that

$$r\rho_0\Omega^2 = p_0' - \rho_0 g, \tag{2.10}$$

where ρ_0 and p_0 are the equilibrium plasma mass density and the equilibrium plasma pressure, respectively, and the prime is the radial derivative.

2.1.3. Linearization of basic equations

We linearize the basic equations assuming each perturbation to depend on t, θ, z as $\exp(-i\omega t + im\theta + ik_z z)$, where ω is the oscillation frequency, m is the azimuthal mode number, and k_z is the parallel projection of the wave vector. In addition to m, we introduce $k_y \equiv m/r$, the azimuthal projection of the wave vector, and in addition to ω, we use the Doppler-shifted oscillation frequency

$$\widetilde{\omega} = \omega - m\Omega. \tag{2.11}$$

The (r, θ, z) projections of the perturbed plasma velocity $\widetilde{\mathbf{V}}$ are $(\widetilde{V}_r,\ \widetilde{V}_\theta,\ \widetilde{V}_z)$. Similarly, the (r, θ, z) components of the perturbed magnetic field $\widetilde{\mathbf{B}}$ are $(\widetilde{B}_r,\ \widetilde{B}_\theta,\ \widetilde{B}_z)$. The perturbed plasma mass density is denoted by $\widetilde{\rho}$.

The (r, θ) projections of the freezing-in condition (2.4) yield

$$-i\widetilde{\omega}\widetilde{B}_r - ik_z B_0\widetilde{V}_r = 0, \tag{2.12}$$

$$-i\widetilde{\omega}\widetilde{B}_\theta - \frac{d\Omega}{d\ln r}\,\widetilde{B}_r - ik_z B_0\widetilde{V}_\theta = 0. \tag{2.13}$$

Maxwell equation (2.5) leads to the following relation between the components of the perturbed magnetic field:

$$ik_z\widetilde{B}_z + ik_y\widetilde{B}_\theta + \frac{1}{r}\,\frac{\partial}{\partial r}\,(r\widetilde{B}_r) = 0. \tag{2.14}$$

The equation of motion of perturbed plasma, Eq. (2.1), yields

$$-i\widetilde{\omega}\widetilde{V}_r - 2\Omega\widetilde{V}_\theta + \frac{1}{\rho_0}\frac{\partial\widetilde{p}}{\partial r} - \frac{iv_A^2 k_z}{B_0}\widetilde{B}_r + \frac{v_A^2}{B_0}\frac{\partial\widetilde{B}_z}{\partial r} - \frac{\widetilde{\rho}p_0'}{\rho_0^2} = 0, \quad (2.15)$$

$$-i\widetilde{\omega}\widetilde{V}_\theta + \frac{\kappa^2}{2\Omega}\widetilde{V}_r + \frac{ik_y\widetilde{p}}{\rho_0} - \frac{iv_A^2 k_z}{B_0}\widetilde{B}_\theta + \frac{iv_A^2}{B_0}k_y\widetilde{B}_z = 0, \quad (2.16)$$

$$-i\widetilde{\omega}\widetilde{V}_z = -ik_z\widetilde{p}/\rho_0, \quad (2.17)$$

where $v_A^2 = B_0^2/4\pi\rho_0$ is the squared Alfvén velocity, $\kappa^2 = (2\Omega/r)d(r^2\Omega)/dr$, and the gravity force is expressed in terms of Ω and p_0' by means of (2.10).

Using (2.14), we express $\widetilde{B}_z$ in terms of $\widetilde{B}_r$ and $\widetilde{B}_\theta$:

$$\widetilde{B}_z = -\frac{k_y}{k_z}\widetilde{B}_\theta + \frac{i}{k_z}\tau_B, \quad (2.18)$$

where

$$\tau_B = \frac{1}{r}\frac{\partial}{\partial r}(r\widetilde{B}_r). \quad (2.19)$$

Equations (2.12) and (2.13) allow expressing the perturbed velocities $\widetilde{V}_r$ and $\widetilde{V}_\theta$ in terms of $\widetilde{B}_r$ and $\widetilde{B}_\theta$:

$$\widetilde{V}_r = -\widetilde{\omega}\widetilde{B}_r/k_zB_0, \quad (2.20)$$

$$\widetilde{V}_\theta = -\frac{\widetilde{\omega}\widetilde{B}_\theta}{k_zB_0} + \frac{i\widetilde{B}_r}{k_zB_0}\frac{d\Omega}{d\ln r}. \quad (2.21)$$

2.2. The Frieman–Rotenberg approach

2.2.1. Basic equations of the FR approach

We introduce the FR variable

$$p_* = \widetilde{p} + \widetilde{B}_zB_0/4\pi. \quad (2.22)$$

Then Eqs. (2.15) and (2.16) become

$$i\left(D_0 - \frac{d\Omega^2}{d\ln r}\right)\widetilde{B}_r + 2\Omega\widetilde{\omega}\widetilde{B}_\theta + \frac{k_zB_0}{\rho_0}\frac{dp_*}{dr} - \frac{k_zB_0\widetilde{\rho}p_0'}{\rho_0^2} = 0, \quad (2.23)$$

$$iD_0\widetilde{B}_\theta - 2\Omega\widetilde{\omega}\widetilde{B}_r + \frac{ik_yk_zB_0}{\rho_0}p_* = 0, \quad (2.24)$$

where

$$D_0 = \alpha_A\widetilde{\omega}^2, \quad (2.25)$$

$$\alpha_A = 1 - k_z^2v_A^2/\widetilde{\omega}^2. \quad (2.26)$$

It follows from (2.6) that the perturbed density satisfies the equation

$$\frac{\widetilde{\rho}}{\rho_0} = \frac{1}{k_zB_0}\left[i\left(\tau_B + \frac{d\ln\rho_0}{dr}\widetilde{B}_r\right) - k_y\widetilde{B}_\theta\right] + \frac{k_z\widetilde{V}_z}{\widetilde{\omega}}. \quad (2.27)$$

In turn, according to (2.7), the perturbed plasma pressure $\widetilde{p}$ is given by

$$\widetilde{p} = \frac{ip_0'}{k_zB_0}\widetilde{B}_r + \frac{\rho_0c_s^2}{k_zB_0}\left(i\tau_B - k_y\widetilde{B}_\theta + \frac{k_z^2B_0}{\widetilde{\omega}}\widetilde{V}_z\right), \quad (2.28)$$

where $c_s^2 = \Gamma p_0/\rho_0$ is the sound velocity squared. Using (2.17), (2.27), and (2.28), we find expressions for $\widetilde{p}$ and $\widetilde{\rho}$ in terms of the perturbed magnetic field:

$$\widetilde{p} = \frac{1}{k_zB_0\alpha_s}\left[ip_0'\widetilde{B}_r + c_s^2\rho_0(i\tau_B - k_y\widetilde{B}_\theta)\right], \quad (2.29)$$

$$\frac{\widetilde{\rho}}{\rho_0} = \frac{1}{k_zB_0\alpha_s} \times \left\{i\left[\tau_B + \left(\alpha_s\frac{d\ln\rho_0}{dr} + \frac{k_z^2p_0'}{\rho_0\widetilde{\omega}^2}\right)\widetilde{B}_r\right] - k_y\widetilde{B}_\theta\right\}, \quad (2.30)$$

where

$$\alpha_s = 1 - k_z^2c_s^2/\widetilde{\omega}^2. \quad (2.31)$$

Substitution of (2.30) in (2.23) yields

$$i\lambda_r\widetilde{B}_r + \lambda_\theta\widetilde{B}_\theta - i\lambda_\tau\tau_B + k_zB_0p_*'/\rho_0 = 0, \quad (2.32)$$

where

$$\lambda_r = D_0 - \frac{d\Omega^2}{d\ln r} - \frac{p_0'}{r\rho_0}\frac{d\ln\rho_0}{d\ln r} - \frac{k_z^2}{\alpha_s\widetilde{\omega}^2}\frac{p_0'^2}{r\rho_0^2}, \quad (2.33)$$

$$\lambda_\theta = 2\Omega\widetilde{\omega} + k_yp_0'/\alpha_s\rho_0, \quad (2.34)$$

$$\lambda_\tau = p_0'/\alpha_s\rho_0. \quad (2.35)$$

2.2.2. Canonical equations in the FR approach (the HBIB-type equations)

It follows from (2.24) that

$$\widetilde{B}_\theta = -\frac{1}{D_0}\left(i2\Omega\widetilde{\omega}\widetilde{B}_r + \frac{k_y k_z B_0}{\rho_0}p_*\right). \quad (2.36)$$

Substitution of (2.36) in (2.18) and (2.29) leads to

$$\widetilde{B}_z = \frac{i}{k_z}\left(\tau_B + \frac{k_y}{D_0}2\Omega\widetilde{\omega}\widetilde{B}_r\right) + \frac{k_y^2}{\rho_0}B_0 p_*, \quad (2.37)$$

$$\widetilde{p} = \frac{1}{k_z B_0 \alpha_s}\left\{ic_s^2\rho_0\tau_B + i\left(p_0' + \frac{c_s^2\rho_0 k_y 2\Omega\widetilde{\omega}}{D_0}\right) + \frac{c_s^2}{D_0}k_y^2 k_z B_0 p_*\right\}. \quad (2.38)$$

Using (2.37) and (2.38), we represent (2.22) in the form

$$D\tau_B = C_1\widetilde{B}_r - i4\pi k_z B_0 C_2 p_*, \quad (2.39)$$

where

$$D = D_0(1+\beta/\alpha_s), \quad (2.40)$$

$$C_1 = -2k_y\Omega\widetilde{\omega}\left(1+\frac{\beta}{\alpha_s}\right) - \frac{D_0}{\alpha_s v_A^2}\frac{p_0'}{\rho_0}, \quad (2.41)$$

$$C_2 = \frac{1}{B_0^2}\left[D_0 - k_y^2 v_A^2\left(1+\frac{\beta}{\alpha_s}\right)\right]. \quad (2.42)$$

Substitution of (2.36) in (2.32) yields

$$-ip_*' = -\frac{k_y\lambda_\theta}{D_0}p_* + \frac{\rho_0\lambda_\tau}{k_z B_0}\tau_B - \frac{\rho_0}{k_z B_0}\left(\lambda_r - \frac{\lambda_\theta}{D_0}2\Omega\widetilde{\omega}\right)\widetilde{B}_r. \quad (2.43)$$

Using (2.39), we eliminate τ_B from (2.43). Then we arrive at

$$i4\pi k_z B_0 D p_*' = -4\pi k_z B_0 \bar{C}_1 p_* + C_3\widetilde{B}_r. \quad (2.44)$$

Here,

$$\bar{C}_1 = -\left[4\pi\rho_0\lambda_\tau C_2 + \lambda_\theta k_y\left(1+\frac{\beta}{\alpha_s}\right)\right], \quad (2.45)$$

$$C_3 = 4\pi\rho_0\left\{D\left(\lambda_r - \frac{\lambda_\theta}{D_0}2\Omega\widetilde{\omega}\right) - \lambda_\tau C_1\right\}. \quad (2.46)$$

By means of (2.41) and (2.42), Eqs. (2.45) and (2.46) reduce to

$$\bar{C}_1 = C_1, \quad (2.47)$$

$$C_3 = 4\pi\rho_0 D_0\left[\left(1+\frac{\beta}{\alpha_s}\right)\left(D_0 - \frac{d\Omega^2}{d\ln r} - \frac{p_0'}{r\rho_0}\frac{d\ln\rho_0}{d\ln r} - \frac{4\Omega^2\widetilde{\omega}^2}{D_0} - \frac{2m\Omega\widetilde{\omega}}{D_0}\frac{p_0'}{r\rho_0}\right) + \frac{D_0}{\alpha_s v_A^2\widetilde{\omega}^2}\frac{p_0'^2}{\rho_0^2}\right]. \quad (2.48)$$

Equations (2.39) and (2.44) can be called the canonical equations of the MRI theory. They were initially obtained in Refs. [11, 12]. Therefore, they can also be called the HBIB-type equations. The values D, C_1, C_2, and C_3 are the primary canonical parameters.

2.2.3. The canonical mode equation and canonical local dispersion relation

To eliminate the value p_* from our problem, we use (2.39) to find

$$4i\pi k_z B_0 p_* = (C_1\widetilde{B}_r - D\tau_B)/C_2. \quad (2.49)$$

Then (2.44) takes the form

$$D(D\tau_B/C_2)' + \Lambda\widetilde{B}_r = 0, \quad (2.50)$$

where

$$\Lambda = a + b, \quad (2.51)$$

$$a = C_3 - C_1^2/C_2, \quad (2.52)$$

$$b = -Dr(C_1/rC_2)'. \quad (2.53)$$

The quantity Λ is the secondary canonical parameter. We call the values a and b the local and differential canonical secondary parameters, respectively.

We take the function $\widetilde{B}_r$ in the form

$$\widetilde{B}_r = \bar{B}_r(r)\exp(ik_r r), \quad (2.54)$$

where $\bar{B}_r(r)$ is a slowly varying amplitude. Then (2.50) leads to

$$-k_r^2 D^2/C_2 + \Lambda = 0, \quad (2.55)$$

which is the canonical local dispersion relation.

3. AXISYMMETRIC PERTURBATIONS IN THE ONE-FLUID PLASMA MODEL

3.1. Reduction of the dispersion relation

Here, we take

$$k_y = 0, \quad (3.1)$$

which corresponds to axisymmetric perturbations. Then Eqs. (2.41)–(2.43) reduce to

$$C_1 = -p_0' D_0/\rho_0 v_A^2 \alpha_s, \tag{3.2}$$

$$C_2 = D_0/B_0^2, \tag{3.3}$$

$$C_3 = 4\pi\rho_0 D_0 \left[\left(1+\frac{\beta}{\alpha_s}\right) \left(D_0 - \frac{d\Omega^2}{d\ln r} - \frac{4\Omega^2\omega^2}{D_0} - \right.\right.$$
$$\left.\left. - \frac{p_0'}{r\rho_0}\frac{d\ln\rho_0}{d\ln r}\right) + \frac{p_0'^2 D_0}{\rho_0^2 v_A^2 \omega^2 \alpha_s}\right]. \tag{3.4}$$

It can be seen that $C_1 \neq 0$ for $p_0' \neq 0$. Turning to (2.52), we conclude that the differential part of the local dispersion relation does not vanish in this case. Therefore, for $p_0' \neq 0$, the correct local dispersion relation cannot be obtained by the standard local approach [3, 5].

3.2. The simplest astrophysical high-β plasma

The problem statement in [1–3] implies the incompressibility approximation $c_s^2 \to \infty$ and the assumption $p_0' = 0$. The condition $c_s^2 \to \infty$ is equivalent to $\beta \to \infty$. Then Eq. (3.3) for C_2 remains in force, while Eqs. (2.40), (3.2), and (3.4) reduce to

$$D = -(\omega^2 - k_z^2 v_A^2)^2/k_z^2 v_A^2, \tag{3.5}$$

$$C_1 = 0, \tag{3.6}$$

$$C_3 = -4\pi\rho_0(\omega^2 - k_z^2 v_A^2)^2 \times$$
$$\times \left(\omega^2 - k_z^2 v_A^2 - \frac{d\Omega^2}{d\ln r} - \frac{4\Omega^2\omega^2}{\omega^2 - k_z^2 v_A^2}\right). \tag{3.7}$$

In this case, in accordance with (2.52) and (2.53),

$$a = C_3, \tag{3.8}$$

$$b = 0. \tag{3.9}$$

It then follows from (2.51) that

$$\Lambda = C_3. \tag{3.10}$$

Substitution of (3.5), (3.6), and (3.8) in (2.55) gives the dispersion relation

$$\omega^2 - k_z^2 v_A^2 - \frac{k_z^2}{k^2}\left(\frac{d\Omega^2}{d\ln r} + \frac{4\Omega^2\omega^2}{\omega^2 - k_z^2 v_A^2}\right) = 0, \tag{3.11}$$

where $k^2 = k_z^2 + k_r^2$. This is the Balbus–Hawley (BH) dispersion relation [3]. It leads to the MRI criterion in (1.1).

Because we assume $p_0' = 0$ in this subsection, dispersion relation (3.11) describes the case where, according to equilibrium condition (2.10), plasma rotation is caused by the gravitational force only,

$$r\Omega^2 = -g. \tag{3.12}$$

This can be called the simplest astrophysical situation or the case of gravitation-dominated plasma.

The result of analysis of (3.11) is well known: this dispersion relation describes the MRI driving for $d\Omega^2/d\ln r < 0$, representing the Velikhov effect. Meanwhile, in accordance with (3.12),

$$\frac{d\Omega^2}{d\ln r} = -\frac{d(g/r)}{d\ln r}. \tag{3.13}$$

In this context, the MRI driving due to the Velikhov effect is revealed as a result of an unstable profile of gravitation force.

We recall that a dimensionless parameter Δ was introduced in [5] by

$$\Delta \equiv -\left(1 + \frac{1}{k^2 v_A^2}\frac{d\Omega^2}{d\ln r}\right). \tag{3.14}$$

Then the instability region following from (3.11) is given by (cf. (1.1))

$$\Delta > 0. \tag{3.15}$$

Near the instability boundary, it follows from (3.11) that $\omega^2 = -\gamma^2$, where

$$\gamma^2 = k_z^2 v_A^2 \Delta/\Delta_1. \tag{3.16}$$

Here, $\Delta_1 \equiv 1 + 4\Omega^2/k^2 v_A^2$ (cf. [5]).

3.3. High-β laboratory plasma

Now we assume the gravitational field to be negligible,

$$g = 0. \tag{3.17}$$

Then, according to equilibrium condition (2.10), the rotation frequency is defined by

$$r\rho_0\Omega^2 = p_0'. \tag{3.18}$$

Therefore, inducing rotation of the laboratory medium requires organizing a region where

$$p_0' > 0. \tag{3.19}$$

The incompressibility approximation $c_s^2 \to \infty$ considered in Sec. 3.2 corresponds to the case where the parameter α_s becomes infinite. We introduce the quasi-incompressible approximation assuming the parameter $k_z^2 c_s^2/\omega^2$ to be large but finite, $k_z^2 c_s^2/\omega^2 \gg 1$. Then we have

$$\alpha_s = -k_z^2 c_s^2/\omega^2. \tag{3.20}$$

As a result, instead of (3.11), we obtain the dispersion relation

$$\omega^4 - k^2 v_A^2 \omega^2 \Delta_1^L - k_z^2 k^2 v_A^4 \Delta^L = 0. \tag{3.21}$$

Here, similarly to (3.14),

$$\Delta^L = \left[1 + \frac{1}{k^2 v_A^2}\left(\frac{d\Omega^2}{d\ln r} - \frac{\Omega^4 r^2}{c_s^2} + \Omega^2 \frac{d\ln\rho_0}{d\ln r}\right)\right] \tag{3.22}$$

and

$$\Delta_1^L = 1 - \Delta^L + \frac{\Omega^2}{v_A^2 k^2} \times \left[4 + \frac{1}{\beta}\left(\frac{d\ln p_0}{d\ln r} - \frac{d\ln\rho_0}{d\ln r}\right)\right]. \tag{3.23}$$

The superscript "L" means "laboratory".

Taking $\omega = 0$, we obtain

$$\Delta^L = 0 \tag{3.24}$$

for the instability boundary (cf. (3.15)) or, in the explicit form,

$$\frac{d\Omega^2}{d\ln r} + k^2 v_A^2 - \frac{\Omega^4 r^2}{c_s^2} + \Omega^2 \frac{d\ln\rho_0}{d\ln r} = 0. \tag{3.25}$$

Comparing (3.25) with (1.1) shows that the MRI driving due to the Velikhov effect remains in force in our model. The stabilizing term $k^2 v_A^2$, describing the magnetoacoustic effect, is also revealed in our analysis. Meanwhile, we obtain one more driving mechanism related to the term with Ω^4 in (3.25). The term with Ω^2 in (3.25) describes an additional driving mechanism for a negative density gradient,

$$d\ln\rho_0/d\ln r < 0. \tag{3.26}$$

Otherwise, the density gradient effect is stabilizing. As a whole, the MRI occurs for

$$\frac{d\Omega^2}{d\ln r} + k^2 v_A^2 - \frac{\Omega^4 r^2}{c_s^2} + \Omega^2 \frac{d\ln\rho_0}{d\ln r} < 0. \tag{3.27}$$

Near the instability boundary, i. e., for small Δ^L, Eq. (3.21) yields

$$\gamma^2 = k_z^2 v_A^2 \Delta^L / \Delta_1^{L(0)}, \tag{3.28}$$

where $\Delta_1^{L(0)}$ is Δ_1^L for $\Delta^L = 0$, i. e.,

$$\Delta_1^{L(0)} = 1 + \frac{\Omega^2}{v_A^2 k^2} \times \left[4 + \frac{1}{\beta}\left(\frac{d\ln p_0}{d\ln r} - \frac{d\ln\rho_0}{d\ln r}\right)\right]. \tag{3.29}$$

Explicitly, Eq. (3.28) means

$$\gamma^2 = k_z^2 v_A^2 \left(\frac{\Omega^4 r^4}{c_s^2} - k^2 v_A^2 - \frac{d\Omega^2}{d\ln r} - \Omega^2 \frac{d\ln\rho_0}{d\ln p_0}\right) \times \left(k^2 v_A^2 + \frac{4}{\beta}\frac{d\ln T_0}{d\ln r}\right)^{-1}, \tag{3.30}$$

where $T_0 = p_0/\rho_0$ is the equilibrium plasma temperature. With (3.18), the MRI criterion in such a medium becomes

$$\frac{d\Omega^2}{d\ln r} + k^2 v_A^2 \left\{1 - \frac{\beta}{k^2 r^2 \Gamma^2}\frac{d\ln p_0}{d\ln r} \times \left[\frac{d\ln T_0}{d\ln r} - (\Gamma - 1)\frac{d\ln\rho_0}{d\ln r}\right]\right\} < 0. \tag{3.31}$$

This instability criterion replaces (1.1).

An important consequence of expression (3.31) is that the MRI in a laboratory medium can be driven even in the absence of the Velikhov effect, i. e., for

$$d\Omega^2/d\ln r = 0. \tag{3.32}$$

In this case, the MRI is possible if

$$\frac{d\ln T_0}{d\ln r} - (\Gamma - 1)\frac{d\ln\rho_0}{d\ln r} > 0. \tag{3.33}$$

The analysis in this subsection shows that behavior of the MRI in the simplest high-β laboratory plasma is essentially different from that in the simplest high-β astrophysical plasma.

4. COLLISIONLESS PLASMA

4.1. Basic equations

In the case of collisionless plasma, the equilibrium condition is (cf. (2.10))

$$-\rho_0 r\Omega^2 = -p'_{\perp 0} + g\rho_0, \tag{4.1}$$

where $p_{\perp 0}$ is the equilibrium perpendicular plasma pressure.

To describe the perturbed plasma dynamics, instead of (2.1), we start from the equation of motion in the form (see [19] and Sec. 19.1 in [45])

$$\left\{\rho\,\frac{d\mathbf{V}}{dt}\right\}^{\sim} = = \left\{-\nabla\cdot\mathbf{p}+\rho\mathbf{g}-\frac{1}{4\pi}\left[\frac{1}{2}\nabla B^2-(\mathbf{B}\cdot\nabla)\mathbf{B}\right]\right\}^{\sim}, \quad (4.2)$$

where $\mathbf{p}=\mathbf{p}_0+\widetilde{\mathbf{p}}$ is the total pressure tensor. According to [19, 45],

$$\nabla\cdot\mathbf{p} = \nabla p_\perp+\frac{p_\parallel-p_\perp}{B^2}\left\{\frac{1}{2}\nabla_\perp B^2+\left[[\nabla\times\mathbf{B}]\times\mathbf{B}\right]\right\} + \frac{\mathbf{B}}{B}(\mathbf{B}\cdot\nabla)\left[\frac{1}{B}(p_\parallel-p_\perp)\right], \quad (4.3)$$

where $p_\perp$ and $p_\parallel$ are the total perpendicular and parallel pressures given by $p_\perp=p_{\perp 0}+\widetilde{p}_\perp$ and $p_\parallel=p_{\parallel 0}+\widetilde{p}_\parallel$, $p_{\parallel 0}$ is the equilibrium parallel plasma pressure, and $\widetilde{p}_\perp$ and $\widetilde{p}_\parallel$ are the perturbed perpendicular and parallel plasma pressures.

The (r,θ,z) projections of Eq. (4.2) are (cf. Eqs. (2.15)–(2.17))

$$\rho_0\left(-i\widetilde{\omega}\widetilde{V}_r-2\Omega\widetilde{V}_\theta\right)-\widetilde{\rho}(\Omega^2 r+g) = -\frac{\partial}{\partial r}\widetilde{p}_\perp + \frac{B_0}{4\pi}\left[ik_z\left(1+\frac{\beta_\perp-\beta_\parallel}{2}\right)\widetilde{B}_r-\frac{\partial\widetilde{B}_z}{\partial r}\right], \quad (4.4)$$

$$\rho_0\left(-i\widetilde{\omega}\widetilde{V}_\theta+\frac{\kappa^2}{2\Omega}\widetilde{V}_r\right) = -ik_y\widetilde{p}_\perp + \frac{iB_0}{4\pi}\left[k_z\left(1+\frac{\beta_\perp-\beta_\parallel}{2}\right)\widetilde{B}_\theta-k_y\widetilde{B}_z\right], \quad (4.5)$$

$$-i\widetilde{\omega}\widetilde{V}_z = -ik_z\widetilde{p}_\parallel/\rho_0, \quad (4.6)$$

where $(\beta_\perp,\beta_\parallel)=8\pi(p_{\perp 0},p_{\parallel 0})/B_0^2$.

Using (4.4) and (4.5) and introducing the modified FR variable

$$p_* = \widetilde{p}_\perp + B_0\widetilde{B}_z/4\pi, \quad (4.7)$$

we arrive at (cf. (2.23), (2.24))

$$\frac{\rho_0}{k_zB_0}\left[i\left(D_0-\frac{d\Omega^2}{d\ln r}\right)\widetilde{B}_r+2\Omega\widetilde{\omega}\widetilde{B}_\theta\right]-p'_{\perp 0}\widetilde{\rho} = = -\frac{\partial p_*}{\partial r}, \quad (4.8)$$

$$\frac{\rho_0}{k_zB_0}\left(iD_0\widetilde{B}_\theta-2\Omega\widetilde{\omega}\widetilde{B}_r\right) = -ik_yp_*, \quad (4.9)$$

where D_0 is given by (2.25), and (cf. (2.27))

$$\alpha_A = 1-\frac{k_z^2v_A^2}{\widetilde{\omega}^2}\left(1+\frac{\beta_\perp-\beta_\parallel}{2}\right). \quad (4.10)$$

We calculate the perturbed pressures $\widetilde{p}_\perp$ and $\widetilde{p}_\parallel$ as (see Eqs. (16.43) and (16.46) in [45])

$$(\widetilde{p}_\perp,\widetilde{p}_\parallel) = M\int\left(\frac{v_\perp^2}{2},v_\parallel^2\right)\widetilde{f}\,d\mathbf{v}, \quad (4.11)$$

where M is the ion mass, $\widetilde{f}$ is the perturbed distribution function, $v_\perp$ and $v_\parallel$ are the perpendicular and parallel particle velocities, and $d\mathbf{v}$ is the volume element in the velocity space. According to Sec. 16.4 in [45] and the appendix in [23], the function $\widetilde{f}$ consists of two physically different parts. One is related to the spatial inhomogeneity of the equilibrium distribution function f_0, and the other is due to the plasma compressibility, with

$$\widetilde{f} = \widetilde{f}^{(1)}+\widetilde{f}^{(2)}, \quad (4.12)$$

where

$$\widetilde{f}^{(1)} = -Xf'_0, \quad (4.13)$$

$$\widetilde{f}^{(2)} = \frac{Mv_\perp^2}{2T_\perp}\left(1-\frac{T_\perp}{T_\parallel}+\frac{\widetilde{\omega}}{\widetilde{\omega}-k_zv_\parallel}\frac{T_\perp}{T_\parallel}\right)f_0\frac{\widetilde{B}_z}{B_0}, \quad (4.14)$$

with $T_\perp$ and $T_\parallel$ being the perpendicular and parallel temperatures. Equation (4.14) implies that the particle distribution is bi-Maxwellian.

As a result, Eqs. (4.11) yield the following expressions for the perturbed plasma pressures:

$$\widetilde{p}_\perp = \widetilde{p}_\perp^{(1)}+\widetilde{p}_\perp^{(2)}, \quad (4.15)$$

$$\widetilde{p}_\parallel = \widetilde{p}_\parallel^{(1)}+\widetilde{p}_\parallel^{(2)}. \quad (4.16)$$

Here, the superscripts "(1)" and "(2)" denote the "MHD" and "kinetic" parts of the these perturbed functions, which are respectively given by

$$\left\{\widetilde{p}_\perp^{(1)},\widetilde{p}_\parallel^{(1)}\right\} = i\{p'_{\perp 0},p'_{\parallel 0}\}\widetilde{B}_r/(k_zB_0), \quad (4.17)$$

$$\left\{\widetilde{p}_\perp^{(2)},\widetilde{p}_\parallel^{(2)}\right\} = \left\{2p_{\perp 0}c_\perp,p_{\parallel 0}c_\parallel\right\}\widetilde{B}_z/B_0. \quad (4.18)$$

The coefficients $c_\perp$ and $c_\parallel$ are

$$c_\perp = 1-\frac{T_\perp}{T_\parallel}\left[1+\frac{i\sqrt{\pi}\,\widetilde{\omega}}{|k_z|v_{T\parallel}}W\left(\frac{\widetilde{\omega}}{|k_z|v_{T\parallel}}\right)\right], \quad (4.19)$$

$$c_\parallel = 1-\frac{T_\perp}{T_\parallel}\left\{1+\frac{2\widetilde{\omega}^2}{k_z^2v_{T\parallel}^2}\times\left[1+\frac{i\sqrt{\pi}\,\widetilde{\omega}}{|k_z|v_{T\parallel}}W\left(\frac{\widetilde{\omega}}{|k_z|v_{T\parallel}}\right)\right]\right\}, \quad (4.20)$$

where $v_{T\parallel} = \sqrt{2T_\parallel/M}$ is the ion parallel thermal velocity and $W(x)$ is the plasma dispersion function defined by [20–23]

$$W(x) = \exp(-x^2)\left(1 + \frac{i}{\sqrt{\pi}}\int\limits_0^x \exp(t^2)\,dt\right). \quad (4.21)$$

The function $W(x)$ has the asymptotic forms [21]

$$W(x) = \begin{cases} \dfrac{i}{\sqrt{\pi}\,x}, & x \gg 1, \\ 1, & x \ll 1. \end{cases} \quad (4.22)$$

Accordingly, Eqs. (4.19) and (4.20) imply

$$c_\perp = \begin{cases} 1, & \widetilde{\omega} \gg |k_z| v_{T\parallel}, \\ 1 - \dfrac{T_\perp}{T_\parallel}\left(1 + \dfrac{i\sqrt{\pi}\,\widetilde{\omega}}{|k_z| v_{T\parallel}}\right), & \widetilde{\omega} \ll |k_z| v_{T\parallel}, \end{cases} \quad (4.23)$$

$$c_\parallel = \begin{cases} 1, & \widetilde{\omega} \gg |k_z| v_{T\parallel}, \\ 1 - \dfrac{T_\perp}{T_\parallel}\left(1 + \dfrac{2i\widetilde{\omega}^3}{|k_z|^3 v_{T\parallel}^3}\right), & \widetilde{\omega} \ll |k_z| v_{T\parallel}. \end{cases} \quad (4.24)$$

Evidently, the expressions for the kinetic parts of the perturbed pressures can be extended to more general particle distributions than the bi-Maxwellian one. The expression for $\widetilde{f}^{(2)}$ in terms of f_0 in [23] can then be used.

4.2. The canonical mode equation and canonical local dispersion relation

Following the procedure explained in Sec. 2, we arrive at Eq. (2.39) with

$$D = (1 + \beta_\perp c_\perp) D_0, \quad (4.25)$$

$$C_1 = -\left[2\Omega\widetilde{\omega} k_y (1 + \beta_\perp c_\perp) + \frac{4\pi p'_{\perp 0}}{B_0^2} D_0\right], \quad (4.26)$$

$$C_2 = \left[D_0 - k_y^2 v_A^2 (1 + \beta_\perp c_\perp)\right] / B_0^2. \quad (4.27)$$

In addition, we have Eq. (2.44) with

$$\bar{C}_1 = -\frac{4\pi p'_{\perp 0} D_0}{B_0^2}\left(1 + \frac{\beta_\parallel c_\parallel}{2}\,\frac{k_z^2 v_A^2}{\widetilde{\omega}^2}\right) - 2k_y \Omega\widetilde{\omega}(1 + \beta_\perp c_\perp), \quad (4.28)$$

$$C_3 = 4\pi\rho_0 \left\{D_0(1 + \beta_\perp c_\perp)\left(D_0 - \frac{d\Omega^2}{d\ln r} - \frac{4\Omega^2\widetilde{\omega}^2}{D_0} - \frac{p'_{\perp 0} c_\rho}{\rho_0 r}\right) + \frac{p'_{\perp 0}}{\rho_0}\left(1 + \frac{\beta_\parallel c_\parallel}{2}\,\frac{k_z^2 v_A^2}{\widetilde{\omega}^2}\right) \times \left[2\Omega\widetilde{\omega} k_y (1 + \beta_\perp c_\perp) + \frac{4\pi p'_{\perp 0} D_0}{B_0^2}\right]\right\}, \quad (4.29)$$

where

$$c_\rho = \frac{d\ln\rho_0}{d\ln r} + \frac{k_z^2 r p'_{\parallel 0}}{\widetilde{\omega}^2 \rho_0} + \frac{2m\Omega\widetilde{\omega}}{D_0}\left(1 + \frac{\beta_\parallel c_\parallel}{2}\,\frac{k_z^2 v_A^2}{\widetilde{\omega}^2}\right). \quad (4.30)$$

We then obtain the mode equation (cf. (2.50))

$$D\left[\left(\frac{D\tau_B}{C_2}\right)' + \frac{\delta C_1}{C_2}\tau_B\right] + \Lambda\widetilde{B}_r = 0, \quad (4.31)$$

where

$$\delta C_1 = \bar{C}_1 - C_1 = -p'_{\perp 0} k_z^2 D_0 \beta_\parallel c_\parallel / 2\rho_0\widetilde{\omega}^2, \quad (4.32)$$

Λ is given by (2.51),

$$a = C_3 - C_1\bar{C}_1/C_2, \quad (4.33)$$

and b is of form (2.53). The local mode equation for the collisionless plasma is given by (2.55).

To some extent, the coefficient Λ plays the role of the potential energy of perturbations. Its form is essential for the problem considered because Λ includes all driving mechanisms of MRI. Turning to (4.33) and (4.29), we can see that a, the local part of Λ, contains the differential term $d\Omega^2/d\ln r$. This term is responsible for the Velikhov effect. Meanwhile, the differential part of Λ, i. e., b, can also include the term with $d\Omega^2/d\ln r$. This term in b describes the anti-Velikhov effect.

One more driving mechanism is related to the term involving c_ρ in (4.29). It describes the cross effect of plasma pressure and density gradients. This effect is also of differential nature. Therefore, as the Velikhov effect, it can also be compensated by a respective term in the coefficient b. An important driving mechanism is described by the term with $p'^2_{\perp 0}$ in (4.29), which is the effect of the squared plasma pressure gradient, see Sec. 3.

4.3. Axisymmetric modes in the simplest astrophysical plasma model

Setting $p'_{\perp 0} = 0$ and $k_y = 0$, we have from (4.26) and (4.28) that

$$C_1 = \bar{C}_1 = 0. \quad (4.34)$$

Then, according to (3.53), $b = 0$ (cf. (3.9)) while, according to (4.33), $a = C_3$ (cf. (3.8)) and (2.51) reduces to $\Lambda = C_3$ (cf. (3.10)).

Turning to (4.29), we find that in the case considered,

$$C_3 = 4\pi\rho_0 D_0(1+\beta_\perp c_\perp) \times \times \left(D_0 - \frac{d\Omega^2}{d\ln r} - \frac{4\Omega^2\omega^2}{D_0}\right). \quad (4.35)$$

On the other hand, we have (3.3). As a result, dispersion relation (2.55) reduces to

$$D_0 - k_r^2 v_A^2(1+\beta_\perp c_\perp) - \frac{d\Omega^2}{d\ln r} - \frac{4\Omega^2\omega^2}{D_0} = 0. \quad (4.36)$$

By means of (2.25), (4.10), and (4.19) this dispersion relation can be represented in the form

$$Q_A(Q_M - d\Omega^2/d\ln r) - 4\Omega^2\omega^2 = 0, \quad (4.37)$$

where Q_A and Q_M are the Alfvén and magnetoacoustic parts of the dispersion relation given by

$$Q_A = \omega^2 - k_z^2 v_A^2\left[(1 + (\beta_\perp - \beta_\|)/2\right], \quad (4.38)$$

$$Q_M = \omega^2 - v_A^2\left\{k^2 + k_z^2\frac{\beta_\perp - \beta_\|}{2} + k_r^2\beta_\perp \times \times \left[1 - \frac{T_\perp}{T_\|}\left(1 + \frac{i\sqrt{\pi}\,\omega}{|k_z|v_{T\|}} W\left(\frac{\omega}{|k_z|v_{T\|}}\right)\right)\right]\right\}. \quad (4.39)$$

In the case of nonrotating plasma, Eq. (4.37) splits into two dispersion relations

$$Q_A = 0, \quad (4.40)$$

$$Q_M = 0. \quad (4.41)$$

These dispersion relations describe the Alfvén and magnetoacoustic oscillations branches.

We see that in the case of axisymmetric modes in astrophysical plasma, in the absence of pressure anisotropy, the only driving mechanism is the Velikhov effect described by the term with $d\Omega^2/d\ln r$ in (4.37). The analysis of (4.37) with the pressure anisotropy taken into account was performed in [19].

4.4. Axisymmetric modes in the simplest laboratory plasma model

In the case of laboratory plasma, we have $g = 0$. For $k_y = 0$, Eqs. (4.26) and (4.28) then yield

$$C_1 = -r\Omega^2 D_0/v_A^2, \quad (4.42)$$

$$\bar{C}_1 = -\frac{r\Omega^2 D_0}{v_A^2}\left(1 + \frac{\beta_\| c_\|}{2}\frac{k_z^2 v_A^2}{\omega^2}\right). \quad (4.43)$$

It follows from (4.30) that in the case considered,

$$c_\rho = \frac{d\ln\rho_0}{d\ln r} + \frac{k_z^2 r p'_{\|0}}{\omega^2\rho_0}. \quad (4.44)$$

Then Eq. (4.29) reduces to

$$C_3 = 4\pi\rho_0 D_0\left\{(1+\beta_\perp c_\perp)\left[D_0 - \frac{d\Omega^2}{d\ln r} - \frac{4\Omega^2\omega^2}{D_0} - \Omega^2\left(\frac{d\ln\rho_0}{d\ln r} + \frac{k_z^2 r p'_{\|0}}{\omega^2\rho_0}\right)\right] + \frac{r^2\Omega^4}{v_A^2}\left(1 + \frac{\beta_\| c_\|}{2}\frac{k_z^2 v_A^2}{\omega^2}\right)\right\}. \quad (4.45)$$

Using (4.45), (4.42), and (3.3), we transform (4.33) to

$$a = 4\pi\rho_0 D_0(1+\beta_\perp c_\perp)\left[D_0 - \frac{d\Omega^2}{d\ln r} - \frac{4\Omega^2\omega^2}{D_0} - \Omega^2\left(\frac{d\ln\rho_0}{d\ln r} + \frac{k_z^2 r p'_{\|0}}{\omega^2\rho_0}\right)\right]. \quad (4.46)$$

With (4.25), (3.3), and (4.42), Eq. (2.53) yields

$$b = 4\pi\rho_0 D_0(1+\beta_\perp c_\perp)\left(\frac{d\Omega^2}{d\ln r} + \Omega^2\frac{d\ln\rho_0}{d\ln r}\right). \quad (4.47)$$

Substituting (4.46) and (4.47) in (2.51), we have

$$\Lambda = 4\pi\rho_0 D_0(1+\beta_\perp c_\perp) \times \times \left(D_0 - \frac{4\Omega^2\omega^2}{D_0} - \Omega^2\frac{k_z^2 r p'_{\|0}}{\omega^2\rho_0}\right). \quad (4.48)$$

With (4.25), (3.3), and (4.48), dispersion relation (2.55) leads to

$$D_0 - k_r^2 v_A^2(1+\beta_\perp c_\perp) - \frac{4\Omega^2\omega^2}{D_0} - \frac{\Omega^2 k_z^2 r p'_{\|0}}{\omega^2\rho_0} = 0. \quad (4.49)$$

It can be seen from (4.49) that in contrast to the one-fluid approach, both the Velikhov effect and the effect of plasma density gradient are not involved in the axisymmetric MRI in the collisionless laboratory plasma. The reason for the difference in predictions of the one-fluid MHD and the kinetics is that the MHD implies an engagement between the perpendicular and parallel plasma motion. This engagement is described by the factor $\alpha_s \equiv 1 - k_z^2 c_s^2/\omega^2$ determined by Eq. (2.31). Therefore, if we formally take $\alpha_s \to 1$ in Eq. (2.51) for the parameter Λ, we arrive at the conclusion that both the Velikhov effect and the effect of plasma density gradient disappear.

Here the question arises: what is reason of the crucial difference between the astrophysical and laboratory situations, with the Velikhov effect presence in (4.36) and absence in (4.49)? Formally, this difference is explained by the fact that $C_1 = 0$ in astrophysics and, as a result, $b = 0$, while $C_1 \neq 0$ and $b \neq 0$ in laboratory conditions. Physically, this difference is a consequence of the fact that in contrast to the laboratory situations, the perturbed mass density plays no role in the astrophysical MRI. As a result, it does not lead to annihilation of the Velikhov effect related to the differential term in (2.52).

Similarly to (4.37), Eq. (4.49) can be represented in the form

$$Q_A\left(Q_M - \frac{k_z^2 p'_{\perp 0} p'_{\parallel 0}}{\omega^2 \rho_0^2}\right) - 4\Omega^2\omega^2 = 0, \tag{4.50}$$

where Q_A and Q_M are given by (4.38) and (4.39). The term involving $p'_{\perp 0} p'_{\parallel 0}$ in (4.50) describes the above-mentioned driving effect due to the squared plasma pressure gradient.

In the case of Maxwellian ions, Eq. (4.49) reduces to

$$(\omega^2 - k_z^2 v_A^2)\left\{\omega^2 - v_A^2\left[k^2 - k_r^2\beta\,\frac{i\sqrt{\pi}}{|k_z|v_T}\,\times\right.\right.$$
$$\left.\left.\times\, W\left(\frac{\omega}{|k_z|v_T}\right)\right] - \frac{{p_0'}^2 k_z^2}{\rho_0^2\omega^2}\right\} -$$
$$- 4\Omega^2\omega^2 = 0, \tag{4.51}$$

where $v_T = \sqrt{2T/M}$ and $T = T_\parallel = T_\perp$ is the equilibrium ion temperature. For $\omega \ll |k_z|v_T$, it hence follows that

$$1 - i\,\frac{\sqrt{\pi}\,k_r^2}{k^2}\,\beta^{1/2}\,\frac{\omega}{|k_z|v_A} +$$
$$+ \frac{k_z^2}{4k^2}\,\beta^2\,\frac{v_A^2}{r^2\omega^2}\left(\frac{d\ln p_0}{d\ln r}\right)^2 = 0. \tag{4.52}$$

For $\beta \gg 1$, Eq. (4.52) yields

$$\left(\frac{\omega}{|k_z|v_T}\right)^3 = -\frac{i}{\sqrt{2\pi}r^2k_r^2}\left(\frac{d\ln p_0}{d\ln r}\right)^2. \tag{4.53}$$

It can be seen that one of the roots of this equation describes unstable perturbations.

For $\beta \ll 1$, Eq. (4.52) becomes

$$\omega^2 = -\frac{k_z^2\beta}{4k^2}\,\frac{v_T^2}{r^2}\left(\frac{d\ln p_0}{d\ln r}\right)^2. \tag{4.54}$$

This dispersion relation describes an aperiodic instability with the growth rate

$$\gamma = \frac{|k_z|\beta^{1/2}}{2k}\,\frac{v_T}{r}\left|\frac{d\ln p_0}{d\ln r}\right|. \tag{4.55}$$

Thus, for arbitrary β, there is an instability with the growth rate independent of β for $\beta > 1$ and decreasing as $\beta^{1/2}$ for $\beta \ll 1$. It is the MRI in collisionless laboratory plasma. Recalling the discussion in [5], we note that the appearance of this MRI is governed by the collisionless gyrorelaxation effect; it can therefore be called the pressure-gradient-driven gyrorelaxation MRI in collisionless laboratory plasma.

5. DUSTY PLASMA

5.1. Basic equations and equilibrium

Turning to Refs. [46, 47], we see that one of the main plasmadynamic equations showing the presence of dust grains is the quasineutrality condition

$$-en_e + en_i - eZ_dn_d = 0, \tag{5.1}$$

where n_e, n_i, and n_d are the electron, ion and dust densities, e is the ion charge, $-eZ_d$ is the charge of the dust grain, and Z_d is the number of excessive ($Z_d > 0$) or deficient ($Z_d < 0$) electrons on the grain. The remaining plasmadynamic equations are taken in the approximation of immobile dust, implying that the mass of each grain is infinite and the dust velocity is zero.

The assumption of immobile dust is valid when the frequencies of interest exceed both the dust cyclotron frequency and the dust plasma frequency. These restrictions, together with knowledge of the dust composition (dust material), determine the corresponding size of grains.

We take the equations of motion of ions and electrons in the standard form

$$\rho_i\frac{d_i\mathbf{V}_i}{dt} = en_i\left(\mathbf{E} + \frac{1}{c}\mathbf{V}_i\times\mathbf{B}\right) - \nabla p_i + \rho_i\mathbf{g}, \tag{5.2}$$

$$0 = -en_e\left(\mathbf{E} + \frac{1}{c}\mathbf{V}_e\times\mathbf{B}\right) - \nabla p_e, \tag{5.3}$$

where $\mathbf{V}_i$ and $\mathbf{V}_e$ are the ion and electron velocities, ρ_i is the ion mass density, $d_i/dt = \partial/\partial t + \mathbf{V}_i\cdot\nabla$, and p_i and p_e are the ion and electron pressures. Adding (5.2) and (5.3), we obtain

$$\rho_i\frac{d_i\mathbf{V}_i}{dt} = e(n_i - n_e)\mathbf{E} +$$
$$+ \frac{e}{c}(n_i\mathbf{V}_i - n_e\mathbf{V}_e)\times\mathbf{B} - \nabla p + \rho_i\mathbf{g}, \tag{5.4}$$

where $p = p_i + p_e$ is the plasma pressure. Because the dust is assumed to be immobile, we have

$$e(n_i\mathbf{V}_i - n_e\mathbf{V}_e) = \mathbf{j}, \tag{5.5}$$

where $\mathbf{j}$ is the electric current density. Then, with (5.1), Eq. (5.4) reduces to

$$\rho\frac{d\mathbf{V}}{dt} = en_dZ_d\mathbf{E} + \frac{1}{c}[\mathbf{j}\times\mathbf{B}] - \nabla p + \rho\mathbf{g}. \tag{5.6}$$

Here, in correspondence with the one-fluid approach, we have changed the notation as $\rho_i \to \rho$, $\mathbf{V}_i \to \mathbf{V}$, $d_i/dt \to d/dt$. By means of the identity

$$\frac{1}{c}[\mathbf{j}\times\mathbf{B}] = -\frac{1}{4\pi}\left\{\nabla\frac{\mathbf{B}^2}{2} - (\mathbf{B}\cdot\nabla)\mathbf{B}\right\}, \tag{5.7}$$

Eq. (5.6) is transformed into (cf. (2.1))

$$\rho\frac{d\mathbf{V}}{dt} = en_dZ_d\mathbf{E} - \frac{1}{4\pi}\left\{\nabla\frac{\mathbf{B}^2}{2} - (\mathbf{B}\cdot\nabla)\mathbf{B}\right\} - \nabla p + \rho\mathbf{g}. \tag{5.8}$$

In the scope of the one-fluid approach, the equation of motion for electrons in (5.3) is transformed as follows. The electron velocity $\mathbf{V}_e$ is taken to be equal to the ion velocity $\mathbf{V}_i$,

$$\mathbf{V}_e = \mathbf{V}_i \equiv \mathbf{V}. \tag{5.9}$$

The term with the electron pressure gradient is neglected. Then Eq. (5.3) takes the form of Eq. (2.3). The Ohm law in form (2.10) leads to the standard freezing-in condition (2.4). In addition, we use Maxwell equation (2.5), plasma continuity equation (2.6), and adiabaticity condition (2.7). Then the presence of the dust is revealed in our model only through the term with the electric field in plasma equation of motion (5.8).

It follows from the equilibrium part of the radial projection of Eq. (5.8) that (cf. (2.10))

$$r\rho_0\Omega(\Omega - \Omega_d) = p_0' - \rho_0 g, \tag{5.10}$$

where Ω_d is the dust-induced effective rotation frequency defined by

$$\Omega_d = \omega_{Bi}Z_dn_d/n_0. \tag{5.11}$$

It seems to be important that, according to (5.10), a dusty plasma column rotates even in the absence of the gravity force and the plasma pressure gradient: at $g = 0$ and $p_0' = 0$, we have

$$\Omega = \Omega_d. \tag{5.12}$$

Another important particular case of the dusty plasma equilibrium is the gravitation-free plasma, $g = 0$, in the presence of a plasma pressure gradient $p_0' \neq 0$. Then Eq. (5.10) gives

$$r\rho_0\Omega(\Omega - \Omega_d) = p_0'. \tag{5.13}$$

The presence of dust allows the equilibrium of such plasma to occur not only for $p_0' > 0$ but also for negative plasma pressure gradient, $p_0' < 0$.

5.2. Derivation of the mode equation and local dispersion relation

By means of the Ohm law, we find the expressions for the perturbed electric fields:

$$\widetilde{E}_r = -(r\Omega\widetilde{B}_z + B_0\widetilde{V}_\theta)/c, \tag{5.14}$$

$$\widetilde{E}_\theta = B_0\widetilde{V}_r/c, \tag{5.15}$$

$$\widetilde{E}_z = r\Omega\widetilde{B}_r/c. \tag{5.16}$$

Then the perturbed plasma equation of motion in (5.8) yields (cf. (2.15)–(2.17))

$$-i\widetilde{\omega}\widetilde{V}_r - 2\Omega\widetilde{V}_\theta + \frac{1}{\rho_0}\frac{\partial\widetilde{p}}{\partial r} - \frac{iv_A^2k_z}{B_0}\widetilde{B}_r + \frac{v_A^2}{B_0}\frac{\partial\widetilde{B}_z}{\partial r} + \Omega_d\left(\widetilde{V}_\theta + r\Omega\frac{\widetilde{B}_z}{B_0}\right) - \frac{\widetilde{\rho}}{\rho_0}\left(\frac{p_0'}{\rho_0} + r\Omega\Omega_d\right) = 0, \tag{5.17}$$

$$-i\widetilde{\omega}\widetilde{V}_\theta + \frac{\kappa^2}{2\Omega}\widetilde{V}_r + \frac{ik_y\widetilde{p}}{\rho_0} - \frac{iv_A^2k_z}{B_0}\widetilde{B}_\theta + \frac{iv_A^2}{B_0}k_y\widetilde{B}_z - \Omega_d\widetilde{V}_r = 0, \tag{5.18}$$

$$-i\widetilde{\omega}\widetilde{V}_z = r\Omega_d\Omega\widetilde{B}_r/B_0 - ik_z\widetilde{p}/\rho_0. \tag{5.19}$$

We introduce the FR variable p_* defined by (2.22). Then Eqs. (5.17) and (5.18) become

$$i\left(D_0 - \frac{d\Omega^2}{d\ln r} + \Omega_d\frac{d\Omega}{d\ln r}\right)\widetilde{B}_r + \widetilde{\omega}(2\Omega - \Omega_d)\widetilde{B}_\theta + \frac{k_zB_0}{\rho_0}\frac{dp_*}{dr} + ir\Omega\Omega_d\tau_B - \frac{k_zrB_0\widetilde{\rho}}{\rho_0}\left(\Omega\Omega_d + \frac{p_0'}{\rho_0}\right) = 0, \tag{5.20}$$

$$iD_0\widetilde{B}_\theta - (2\Omega - \Omega_d)\widetilde{\omega}\widetilde{B}_r + \frac{ik_yk_zB_0}{\rho_0}p_* = 0, \tag{5.21}$$

where D_0 is given by (2.25) and (2.26).

The expressions for $\widetilde{p}$ and $\widetilde{\rho}$ in terms of the perturbed magnetic field are (cf. (2.29), (2.30))

$$\widetilde{p} = \frac{1}{k_z B_0 \alpha_s} \left[i \left(p_0' + \frac{k_z^2 c_s^2 \rho_0 r \Omega_d \Omega}{\widetilde{\omega}^2} \right) \widetilde{B}_r + \right. \\ \left. + c_s^2 \rho_0 \left(i\tau_B - k_y \widetilde{B}_\theta \right) \right], \quad (5.22)$$

$$\frac{\widetilde{\rho}}{\rho_0} = \frac{1}{k_z B_0 \alpha_s} \left\{ i \left[\tau_B + \left(\alpha_s \frac{d \ln \rho_0}{dr} + \frac{k_z^2 p_0'}{\rho_0 \widetilde{\omega}^2} + \right. \right. \right. \\ \left. \left. \left. + \frac{k_z^2 r \Omega \Omega_d}{\widetilde{\omega}^2} \right) \widetilde{B}_r \right] - k_y \widetilde{B}_\theta \right\}, \quad (5.23)$$

where α_s is given by (2.31). We see that $\widetilde{p}$ and $\widetilde{\rho}$ depend on the dust presence through their dependence on $\widetilde{V}_z$, which, in accordance with (5.19), depends on Ω_d.

Substitution of (5.23) in (5.20) yields (2.32) with

$$\lambda_r = D_0 - \frac{d\Omega^2}{d \ln r} + \Omega_d \frac{d\Omega}{d \ln r} - \left(\Omega \Omega_d + \frac{p_0'}{r \rho_0} \right) \frac{d \ln \rho_0}{d \ln r} - \\ - \frac{r k_z^2}{\alpha_s \widetilde{\omega}^2} \left(\Omega \Omega_d + \frac{p_0'}{r \rho_0} \right)^2, \quad (5.24)$$

$$\lambda_\theta = \widetilde{\omega}(2\Omega - \Omega_d) + \frac{m}{\alpha_s} \left(\Omega \Omega_d + \frac{p_0}{r \rho_0} \right), \quad (5.25)$$

$$\lambda_\tau = \frac{1}{\alpha_s} \left(\frac{p_0'}{\rho_0} + \frac{k_z^2 c_s^2}{\widetilde{\omega}^2} r \Omega \Omega_d \right). \quad (5.26)$$

We note that according to (5.26), in the case of pure plasma with $p_0' = 0$, the coefficient λ_τ vanishes. The presence of dust leads to a nonvanishing λ_τ even for $p_0' = 0$ if the plasma temperature is finite, $c_s^2 \neq 0$.

It follows from (5.21) that (cf. (2.36))

$$\widetilde{B}_\theta = -\frac{1}{D_0} \left[i \widetilde{\omega}(2\Omega - \Omega_d) \widetilde{B}_r + \frac{k_y k_z B_0}{\rho_0} p_* \right]. \quad (5.27)$$

Substitution of (5.27) in (2.18) and (5.22) leads to

$$\widetilde{B}_z = \frac{i}{k_z} \left[\tau_B + \frac{k_y}{D_0} \widetilde{\omega}(2\Omega - \Omega_d) \widetilde{B}_r \right] + \\ + \frac{k_y^2}{\rho_0} B_0 p_*, \quad (5.28)$$

$$\widetilde{p} = \frac{1}{k_z B_0 \alpha_s} \left\{ i c_s^2 \rho_0 \tau_B + i \left[p_0' + \frac{k_z^2 c_s^2 \rho_0 r \Omega \Omega_d}{\widetilde{\omega}^2} + \right. \right. \\ \left. \left. + \frac{c_s^2 \rho_0 k_y \widetilde{\omega}}{D_0} (2\Omega - \Omega_d) \right] + \frac{c_s^2}{D_0} k_y^2 k_z B_0 p_* \right\}. \quad (5.29)$$

With (5.28) and (5.29), Eq. (2.22) is represented in form (2.39) with D and C_2 given by (2.40), (2.42), and

$$C_1 = -k_y \widetilde{\omega}(2\Omega - \Omega_d) \left(1 + \frac{\beta}{\alpha_s} \right) - \\ - \frac{D_0}{\alpha_s v_A^2} \left(\frac{p_0'}{\rho_0} + \frac{c_s^2 k_z^2 r \Omega \Omega_d}{\widetilde{\omega}^2} \right). \quad (5.30)$$

It follows from (5.30) that, in accordance with (2.41), in the case of axisymmetric modes ($k_y = 0$) and the vanishing plasma pressure gradient ($p_0' = 0$), we have $C_1 = 0$ in the pure plasma. In contrast, in the dusty finite-temperature plasma, this is a finite quantity:

$$(C_1)_{k_y = 0,\, p_0' = 0} = -\frac{D_0 c_s^2 k_z^2 r \Omega \Omega_d}{\alpha_s v_A^2 \widetilde{\omega}^2}. \quad (5.31)$$

This is a consequence of a nonzero λ_τ (see the comment after (5.26)).

Substitution of (5.27) in (2.32) yields (cf. (2.43))

$$-ip_*' = -\frac{k_y \lambda_\theta}{D_0} p_* + \frac{\rho_0 \lambda_\tau}{k_z B_0} \tau_B - \\ - \frac{\rho_0}{k_z B_0} \left[\lambda_r - \frac{\lambda_\theta}{D_0} \widetilde{\omega} (2\Omega - \Omega_d) \right] \widetilde{B}_r. \quad (5.32)$$

Then we arrive at (2.44), with $\bar{C}_1 = C_1$ (cf. Eq. (2.47)) and

$$C_3 = 4\pi \rho_0 D_0 \left\{ \left(1 + \frac{\beta}{\alpha_s} \right) \left[D_0 - \frac{d\Omega^2}{d \ln r} + \Omega_d \frac{d\Omega}{d \ln r} - \right. \right. \\ - \left(\frac{p_0'}{r \rho_0} + \Omega \Omega_d \right) \frac{d\rho_0}{d \ln r} - \frac{(2\Omega - \Omega_d)^2}{D_0} \widetilde{\omega}^2 - \\ \left. - \frac{m \widetilde{\omega}}{D_0} (2\Omega - \Omega_d) \Omega \Omega_d \right] + \frac{r^2}{\alpha_s v_A^2 \widetilde{\omega}^2} \left[D_0 \frac{p_0'}{\rho_0^2} - 2\Omega \Omega_d \times \right. \\ \left. \left. \times \frac{p_0'}{r \rho_0} k_z^2 v_A^2 - k_z^2 v_A^2 (1 + \beta) \Omega^2 \Omega_d^2 \right] \right\}. \quad (5.33)$$

The canonical mode equation in the dusty plasma turns out to be the same as in the case of pure plasma (see (2.50)) with Λ given by (2.51)–(2.53). The same applies to the canonical local dispersion relation (see (2.55)).

5.3. Axisymmetric modes in a high-β dusty plasma with $p_0' = 0$

For $\beta \gg 1$, $k_y = 0$, $p_0' = 0$, and $\rho_0 = \text{const}$, we have the expressions for D and C_2 in (3.5) and (3.3), and Eq. (5.33) then yields

$$C_3 = -4\pi \rho_0 D_0 \left\{ \frac{D_0}{k_z^2 v_A^2} \left[D_0 - \frac{d\Omega^2}{d \ln r} + \right. \right. \\ \left. \left. + \Omega_d \frac{d\Omega}{d \ln r} - \frac{(2\Omega - \Omega_d)^2 \omega^2}{D_0} \right] + \frac{r^2}{v_A^2} \Omega^2 \Omega_d^2 \right\}, \quad (5.34)$$

while C_1 is given by (5.31). Then, according to (2.52),

$$a = -\frac{4\pi\rho_0}{k_z^2 v_A^2}(\omega^2 - k_z^2 v_A^2)^2 \left[\omega^2 - k_z^2 v_A^2 - \frac{d\Omega^2}{d\ln r} + \right.$$
$$\left. + \Omega_d \frac{d\Omega}{d\ln r} - \frac{(2\Omega - \Omega_d)^2\omega^2}{\omega^2 - k_z^2 v_A^2}\right]. \quad (5.35)$$

Next, in accordance with (2.53),

$$b = \frac{4\pi\rho_0}{k_z^2 v_A^2}(\omega^2 - k_z^2 v_A^2)^2 \frac{d(\Omega\Omega_d)}{d\ln r}. \quad (5.36)$$

It then follows from (2.51) that

$$\Lambda = -\frac{4\pi\rho_0}{k_z^2 v_A^2}(\omega^2 - k_z^2 v_A^2)^2 \left[\omega^2 - k_z^2 v_A^2 - \frac{d\Omega^2}{d\ln r} - \right.$$
$$\left. - \Omega \frac{d\Omega_d}{d\ln r} - \frac{(2\Omega - \Omega_d)^2\omega^2}{\omega^2 - k_z^2 v_A^2}\right]. \quad (5.37)$$

As a result, we obtain the local dispersion relation

$$\omega^2 - k_z^2 v_A^2 - \frac{k_z^2}{k^2}\left[\frac{d\Omega^2}{d\ln r} + \Omega \frac{d\Omega_d}{d\ln r} + \right.$$
$$\left. + \frac{(2\Omega - \Omega_d)^2\omega^2}{\omega^2 - k_z^2 v_A^2}\right] = 0. \quad (5.38)$$

Taking $\Omega_d = 0$ in (5.38) yields the standard MRI criterion, i. e., the VHB instability criterion given by (1.1). It follows from (5.38) that for $\Omega_d \neq 0$, the instability occurs for

$$\frac{d\Omega^2}{d\ln r} + \Omega \frac{d\Omega_d}{d\ln r} + k^2 v_A^2 < 0. \quad (5.39)$$

This is a generalization of the VHB instability criterion to the case of dusty plasma.

In the case of gravitation-free dusty plasma ($g = 0$ and $\Omega = \Omega_d$), Eq. (5.39) reduces to

$$\frac{3}{2}\frac{d\Omega_d^2}{d\ln r} + k^2 v_A^2 < 0. \quad (5.40)$$

This instability criterion looks as being independent of β, but, in accordance with the above-said, it is valid for high β only. It shows that for the instability to develop, the profile of $\Omega_d(r)$ must be decreasing, $d\Omega_d^2/d\ln r < 0$.

5.4. Axisymmetric modes in a low-β dusty plasma with $p_0' = 0$

For $k_y = 0$, $\beta = 0$, $p_0' = 0$, and $\rho_0 = \text{const}$, dispersion relation (2.55) reduces to [44]

$$(\omega^2 - k_z^2 v_A^2)\left(\omega^2 - k^2 v_A^2 - \frac{d\Omega^2}{d\ln r} + \Omega_d \frac{d\Omega}{d\ln r} - \right.$$
$$\left. - \frac{r^2 k_z^2 \Omega^2 \Omega_d^2}{\omega^2}\right) - \omega^2(2\Omega - \Omega_d)^2 = 0. \quad (5.41)$$

With (5.41), for $\Omega_d \neq 0$, it seems natural to introduce the parameter

$$\Delta_d \equiv -\left[1 + \frac{1}{k^2 v_A^2}\left(\frac{d\Omega^2}{d\ln r} - \Omega_d \frac{d\Omega}{d\ln r}\right)\right] \quad (5.42)$$

instead of Δ given by (3.14). Then (5.41) becomes

$$(\omega^2 - k_z^2 v_A^2)\left(\omega^2 + k^2 v_A^2 \Delta_d - \frac{r^2 k_z^2 \Omega^2 \Delta_d^2}{\omega^2}\right) -$$
$$- \omega^2(2\Omega - \Omega_d)^2 = 0. \quad (5.43)$$

In the approximation $(\Omega, \omega) \ll k_z v_A$ and $\Delta_d \ll k_z v_A$, Eq. (5.43) becomes quadratic in ω^2:

$$\omega^4 + \omega^2 k^2 v_A^2 \Delta_d - r^2 k_z^2 \Omega^2 \Omega_d^2 = 0. \quad (5.44)$$

It can be seen that this equation describes unstable perturbations at any sign of Δ_d. Their growth rate is given by

$$\gamma^2 \equiv -\omega^2 =$$
$$= \frac{1}{2}\left(k^2 v_A^2 \Delta_d + \sqrt{k^4 v_A^4 \Delta_d^2 + 4r^2 k_z^2 \Omega^2 \Omega_d^2}\right). \quad (5.45)$$

The case $\Delta_d < 0$ corresponds to the dust-induced rotational instability (DRI). For small Ω_d, it follows from (5.45) that the growth rate of this instability is determined by [44]

$$\gamma^2 = -r^2 k_z^2 \Omega^2 \Omega_d^2 / k^2 v_A^2 \Delta_d. \quad (5.46)$$

It seems important that the DRI is driven even for sufficiently large wave numbers, in particular, for $k^2 v_A^2 \gg |d\Omega^2/d\ln r|$.

In the absence of gravitation force, the rotation frequency Ω is equal to the effective dust-induced rotation frequency Ω_d (see (5.12)). Then dispersion relation (5.41) reduces to

$$(\omega^2 - k_z^2 v_A^2)\left(\omega^2 - k^2 v_A^2 - \frac{1}{2}\frac{d\Omega_d^2}{d\ln r} - \frac{r^2 k_z^2 \Omega_d^4}{\omega^2}\right) -$$
$$- \omega^2 \Omega_d^2 = 0. \quad (5.47)$$

For $\omega \ll (k_z v_A, k v_A)$, this gives (cf. (5.44))

$$\left(\frac{\omega}{k_z v_A}\right)^4 - \left(\frac{k^2 v_A^2}{\Omega_d^2} + \frac{1}{2}\frac{d\ln\Omega_d^2}{d\ln r}\right)\left(\frac{\omega}{k_z v_A}\right)^2 -$$
$$- \frac{r^2 \Omega_d^2}{v_A^2} = 0. \quad (5.48)$$

For a uniform dust-induced rotation frequency, $d\Omega_d/dr = 0$, this yields

$$\left(\frac{\omega}{k_z v_A}\right)^4 - \frac{\omega^2 k^2}{k_z^2 \Omega_d^2} - \frac{r^2 \Omega_d^2}{v_A^2} = 0. \quad (5.49)$$

It can be seen that the two roots of this dispersion relation are imaginary, and hence $\omega^2 = -\gamma^2$, where γ is the growth/decay rate. These roots are given by (cf. (5.45))

$$\frac{\gamma^2}{k_z^2} = \sqrt{r^2\Omega_d^2 v_A^2 + \left(\frac{k^2 v_A^4}{2\Omega_d^2}\right)^2} - \frac{k^2 v_A^4}{2\Omega_d^2}. \qquad (5.50)$$

They are relevant to the simplest case of the above-mentioned DRI. In the long-wavelength limit $kv_A \ll \Omega_d$, it hence follows that

$$\gamma^2 = k_z^2 r v_A |\Omega_d|. \qquad (5.51)$$

In the opposite short-wavelength limit $kv_A \gg \Omega_d$, Eq. (5.50) leads to (cf. (5.46))

$$\gamma^2 = k_z^2 r^2 \Omega_d^4 / v_A^2 k^2. \qquad (5.52)$$

Therefore, differential plasma rotation is not needed for appearance of the DRI.

6. DISCUSSION

The present paper shows that the MRI properties are in general different in astrophysical and laboratory cases because the respective equilibrium conditions are different. In the simplest astrophysical situation, the rotation is caused by the gravitation force, but a positive plasma pressure gradient is required for producing a similar laboratory equilibrium. Therefore, these two cases must be distinguished in the MRI theory.

We have derived the mode equation describing perturbations in a one-fluid rotating plasma cylinder immersed in a parallel uniform magnetic field (see Eq. (2.50)). This equation is expressed in terms of the primary canonical parameters D, C_1, C_2, and C_3 given by (2.40)–(2.42), and (2.48), and the secondary canonical parameters Λ, a, and b (see (2.51)–(2.53)). Using this equation, we obtained the canonical local dispersion relation (2.55).

The parameter a contains the term involving $d\Omega^2/dr$, which describes the Velikhov effect as well as the effects of the pressure and density gradients. The most important feature of the parameter b is that it involves the derivative $(p_0'/r\rho_0)'$, which describes the anti-Velikhov effect in laboratory plasma. The parameter b also contains the terms proportional to the pressure and density gradients, which contribute to the MRI drive, and the terms nonvanishing for $m \neq 0$, which are important for studying the nonaxisymmetric MRI in both astrophysical and laboratory conditions. For axisymmetric perturbations in the simplest astrophysical plasma, we obtain zero b (see (3.9)).

The axisymmetric perturbations in the simplest astrophysical and laboratory plasmas are described by the respective dispersion relations (3.11) and (3.21). They accordingly lead to instability boundaries given by (1.1) and (3.25). Equation (3.25) includes three mechanisms of the MRI drive. These mechanisms are shown explicitly in (3.31).

The axisymmetric MRI in the simple astrophysical plasma, in both high-β and low-β cases, can be characterized by the dimensionless parameter Δ introduced in [5] (see (3.14)). In contrast, the laboratory high-β plasma is described by the parameter Δ^L defined by (3.22).

We have shown that description of MRI within the collisionless plasma model requires calculation of the primary canonical parameters D, C_1, $\bar{C}_1$, C_2, and C_3. They are given by (4.25)–(4.29). With these parameters, we obtain canonical mode equation (4.31) with the secondary canonical parameter Λ expressed in terms of the above quantities. In turn, Λ is a sum of local and differential parts expressed in terms of the secondary canonical parameters a and b. The parameter a represents the main driving mechanism of MRI, the Velikhov effect, related to the derivative $d\Omega^2/d\ln r$. Besides, it includes two additional driving mechanisms, the perpendicular plasma pressure gradient squared and the product of the pressure and density gradients (see (3.15)). It turns out that the same effects but with the opposite sign are contained in the parameter b, which can be called "the anti-driving term". In particular, b includes the anti-Velikhov effect, which weakens or can completely suppress the $d\Omega^2/d\ln r$ drive.

Comparing the obtained kinetic canonical mode equation with the one-fluid equation presented in Sec. 2, we see that in contrast to the one-fluid model dealing with the canonical parameters D, C_1, C_2, and C_3, the kinetics contains the additional canonical parameter $\bar{C}_1 \neq C_1$.

The parameter Λ is crucial for the problem considered because it includes all the driving mechanisms of MRI. We have analyzed them for the axisymmetric modes in astrophysical and laboratory plasmas. It is shown that the axisymmetric MRI in the astrophysical plasma behaves the same as predicted by the electrodynamic approach [5]. Then, with the mechanisms due to the plasma inhomogeneity (i. e., without the plasma pressure anisotropy effects), the only reason for this instability can be the Velikhov effect [1, 3].

For laboratory plasma, the axisymmetric modes are described by dispersion relation (4.50). The Velikhov effect is absent in this dispersion relation. Instead, the effect of squared plasma pressure gradient is demonstrated.

According to our analysis, in the laboratory plasma with isotropic pressure, a pressure-gradient-driven MRI can occur. There are two varieties of this instability characterized by dispersion relations (4.53) and (4.55).

We have elaborated a mathematical technique to analyze the MRI in a dusty plasma in the approximation of immobile dust grains. A basis of our analysis is the plasma equation of motion in (5.8). This equation differs from the standard one describing the pure plasma by the presence of the electric field. This field modifies the equilibrium plasma rotation (see (5.10)) and influences the perturbations (see (5.17)–(5.19)). Both these effects lead to modification of the primary canonical parameters D, C_1, C_2, and C_3 (see (5.30) and (5.33)). We have derived the canonical mode equation in the presence of dust and the local dispersion relation.

We have restricted our analysis to only the local modes, assuming them to be axisymmetric. The nonlocal modes and the nonaxisymmetric variety of the local modes have been studied in Refs. [48] and [49], respectively.

As a whole, we have advanced the MRI theory towards more complete understanding of the relevant phenomena and indicated a new, modern trend in this theory. The present paper gives a rather broad basis for further theoretical study of MRI in astrophysical and laboratory plasmas.

In addition to the unified MHD theory of MRI and related instabilities in a rotating plasma, it seems interesting to elaborate the unified electromagnetic theory of such instabilities. This was the topic of Ref. [50].

According to [16], the MRI and related instabilities in the kinetic plasma model are important, in particular, for understanding the mechanisms of the radio and X-ray source Sagitarius A* in our Galaxy. The one-fluid and kinetic instabilities also seem relevant to laboratory experiments aimed at reproducing the astrophysical instabilities [6, 51].

In the present paper, both the one-fluid and kinetic regimes are considered for the magnetized plasma with the ion cyclotron frequency larger than the oscillation frequency and the plasma rotation frequency. A weak magnetization implies the so-called Hall regime, which was broadly analyzed in astrophysics [52–63] within both the MHD and electrodynamic approaches.

According to Ref. [64], the electron inertia should in some cases be allowed in astrophysics. The electrodynamic theory of axisymmetric modes with the electron inertia has been developed in Refs. [63, 65].

We have restricted ourselves to the linear approximation. It seems that inclusion of the three-wave interaction [66] and the nonlinear zonal flow generation [67] may be an important generalization of our theory.

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