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CONTRIBUTIONS TO THE THEORY OF MAGNETOROTATIONAL INSTABILITY AND WAVES IN A ROTATING PLASMA

A. B. Mikhailovskii [a*], ***J. G. Lominadze*** [b**], ***A. P. Churikov*** [c], ***V. S. Tsypin*** [d], ***N. N. Erokhin*** [a], ***N. S. Erokhin*** [e], ***S. V. Konovalov*** [a], ***E. A. Pashitskii*** [f,g], ***A. V. Stepanov*** [h], ***S. V. Vladimirov*** [i], ***R. M. O. Galvão*** [d,j]

[a] *Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute*
123182, Moscow, Russia

[b] *Georgian National Astrophysical Observatory*
0160, Tbilisi, Georgia

[c] *Syzran Branch of Samara Technical University*
446001, Syzran, Samara Region, Russia

[d] *Brazilian Center for Physics Research*
BR-22290180, Rio de Janeiro, Brazil

[e] *Space Research Institute, Russian Academy of Sciences*
117810, Moscow, Russia

[f] *Institute of Physics, National Academy of Sciences of Ukraine*
03028, Kiev, Ukraine

[g] *Taras Shevchenko Kiev National University*
03022, Kiev, Ukraine

[h] *Pulkovo Observatory, Russian Academy of Sciences*
196140, St. Petersburg, Russia

[i] *School of Physics, University of Sydney*
2006, Sydney, Australia

[j] *Institute of Physics, University of São Paulo, 05508-900, São Paulo, Brazil*

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One-fluid magnetohydrodynamic (MHD) theory of magnetorotational instability (MRI) in an ideal plasma is presented. The theory predicts a possibility of MRI for arbitrary β, where β is the ratio of the plasma pressure to the magnetic field pressure. The kinetic theory of MRI in a collisionless plasma is developed. It is demonstrated that as in the ideal MHD, MRI can occur in such a plasma for arbitrary β. The mechanism of MRI is discussed; it is shown that the instability appears because of a perturbed parallel electric field. The electrodynamic description of MRI is formulated under the assumption that the dispersion relation is expressed in terms of the permittivity tensor; general properties of this tensor are analyzed. It is shown to be separated into the nonrotational and rotational parts. Thereby, a first step for incorporation of MRI into the general theory of plasma instabilities is made. The rotation effects on Alfvén waves are considered.

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[*] E-mail: abmikh@list.ru
[**] E-mail: j.lominadze@astro-ge.org

1. INTRODUCTION AND OVERVIEW

In 1959, Velikhov [1] demonstrated that a nondissipative Couette flow (the flow of an ideally conducting liquid between rotating cylinders) can be destabilized by a vertical magnetic field. A particular case of the rotation frequency profile (the Velikhov profile, see the details below) was studied there. In 1960, the analysis in [1] was extended to a general rotation frequency profile in [2]. For a long time after that, this destabilization effect was almost not claimed. The situation changed radically in 1991, when paper [3] was published. It addressed the problem of an anomalous viscosity in accretion disks, going back to the study in [4]. (According to the definition in [5], "accretion" refers to the accumulation of matter onto a massive central body.) The authors of [3] suggested that explaining the anomalous viscosity in the disks should invoke an instability resulting in electromagnetic turbulence. The essence of paper [3] was the idea that the role of such an instability is played by the instability analyzed in [1, 2].

After [3], this instability was called the magnetorotational instability (MRI). It was a basis for numerous astrophysical studies forming the astrophysical trend in the theory of MRI. The original papers of this trend are [6–35]. The first stage of astrophysical applications of MRI was summarized in review [5]. At present, MRI is mentioned in more than a thousand papers; the overwhelming majority of these papers belongs to the astrophysical trend. In addition to astrophysics, MRI has been suggested to be significant for magnetic geodynamics [36]. There are also relatively narrow trends of experimental and theoretical studies of MRI in liquid metals [37–44] and plasmaphysical theoretical investigations of this instability (in particular, equilibria) [45–47].

There are many books and reviews summarizing advantages of the theory of plasma instabilities [48–59]. This theory contains rather broad information on several types of instabilities in a rotating plasma. The best known among them is the Rosenbluth–Simon instability [60]. At the same time, all these books and reviews contain no information on MRI. This shows that MRI is not yet incorporated into the general theory of plasma instabilities. One of the goals of the present paper is to provide such an incorporation.

The reasonable question is: why is MRI not yet in the general theory of plasma instabilities, although it was discovered almost 50 years ago? The answer is that papers [1, 2] treated it as a magnetohydrodynamic (MHD) but not a plasmaphysical phenomenon (e. g., the Couette flow is not a plasmaphysical notion). In addition, papers [1, 2] as well as [3] suggested that the considered liquid is incompressible. Meanwhile, incompressible perturbations are often identified in plasma theory as corresponding to high-β plasmas, where β is the ratio of the plasma pressure to the magnetic field pressure. Such a plasma is suggested to be the most interesting for astrophysics, while plasma physics typically deals with the opposite case of low-β plasmas. That MRI is actually possible for arbitrary β was pointed out in a relatively recent paper [61].

The fact that MRI, being one of a variety of electromagnetic instabilities, can develop in a low-β plasma is of principal interest for fundamental plasma physics. There is a broadly shared standpoint in plasma physics that only electrostatic instabilities are important for low values of β. This is the main reason why MRI is not included in the general theory of plasma instabilities.

But such a situation is unsatisfactory for a number of reasons. First, the plasma theory has elaborated very powerful methods for studying instabilities and related nonlinear processes. These techniques are not involved in the MRI theory. Second, plasma physics is a source of ideas for a series of applied branches of physics including magnetic nuclear fusion and space physics. Therefore, the absence of plasmaphysical information on the MRI theory is detremental to these areas of research. In addition, the discussed drawback deprives the plasma theory predictions of sufficient completeness.

In this paper, we collect the existing results on MRI necessary for further developing the theory of this instability and complement them by a number of new results. We then develop the electrodynamic theory of MRI, thereby incorporating this instability into the general theory of plasma instabilities.

In addition to the main MRI problem in a rotating plasma, there is a subsidiary problem to elucidate the rotation effects on oscillation branches of nonrotating plasma. We study this problem by considering the rotational effects on Alfvén waves. In other words, our paper not only addresses MRI processes but also provides more careful investigation of wave properties of a rotating plasma.

The general theory of plasma instabilities is separated into two main areas: the first can be formulated as the theory of microinstabilities and the second deals with stability problems of fusion-oriented magnetic confinement systems of the tokamak type. The approaches used by these branches are different. The approach of the theory of microinstabilities is based on investigation of the permittivity tensor, while the stability analysis in confinement systems is often performed by means of certain stability criteria, the simplest one being the Suydam stability criterion (see [54] for the details). The present paper is oriented towards further development of the theory of microinstabilities. Therefore, the notion of plasma permittivity is central in our investigation.

In accordance with the above, papers [1–3] studied MRI in the scope of the one-fluid MHD approach invoking the approximation of an incompressible medium, and therefore their results are valid only for high β. In Sec. 2, we follow the same approach but take the plasma compressibility into account. Thereby, we obtain results valid for arbitrary β.

In Sec. 3, we develop the kinetic theory of MRI relevant to collisionless plasmas. Section 4 addresses the explanation of the MRI mechanism. Section 5 is aimed at the development of the electrodynamic theory of MRI. This involves introducing and calculating the above permittivity tensor for the rotating plasma, which is the essence of Sec. 5. We explain the structure of this tensor and represent the corresponding dispersion relations allowing us to calculate the oscillation frequency. Section 6 is devoted to the analysis of rotation effects on Alfvén modes. Discussions are given in Sec. 7.

2. ONE-FLUID MHD THEORY OF MRI IN IDEAL PLASMA

2.1. Dispersion relation for MRI

2.1.1. The original one-fluid MHD equations

We consider an axisymmetric plasma cylinder placed in the magnetic field

$$\mathbf{B}_0 = (0, 0, B_0) \tag{2.1}$$

directed along its axis. We use the cylindrical coordinates (R, ϕ, z) and call ϕ the azimuthal coordinate. For simplicity, the field B_0 is assumed to be uniform, $dB_0/dR = 0$. We suppose that plasma rotates in the azimuthal direction, such that its equilibrium velocity $\mathbf{V}_0$ is given by

$$\mathbf{V}_0 = (0, V_0, 0)\,, \tag{2.2}$$

where $V_0 = R\Omega$ and $\Omega = \Omega(R)$ is the rotation frequency dependent on the radial coordinate.

To describe the perturbed plasma dynamics, we start from the equation of motion in the form

$$\rho\frac{d\mathbf{V}}{dt} = -\boldsymbol{\nabla}\left(p + \frac{\mathbf{B}^2}{8\pi}\right) + \frac{1}{4\pi}\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{B}, \tag{2.3}$$

where $d/dt = \partial/\partial t + \mathbf{V}\cdot\boldsymbol{\nabla}$, and $\rho = \rho_0 + \delta\rho$, $\mathbf{V} = \mathbf{V}_0 + \delta\mathbf{V}$, $p = p_0 + \delta p$, and $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$ are respectively the total mass density, velocity, pressure, and magnetic field, with δ denoting the perturbations. Thus, we deal with the perturbed mass density $\delta\rho$, the perturbed plasma pressure δp, the perturbed velocity $\delta\mathbf{V}$, and the perturbed magnetic field $\delta\mathbf{B}$. The vectors $\delta\mathbf{V}$ and $\delta\mathbf{B}$ are represented as

$$\delta\mathbf{V} = (\delta V_R, \delta V_\phi, \delta V_z)\,, \quad \delta\mathbf{B} = (\delta B_R, \delta B_\phi, \delta B_z)\,.$$

The function $\delta\rho$ is governed by the continuity equation

$$\frac{\partial}{\partial t}\delta\rho + \rho_0\boldsymbol{\nabla}\cdot\delta\mathbf{V} = 0. \tag{2.4}$$

We assume that the perturbations are independent of the azimuthal coordinate ϕ. The dependence of each perturbed value $\delta F(\mathbf{r}, t)$ can then be written as

$$\delta F = \delta F(R)\exp\left(-i\omega t + ik_R R + ik_z z\right), \tag{2.5}$$

where ω is the mode frequency and k_R and k_z are the perpendicular and parallel projections of the wave vector. The radial dependence of the functions $F(R)$ is assumed to be negligibly weak. Approximately, we then have

$$\boldsymbol{\nabla}\cdot\delta\mathbf{V} = ik_R\delta V_R + ik_z\delta V_z. \tag{2.6}$$

Using Eq. (2.5), we reduce Eq. (2.4) to

$$-i\omega\delta\rho + i\rho_0\left(k_R\delta V_R + k_z\delta V_z\right) = 0. \tag{2.7}$$

The perturbed plasma pressure δp is found by invoking the adiabatic condition,

$$\frac{d}{dt}\left(\frac{p}{\rho^\Gamma}\right) = 0, \tag{2.8}$$

where Γ is the adiabatic exponent. It follows from Eq. (2.8) that

$$\delta p = \frac{\Gamma p_0}{\omega}\left(k_R\delta V_R + k_z\delta V_z\right). \tag{2.9}$$

2.1.2. Derivation of the general dispersion relation

As a consequence of Eq. (2.3), the perturbed velocity $\delta\mathbf{V}$ is governed by the equation of motion with the components

$$-i\omega\delta V_R + \frac{ik_R c_s^2}{\omega}\left(k_R\delta V_R + k_z\delta V_z\right) - \\ - 2\Omega\delta V_\phi + \frac{iv_A^2}{B_0}\left(k_R\delta B_z - k_z\delta B_R\right) = 0, \tag{2.10}$$

$$-i\omega\delta V_\phi + \frac{\kappa^2}{2\Omega}\delta V_R - \frac{iv_A^2}{B_0}k_z\delta B_\phi = 0, \qquad (2.11)$$

$$-i\omega\delta V_z + \frac{ik_z c_s^2}{\omega}\left(k_R\delta V_R + k_z\delta V_z\right) = 0, \qquad (2.12)$$

where $v_A^2 = B_0^2/4\pi\rho_0$ is the squared Alfvén velocity and $c_s^2 = \Gamma p_0/\rho_0$ is the squared sound velocity. The parameter κ^2 is introduced by

$$\kappa^2 = \frac{2\Omega}{R}\frac{d\left(R^2\Omega\right)}{dR} \equiv 4\Omega^2 + \frac{d\Omega^2}{d\ln R}. \qquad (2.13)$$

To describe behavior of the perturbed magnetic field, we use the freezing condition

$$\partial\mathbf{B}/\partial t - \boldsymbol{\nabla}\times[\mathbf{V}\times\mathbf{B}] = 0. \qquad (2.14)$$

Then we find

$$-i\omega\delta B_R - ik_zB_0\delta V_R = 0, \qquad (2.15)$$

$$-i\omega\delta B_\phi - \frac{d\Omega}{d\ln R}\delta B_R - ik_zB_0\delta V_\phi = 0. \qquad (2.16)$$

Additionally using the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we arrive at

$$\delta B_z = -k_R\delta B_R/k_z. \qquad (2.17)$$

Turning to (2.12), we obtain

$$\delta V_z = \frac{k_zk_Rc_s^2}{\omega^2\alpha_S}\delta V_R, \qquad (2.18)$$

where

$$\alpha_S = 1 - k_z^2c_s^2/\omega^2. \qquad (2.19)$$

Using (2.11) and (2.15), we obtain

$$\delta V_\phi = \frac{i}{\omega\alpha_A^{ID}}\left(-\frac{\kappa^2}{2\Omega} + \frac{k_z^2v_A^2}{\omega^2}\frac{d\Omega}{d\ln R}\right)\delta V_R, \qquad (2.20)$$

where

$$\alpha_A^{ID} = 1 - k_z^2v_A^2/\omega^2. \qquad (2.21)$$

The superscript "ID" means "ideal" and the subscript "A" denotes the Alfvén oscillation branches.

Substituting (2.18) and (2.20) in (2.10) and using (2.15) and (2.16), we obtain the dispersion relation

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = 0, \qquad (2.22)$$

where

$$\alpha_{11} = \alpha_A^{ID}, \qquad (2.23)$$

$$\alpha_{12} = -\alpha_{21} = -2i\Omega/\omega, \qquad (2.24)$$

$$\alpha_{22} = \alpha_M^{ID} - \frac{1}{\omega^2}\frac{d\Omega^2}{d\ln R}, \qquad (2.25)$$

with

$$\alpha_M^{ID} = 1 - \frac{k^2v_A^2}{\omega^2} - \frac{k_R^2c_s^2}{\omega^2\alpha_S} \qquad (2.26)$$

and $k^2 = k_R^2 + k_z^2$; the subscript "M" means "magnetoacoustic". We also note that dispersion relation (2.22) can be represented as

$$\omega^4\alpha_A^{ID}\alpha_M^{ID} - \omega^2\kappa^2 + k_z^2v_A^2\frac{d\Omega^2}{d\ln R} = 0. \qquad (2.27)$$

Substituting (2.21) and (2.26) in (2.27) yields

$$\left(\omega^2 - k_z^2v_A^2\right)\left[\omega^4 - \omega^2k^2\left(v_A^2 + c_s^2\right) + k_z^2k^2c_s^2v_A^2\right] + \\ + \left(\omega^2 - k_z^2c_s^2\right)\left(-\omega^2\kappa^2 + k_z^2v_A^2\frac{d\Omega^2}{d\ln R}\right) = 0. \qquad (2.28)$$

We introduce the dimensionless parameters

$$\Delta = -\left(1 + \frac{1}{k^2v_A^2}\frac{d\Omega^2}{d\ln R}\right), \qquad (2.29)$$

$$\Delta_1 = 2\left[1 + \frac{1}{k^2v_A^2}\frac{\Omega}{R}\frac{d\left(R^2\Omega\right)}{dR}\right] \equiv 2 + \frac{\kappa^2}{k^2v_A^2}. \qquad (2.30)$$

Then Eq. (2.28) is represented as

$$\omega^6 - \omega^4k^2v_A^2\left(2\Delta_1 - \frac{k_R^2}{k^2} + \frac{c_s^2}{v_A^2}\right) + \\ + k_z^2k^2c_s^2v_A^2\left[\omega^2\left(\Delta_1 - \frac{v_A^2}{c_s^2}\Delta\right) + k_z^2v_A^2\Delta\right] = 0. \qquad (2.31)$$

In the particular case where $\beta\to\infty$ $\left(c_s^2\to\infty\right)$ with $\beta = 8\pi p_0/B_0^2$, Eq. (2.31) becomes

$$\omega^4 - k_z^2v_A^2\left(\omega^2\Delta_1 + k_z^2v_A^2\Delta\right) = 0. \qquad (2.32)$$

Similarly to [3], we introduce

$$\tilde\omega^2 \equiv \omega^2 - k_z^2v_A^2. \qquad (2.33)$$

In terms of $\tilde\omega$, Eq. (2.32) is written as

$$\tilde\omega^4 + \frac{k_z^2}{k^2}\left(k_z^2v_A^2\frac{d\Omega^2}{d\ln R} - \kappa^2\tilde\omega^2\right) = 0. \qquad (2.34)$$

This is the same as the Balbus–Hawley dispersion relation [3] for an incompressible medium. It is therefore reasonable to call Eq. (2.34) or Eq. (2.32) the Balbus–Hawley dispersion relation.

Dealing with an arbitrary compressibility, Kim and Ostriker [61] have derived the dispersion relation

$$\begin{aligned}\left(\omega^2 - k_z^2 v_A^2\right)\left[\omega^4 - \omega^2 k_z^2\left(v_A^2 + c_s^2\right) + k_z^4 c_s^2 v_A^2\right] = \\ = k_R^2(\omega^2 - k_z^2 v_A^2)\left[\left(v_A^2 + c_s^2\right)\omega^2 - k_z^2 c_s^2 v_A^2\right] + \\ + \left[4\Omega^2\omega^2 + (\omega^2 - k_z^2 v_A^2)\frac{d\Omega^2}{d\ln R}\right]\left(\omega^2 - k_z^2 c_s^2\right). \quad (2.35)\end{aligned}$$

It can be seen that Eqs. (2.35) and (2.28) are identical. Therefore, it is reasonable to call Eq. (2.35) or (2.28) the Kim–Ostriker dispersion relation.

2.2. Analysis of MRI

2.2.1. General instability criterion

We assyme that Δ is a small value. Then Eq. (2.31) reduces to

$$\Delta_1\omega^2 + k_z^2 v_A^2 \Delta = 0. \quad (2.36)$$

We assume that

$$\Delta_1 > 0. \quad (2.37)$$

It then follows that for

$$\Delta > 0, \quad (2.38)$$

Eq. (2.36) describes unstable perturbations,

$$\omega^2 < 0, \quad (2.39)$$

characterized by $\mathrm{Re}\,\omega = 0$ and $\mathrm{Im}\,\omega = \gamma$, where γ is the growth rate, which, according to [1, 3], is given by

$$\gamma^2 = k_z^2 v_A^2 \Delta/\Delta_1. \quad (2.40)$$

These unstable modes correspond to MRI. It follows from condition (2.38) that MRI occurs if the wave vector is smaller than a critical value,

$$k^2 < k_{crit}^2, \quad (2.41)$$

where

$$v_A^2 k_{crit}^2 = -d\Omega^2/d\ln R. \quad (2.42)$$

It is remarkable that Eq. (2.36) is independent of β. Therefore, both instability criterion (2.38) and growth rate (2.40) near the stability boundary are valid for an arbitrary β.

2.2.2. MRI in the case of the Velikhov rotation frequency profile

It was assumed in [1] that

$$\Omega\,(R) = a + e/R^2, \quad (2.43)$$

where a and e are constants. Then (2.29) becomes

$$\Delta = \frac{2}{k^2 v_A^2}\left[\frac{e}{R^3}\left(a + \frac{e}{R^2}\right) - \frac{1}{2}k^2 v_A^2\right]. \quad (2.44)$$

Hence, the result in [1] implies that MRI is possible only if

$$e > 0. \quad (2.45)$$

The expression (2.42) for the critical wave vector of unstable modes in this case reduces to

$$k_{crit}^2 = \frac{2e}{R^3 v_A^2}\left(a + \frac{e}{R^2}\right). \quad (2.46)$$

3. KINETIC THEORY OF MRI

3.1. Kinetic approach

Turning to the case of a collisionless plasma, we begin with modifying Eq. (2.3) as [54]

$$\rho\frac{d\mathbf{V}}{dt} = -\boldsymbol{\nabla}\cdot\mathbf{p} - \frac{1}{4\pi}\left[\frac{1}{2}\boldsymbol{\nabla}\mathbf{B}^2 - (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}\right], \quad (3.1)$$

where

$$\mathbf{p} = \mathbf{I}p_0 + \delta\mathbf{p}, \quad (3.2)$$

is the total pressure tensor, $\mathbf{I}$ is the unit tensor, and $\delta\mathbf{p}$ is the perturbed pressure tensor.

According to [54],

$$\boldsymbol{\nabla}\cdot\mathbf{p} = \boldsymbol{\nabla}p_\perp, \quad (3.3)$$

where $p_\perp = p_0 + \delta p_\perp$ with $\delta p_\perp$ being the perpendicular (with respect to the equilibrium magnetic field) perturbed-plasma pressure. Then Eq. (2.10) is modified as

$$\begin{aligned}-i\omega\delta V_R\left(\alpha_A^{ID} + \frac{k_z^2 v_A^2}{\omega^2}\right) + ik_R\frac{\delta p_\perp}{\rho_0} - \\ - 2\Omega\delta V_\phi - \frac{iv_A^2}{B_0}\frac{k^2}{k_z}\delta B_R = 0. \quad (3.4)\end{aligned}$$

The value $\delta p_\perp$ is expressed in terms of the perturbed distribution function δf as

$$\delta p_\perp = M\int\frac{v_\perp^2}{2}\delta f\,d\mathbf{v}, \quad (3.5)$$

where $v_\perp$ is the perpendicular particle velocity, $d\mathbf{v}$ is the volume element in the velocity space, and M is the ion mass.

According to [52], the function δf is equal to

$$\delta f = \frac{Mv_\perp^2}{2T} \frac{\omega}{\omega - k_z v_\parallel} f_0 \frac{\delta B_z}{B_0}, \tag{3.6}$$

where T is the ion equilibrium temperature, f_0 is the equilibrium distribution function, and $v_\parallel$ is the parallel particle velocity. Substituting (3.6) in (3.5), we obtain

$$\delta p_\perp = -p_0 \frac{i\sqrt{\pi}\omega}{|k_z|\, v_T} W\left(\frac{\omega}{|k_z|\, v_T}\right) \frac{\delta B_z}{B_0}, \tag{3.7}$$

where $v_T = \sqrt{2T/M}$ is the ion thermal velocity and

$$W(x) = \exp(-x^2)\left(1 + \frac{2i}{\sqrt{\pi}} \int\limits_0^x \exp(t^2)\, dt\right) \tag{3.8}$$

is the plasma dispersion function [50].

As a result, we obtain dispersion relation (2.22) characterized by the tensor α_{ik} $(i, k = 1, 2)$ given by Eqs. (2.23)–(2.25) with the substitution

$$\alpha_M^{ID} \to \alpha_M^{kin}, \tag{3.9}$$

where

$$\alpha_M^{kin} = 1 - \frac{k^2 v_A^2}{\omega^2}\left(1 + c_M^{kin}\right), \tag{3.10}$$

$$c_M^{kin} = -i\sqrt{\pi} \frac{k_R^2}{k^2} \beta \frac{\omega}{|k_z|\, v_T} W\left(\frac{\omega}{|k_z|\, v_T}\right). \tag{3.11}$$

With the known asymptotic form of the function $W(x)$ [50], the limit expressions for c_M^{kin} are as follows:

$$c_M^{kin} = \beta \frac{k_R^2}{k^2} \begin{cases} 1, & \omega \gg |k_z|\, v_T, \\ -i\dfrac{\sqrt{\pi}\omega}{|k_z|\, v_T}, & \omega \ll |k_z|\, v_T. \end{cases} \tag{3.12}$$

By means of Eqs. (2.22)–(2.25) and (3.9), we obtain that MRI in a collisionless plasma is described by the dispersion relation

$$\left(\omega^2 - k_z^2 v_A^2\right)\left[\omega^2 + k^2 v_A^2 \Delta + i\sqrt{\pi}\beta \frac{k_R^2 v_T \omega}{|k_z|} \times \right.$$
$$\left. \times\, W\left(\frac{\omega}{|k_z|\, v_T}\right)\right] -$$
$$- 4\omega^2\Omega^2 = 0. \tag{3.13}$$

3.2. Hydrodynamic MRI in collisionless plasma

For $\omega \gg |k_z|\, v_T$, in accordance with (3.10) and (3.12), Eq. (3.13) can be written as

$$\left(\omega^2 - k_z^2 v_A^2\right)\left(\omega^2 + k^2 v_A^2 \Delta - k_R^2 v_T^2\right) - 4\omega^2\Omega^2 = 0. \tag{3.14}$$

It hence follows that MRI occurs for

$$\Delta \geqslant k_R^2 \beta / k^2. \tag{3.15}$$

In contrast to one-fluid instability condition (2.38), the perturbations considered are unstable only if the parameter Δ exceeds a threshold value.

3.3. Kinetic MRI in collisionless plasma

For $\omega \ll |k_z|\, v_T$, it follows from Eq. (3.13) with (3.12) taken into account that

$$\left(\omega^2 - k_z^2 v_A^2\right)\left(\omega^2 + k^2 v_A^2 \Delta + i\sqrt{\pi}\beta \frac{k_R^2 v_A^2 \omega}{|k_z|\, v_T}\right) -$$
$$- 4\omega^2\Omega^2 = 0. \tag{3.16}$$

For small Δ, this dispersion relation has a small root given by

$$\omega = \frac{i}{\sqrt{\pi}} \frac{|k_z|\, v_A^2}{v_T \beta} \frac{k^2}{k_R^2} \Delta. \tag{3.17}$$

We can see that the perturbations considered are unstable under condition (2.38). We have thus shown that MRI can occur in a collisionless plasma for an arbitrary β if instability condition (2.38) is satisfied.

4. MECHANISM OF MRI

It follows from the Ohm law

$$\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B} = 0 \tag{4.1}$$

that

$$\mathbf{B} \cdot \mathbf{E} = 0. \tag{4.2}$$

The equilibrium part of (4.2) shows that there is an equilibrium electric field E_{0R} given by

$$E_{0R} = -\frac{\Omega R}{c} B_0. \tag{4.3}$$

In the presence of such an equilibrium electric field, the perturbed part of (4.2) means that there is the perturbed parallel electric field δE_z in our problem, determined by

$$B_0 \delta E_z + \delta B_R E_{0R} = 0. \tag{4.4}$$

Hence, we obtain

$$\delta E_z = \frac{\Omega R}{c}\delta B_R. \tag{4.5}$$

Next, we take the ϕ-projection of the Maxwell equation

$$\frac{\partial \delta \mathbf{B}}{\partial t} = -c\boldsymbol{\nabla}\times\delta\mathbf{E} \tag{4.6}$$

to obtain

$$\delta B_\phi = \frac{ck_z}{\omega}\delta E_R + \frac{ic}{\omega}\frac{\partial \delta E_z}{\partial R}, \tag{4.7}$$

where δE_R is the radial component of the perturbed electric field. It follows from (4.5) that

$$c\frac{\partial \delta E_z}{\partial R} = \frac{d\Omega}{d\ln R}\delta B_R + \Omega\frac{\partial}{\partial R}(R\delta B_R). \tag{4.8}$$

Now, we take into account that according to the Maxwell equation $\boldsymbol{\nabla}\cdot\delta\mathbf{B} = 0$ (cf. (2.17)),

$$\frac{1}{R}\frac{\partial}{\partial R}(R\delta B_R) = -ik_z\delta B_z. \tag{4.9}$$

Then Eq. (4.8) becomes

$$c\frac{\partial \delta E_z}{\partial R} = \frac{d\Omega}{d\ln R}\delta B_R - i\Omega R k_z \delta B_z. \tag{4.10}$$

It follows from Ohm law (4.1) that the perturbed radial electric field is expressed in terms of the perturbed velocity and perturbed magnetic field as

$$\delta E_R = \frac{1}{c}(B_0\delta V_\phi + \Omega R\delta B_z). \tag{4.11}$$

Substituting (4.10) and (4.11) in (4.7) leads to (2.16).

We have thus shown that the mechanism of MRI is explained by involving the perturbed parallel electric field δE_z.

5. INCORPORATION OF MRI INTO THE GENERAL THEORY OF PLASMA INSTABILITIES

5.1. Permittivity of rotating plasma

We use the identity

$$\frac{1}{4\pi}\left[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B} - \frac{1}{2}\boldsymbol{\nabla}\mathbf{B}^2\right] = \frac{1}{c}\mathbf{j}\times\mathbf{B}, \tag{5.1}$$

where $\mathbf{B}$ and $\mathbf{j}$ are the total magnetic field and the electric-current density. We then find from (2.10) and (2.11) that

$$\delta j_R = -\frac{c\rho_0}{B_0}\left(-i\omega\delta V_\phi + \frac{\kappa^2}{2\Omega}\delta V_R\right), \tag{5.2}$$

$$\delta j_\phi = -\frac{c\rho_0}{B_0}\left[i\omega\delta V_R\left(1 - \frac{k_R^2 c_s^2}{\omega^2\alpha_S}\right) + 2\Omega\delta V_\phi\right]. \tag{5.3}$$

It follows from (2.15) and (2.16) that

$$\delta V_\phi = -\frac{1}{k_z B_0}\left(\omega\delta B_\phi - i\frac{d\Omega}{d\ln R}\delta B_R\right), \tag{5.4}$$

$$\delta V_R = -\frac{\omega}{k_z B_0}\delta B_R. \tag{5.5}$$

With (5.4) and (5.5), Eqs. (5.2) and (5.3) become

$$\delta j_R = \frac{\omega c\rho_0}{k_z B_0^2}(-i\omega\delta B_\phi + 2\Omega\delta B_R), \tag{5.6}$$

$$\delta j_\phi = \frac{\omega c\rho_0}{k_z B_0^2}\times \left[i\omega\left(1 - \frac{k_R^2 c_s^2}{\omega^2\alpha_S} - \frac{1}{\omega^2}\frac{d\Omega^2}{d\ln R}\right)\delta B_R + 2\Omega\delta B_\phi\right]. \tag{5.7}$$

The theory of instabilities in a homogeneous plasma [52, 55–58] deals with the permittivity tensor ε_{ik} $(i,k = 1,2,3)$ related to the conductivity tensor σ_{ik} by

$$\varepsilon_{ik} = \delta_{ik} + 4\pi i\sigma_{ik}/\omega. \tag{5.8}$$

The conductivity tensor σ_{ik} is determined by the perturbed electric current

$$\delta j_i = \sigma_{ik}\delta E_k. \tag{5.9}$$

Therefore,

$$\delta j_i = \frac{\omega}{4\pi i}(\varepsilon_{ik} - \delta_{ik})\delta E_k. \tag{5.10}$$

We next take into account that in the general case,

$$\delta\mathbf{E} = \delta\mathbf{E}^{(1)} + \delta\mathbf{E}^{(2)}, \tag{5.11}$$

where $\delta\mathbf{E}^{(1)}$ and $\delta\mathbf{E}^{(2)}$ are the electromagnetic and electrostatic parts of the perturbed electric field, respectively. The field $\delta\mathbf{E}^{(1)}$ is governed by the Maxwell equation (cf. (4.6))

$$\frac{\partial\delta\mathbf{B}}{\partial t} = -c\boldsymbol{\nabla}\times\delta\mathbf{E}^{(1)}. \tag{5.12}$$

Then we have

$$\delta E_R^{(1)} = \omega\delta B_\phi/ck_z, \tag{5.13}$$

$$\delta E_\phi^{(1)} = -\omega\delta B_R/ck_z. \tag{5.14}$$

The field $\delta\mathbf{E}^{(2)}$ is defined by the perturbed electrostatic potential $\delta\Psi$ via

$$\delta\mathbf{E}^{(2)} = -\nabla\delta\Psi. \quad (5.15)$$

Because we have restricted ourselves to the case $\mathbf{k} = (k_R, 0, k_z)$, it follows from (5.15) that

$$\delta\mathbf{E}^{(2)} = (-ik_R\delta\Psi, 0, -ik_z\delta\Psi)\,. \quad (5.16)$$

With (5.11) and (5.13)–(5.15), Eq. (5.10) yields

$$\delta j_R = \frac{\omega^2}{4\pi ick_z}(\varepsilon_{11}\delta B_\phi - \varepsilon_{12}\delta B_R) - \frac{\omega k}{4\pi}\varepsilon_{10}\delta\Psi, \quad (5.17)$$

$$\delta j_\phi = \frac{\omega^2}{4\pi ick_z}(\varepsilon_{21}\delta B_\phi - \varepsilon_{22}\delta B_R) - \frac{\omega k}{4\pi}\varepsilon_{20}\delta\Psi, \quad (5.18)$$

$$\delta j_z = \frac{\omega^2}{4\pi ick_z}(\varepsilon_{31}\delta B_\phi - \varepsilon_{32}\delta B_R) - \frac{\omega k}{4\pi}\varepsilon_{30}\delta\Psi, \quad (5.19)$$

where

$$\varepsilon_{10} = (k_R\varepsilon_{11} + k_z\varepsilon_{13})/k, \quad (5.20)$$

$$\varepsilon_{20} = (k_R\varepsilon_{21} + k_z\varepsilon_{23})/k, \quad (5.21)$$

$$\varepsilon_{30} = (k_R\varepsilon_{31} + k_z\varepsilon_{33})/k. \quad (5.22)$$

Comparing (5.17) and (5.18) with (5.6) and (5.7) yields $(i, k = 1, 2)$

$$\varepsilon_{ik} = \frac{c^2}{v_A^2}\begin{pmatrix} 1 & -2i\Omega/\omega \\ \dfrac{2i\Omega}{\omega} & 1 - \dfrac{k_R^2 c_s^2}{\omega^2\alpha_S} - \dfrac{1}{\omega^2}\dfrac{d\Omega^2}{d\ln R}\end{pmatrix}. \quad (5.23)$$

As is known [52, 53], the dispersion relation in the approximation $\varepsilon_{33} \to \infty$ is of the form

$$\begin{vmatrix} \varepsilon_{11} - c^2k_z^2/\omega^2 & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} - c^2k^2/\omega^2 \end{vmatrix} = 0. \quad (5.24)$$

Substituting (5.23) here, we arrive at dispersion relation (2.22).

We can see from (5.23) that

$$\varepsilon_{ik} = \varepsilon_{ik}^{(0)} + \varepsilon_{ik}^{(r)} \quad (i, k = 1, 2)\,, \quad (5.25)$$

where $\varepsilon_{ik}^{(0)}$ is the “nonrotational” part of the permittivity tensor, i.e., the part corresponding to the case of nonrotating plasma, while $\varepsilon_{ik}^{(r)}$ is its “rotational” part. In the scope of the one-fluid MHD approach considered in Sec. 2, in accordance with (5.23), we have

$$\varepsilon_{ik}^{(0)MHD} = \frac{c^2}{v_A^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 - \dfrac{k_R^2 c_s^2}{\omega^2\alpha_S}\end{pmatrix} \quad (5.26)$$

and the rotational part of ε_{ik} $(i, k = 1, 2)$ is given by

$$\varepsilon_{ik}^{(r)} = \frac{c^2}{v_A^2}\begin{pmatrix} 0 & -2i\Omega/\omega \\ \dfrac{2i\Omega}{\omega} & -\dfrac{1}{\omega^2}\dfrac{d\Omega^2}{d\ln R}\end{pmatrix}. \quad (5.27)$$

The kinetic approach leads to the same expressions for $\varepsilon_{11}^{(0)}$, $\varepsilon_{12}^{(0)}$, and $\varepsilon_{21}^{(0)}$ and the following expression for $\varepsilon_{22}^{(0)kin}$:

$$\varepsilon_{22}^{(0)kin} = \frac{c^2}{v_A^2}\left[1 + i\frac{\sqrt{\pi}}{\omega}\frac{k_R^2 v_T}{|k_z|}W\left(\frac{\omega}{|k_z|\,v_T}\right)\right]. \quad (5.28)$$

It can be seen that the elements $\varepsilon_{ik}^{(r)}$ are independent of the detailed plasma properties. In this context, each element $\varepsilon_{ik}^{(r)}$ is an invariant. In contrast to this, the values $\varepsilon_{ik}^{(0)}$ depend on the detailed plasma properties.

5.2. General dispersion relation

To obtain the general dispersion relation, we recall the Ampere law

$$\nabla \times \delta\mathbf{B} = 4\pi\delta\mathbf{j}/c. \quad (5.29)$$

With (2.17), projections of (5.29) are given by

$$k_z\delta B_\phi = i4\pi\delta j_R/c, \quad (5.30)$$

$$k^2\delta B_R = -i4\pi k_z\delta j_\phi/c, \quad (5.31)$$

$$k_R\delta B_\phi = -i4\pi\delta j_z/c. \quad (5.32)$$

In addition, the current continuity equation implies that

$$k_R\delta j_R + k_z\delta j_z = 0. \quad (5.33)$$

Substituting (5.17) and (5.18) in (5.30) and (5.31), we obtain

$$\left(\varepsilon_{11} - \frac{c^2k_z^2}{\omega^2}\right)\delta B_\phi - \varepsilon_{12}\delta B_R - \frac{ick_zk}{\omega}\varepsilon_{10}\delta\Psi = 0, \quad (5.34)$$

$$-\varepsilon_{12}\delta B_\phi + \left(\varepsilon_{22} - \frac{c^2k^2}{\omega^2}\right)\delta B_R + \frac{ick}{k_z}\varepsilon_{20}\delta\Psi = 0. \quad (5.35)$$

Next, substituting (5.17) and (5.19) in (5.33) yields

$$\varepsilon_{01}\delta B_\phi - \varepsilon_{02}\delta B_R - \frac{ick_z k}{\omega}\varepsilon_{00}\delta\Psi = 0, \qquad (5.36)$$

where

$$\varepsilon_{01} = \left(k_R\varepsilon_{11} + k_z\varepsilon_{31}\right)/k, \qquad (5.37)$$

$$\varepsilon_{02} = \left(k_R\varepsilon_{12} + k_z\varepsilon_{32}\right)/k, \qquad (5.38)$$

$$\varepsilon_{00} = \frac{1}{k^2}\left[k_R^2\varepsilon_{11} + k_R k_z\left(\varepsilon_{13} + \varepsilon_{31}\right) + k_z^2\varepsilon_{33}\right]. \qquad (5.39)$$

Equations (5.34)–(5.36) yield the dispersion relation

$$\begin{vmatrix} \varepsilon_{11} - c^2k_z^2/\omega^2 & \varepsilon_{12} & \varepsilon_{10} \\ \varepsilon_{21} & \varepsilon_{22} - c^2k^2/\omega^2 & \varepsilon_{20} \\ \varepsilon_{01} & \varepsilon_{02} & \varepsilon_{00} \end{vmatrix} = 0. \qquad (5.40)$$

On the other hand, the theory of oscillations of a homogeneous nonrotating plasma deals with the general dispersion relation of the form

$$\begin{vmatrix} \varepsilon_{11} - c^2k_z^2/\omega^2 & \varepsilon_{12} & \varepsilon_{13} + c^2k_zk_R/\omega^2 \\ \varepsilon_{21} & \varepsilon_{22} - c^2k^2/\omega^2 & \varepsilon_{23} \\ \varepsilon_{31} + c^2k_zk_R/\omega^2 & \varepsilon_{32} & \varepsilon_{33} - c^2k_R^2/\omega^2 \end{vmatrix} = 0. \qquad (5.41)$$

Then the question arises whether the structures of (5.40) and (5.41) are identical or plasma rotation essentially modifies the fundamental plasma properties. To answer this question, we multiply the first row of matrix (5.41) by k_R^2/k^2 and add it to the third row multiplied by k_zk_R/k^2. Similarly, we multiply the first column of (5.41) by k_R^2/k^2 and add it to the third column multiplied by k_zk_R/k^2. We then obtain that matrices (5.40) and (5.41) are identical.

5.3. Electrodynamic theory of MRI allowing for finite electron temperature in the scope of MHD approach

Freezing condition (2.14) is a consequence of Ohm law (4.1). To justify this, we act on Eq. (4.1) with the operator $\nabla\times$ and use the Maxwell equation (5.12).

Evidently, Eq. (4.1) is relevant to the plasma with cold electrons. In the case of a finite electron temperature, it modifies as

$$\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B} + \frac{\nabla p_e}{en_0} = 0, \qquad (5.42)$$

where p_e is the electron pressure, e is the ion charge, and n_0 is the equilibrium number density. It follows that in the case of a homogeneous plasma, the appearance of the term with the electron pressure does not lead to modification of freezing condition (2.14). However, it contributes to the parallel Ohm law, leading to

$$\mathbf{B}\cdot\mathbf{E} + \frac{\mathbf{B}\cdot\nabla p_e}{en_0} = 0. \qquad (5.43)$$

The perturbed part of (5.43) implies that

$$\delta E_z + \Omega R\delta B_R + \frac{ik_z\delta p_e}{en_0} = 0. \qquad (5.44)$$

We also note that freezing condition (2.14) contains the electron velocity $\mathbf{V}$, while $\mathbf{V}$ entering the parallel plasma motion equation (2.12) is the ion velocity. In other words, the one-fluid MHD is based on the assumption that

$$\mathbf{V}_i = \mathbf{V}_e. \qquad (5.45)$$

We now allow a difference between δV_{iz} and δV_{ez},

$$\delta V_{iz} \neq \delta V_{ez}. \qquad (5.46)$$

Generally,

$$\delta V_{iz} = \delta V_{ez} + \delta j_z/en_0, \qquad (5.47)$$

where δj_z is the perturbed parallel electric current.

The perturbed ion parallel motion equation (2.12) is modified as

$$\delta j_z = -en_0\left(\delta V_{ez} + \frac{k_zk_Rc_s^2}{\omega^2\alpha_S}\delta V_R\right). \qquad (5.48)$$

We seek the perturbed electron pressure δp_e using the electron adiabatic condition similar to (2.8) with $\Gamma_e = 1$, where Γ_e is the electron adiabatic exponent. Then we obtain

$$\delta p_e = \frac{p_{0e}}{\omega}\left(k_R\delta V_R + k_z\delta V_{ez}\right). \qquad (5.49)$$

Substituting (5.49) in (5.44) yields

$$\delta V_{ez} = -\frac{k_R}{k_z}\delta V_R + \frac{i\omega e}{k_z^2T_e}\delta E_z, \qquad (5.50)$$

where T_e is the equilibrium electron temperature. With (5.50), Eq. (5.48) becomes

$$\delta j_z = -\frac{i\omega e^2 n_0}{k_z^2 T_e}\delta E_z + \frac{e n_0 k_R}{k_z \alpha_S}\delta V_R + + \frac{i\omega e^2 n_0 \Omega R}{k_z^2 T_e}\delta B_R. \quad (5.51)$$

We now use representation (5.11), thereby introducing the perturbed electrostatic potential $\delta\Psi$. Then Eq. (5.51) becomes

$$\delta j_z = -\frac{\omega e^2 n_0}{k_z T_e}\delta\Psi + \frac{e n_0 k_R \omega}{k_z^2 \alpha_S B_0}\delta B_R. \quad (5.52)$$

Comparing (5.6), (5.7), and (5.52) with (5.17)–(5.19) gives

$$(\varepsilon_{10}, \varepsilon_{20}, \varepsilon_{31}) = 0, \quad (5.53)$$

$$\varepsilon_{32} = -\frac{4\pi i c e n_0 k_R}{k_z \alpha_S \omega B_0}, \quad (5.54)$$

$$\varepsilon_{30} = \frac{4\pi e^2 n_0}{k k_z T_e}. \quad (5.55)$$

To use dispersion relation (5.40), it is necessary to know the values ε_{01}, ε_{02}, and ε_{00} defined by Eqs. (5.37)–(5.39). Using (5.23) and (5.53)–(5.55), we obtain

$$\varepsilon_{01} = \frac{c^2}{v_A^2}\frac{k_R}{k}, \quad (5.56)$$

$$\varepsilon_{02} = -\frac{i k_R}{k}\frac{c^2}{v_A^2 \omega}\left(2\Omega + \frac{\omega_{Bi}}{\alpha_S}\right), \quad (5.57)$$

$$\varepsilon_{02} = \frac{1}{k^2}\frac{c^2}{v_A^2}\left(k_R^2 + \frac{1}{\rho_s^2}\right), \quad (5.58)$$

where $\rho_s^2 = T_e/M\omega_{Bi}^2$ is the squared ion Larmor radius calculated for the electron temperature and $\omega_{Bi} = eB_0/Mc$ is the ion cyclotron frequency.

5.4. Heuristic kinetic electrodynamic theory of MRI allowing for finite electron temperature and effects of the finite ion Larmor radius

The idea that the permittivity tensor ε_{ik} in the rotating plasma can be represented as a sum of nonrotational $\varepsilon_{ik}^{(0)}$ and rotational $\varepsilon_{ik}^{(r)}$ parts (see Eq. (5.25)) allows suggesting a heuristic electrodynamic theory of MRI allowing for finite electron temperature. It is then convenient to use the general dispersion relation in form (5.41) because the values $\varepsilon_{ik}^{(0)}$ $(i,k = 1,2,3)$ are well-known. In such a problem statement, the effects of a finite ion Larmor radius can simultaneously be taken into account.

From [49], we have

$$\varepsilon_{11}^{(0)} = \frac{c^2}{v_A^2}\left(1 - \frac{3}{4}k_R^2\rho_i^2\right), \quad (5.59)$$

$$\varepsilon_{12}^{(0)} = -\varepsilon_{21}^{(0)} = -\frac{3}{2}i\frac{c^2}{v_A^2}\frac{\omega_{Bi}}{\omega}k_R^2\rho_i^2, \quad (5.60)$$

$$\varepsilon_{22}^{(0)} = \frac{c^2}{v_A^2}\left[1 - \frac{3}{4}k_R^2\rho_i^2 + i\frac{\sqrt{\pi}}{\omega}\frac{k_R^2 v_T}{|k_z|} \times \times \beta W\left(\frac{\omega}{|k_z| v_T}\right)\right], \quad (5.61)$$

where $\rho_i^2 = T_i/M\omega_{Bi}^2$ is the squared ion Larmor radius. The remaining components of the permittivity tensor are

$$\varepsilon_{13}^{(0)} = \varepsilon_{31}^{(0)} = 0, \quad (5.62)$$

$$\varepsilon_{23}^{(0)} = -\varepsilon_{32}^{(0)} = \frac{c^2}{v_A^2}\frac{k_R}{k_z}\frac{\omega_{Bi}}{|k_z| v_{Ti}}W\left(\frac{\omega}{|k_z| v_{Ti}}\right), \quad (5.63)$$

$$\varepsilon_{33}^{(0)} = \frac{1}{k_z^2 d_e^2} + \frac{1}{k_z^2 d_i^2}\left[1 + i\sqrt{\pi}\frac{\omega}{|k_z| v_{Ti}} \times \times W\left(\frac{\omega}{|k_z| v_{Ti}}\right)\right], \quad (5.64)$$

where $d_\alpha^2 = T_\alpha/4\pi e^2 n_0$ $(\alpha = e, i)$ is the squared Debye length. In deriving (5.63) and (5.64), we have assumed $\omega \ll |k_z| v_{Te}$, where $v_{Te} = \sqrt{2T_e/M_e}$ is the electron thermal velocity and M_e is the electron mass.

6. ROTATION EFFECTS ON ALFVÉN WAVES

6.1. Dispersion relation

Alfvén waves in a rotating plasma can be studied by using the following particular case of dispersion relation (5.41):

$$\begin{vmatrix} \varepsilon_{11} - c^2 k_z^2/\omega^2 & \varepsilon_{12} & c^2 k_z k_x/\omega^2 \\ \varepsilon_{21} & \varepsilon_{22} - c^2 k_x^2/\omega^2 & 0 \\ c^2 k_z k_x/\omega^2 & 0 & \varepsilon_{33} - c^2 k_x^2/\omega^2 \end{vmatrix} = 0, \quad (6.1)$$

where (cf. (5.25)–(5.27)),

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \frac{c^2}{v_A^2} \begin{pmatrix} 1 & -2i\Omega/\omega \\ \dfrac{2i\Omega}{\omega} & -\dfrac{1}{\omega^2}\dfrac{d\Omega^2}{d\ln R} \end{pmatrix}, \quad (6.2)$$

and the value ε_{33} is taken from the wave theory of a homogeneous plasma [52] as

$$\varepsilon_{33} = c_\parallel / k_z^2 d_e^2, \quad (6.3)$$

where

$$c_\parallel = 1 + i\sqrt{\pi}\frac{\omega}{|k_z|\, v_{Te}} W\left(\frac{\omega}{|k_z|\, v_{Te}}\right). \quad (6.4)$$

Equation (6.1) yields

$$\omega^2 - k_z^2 v_A^2 \left(\frac{k_x^2 v_A^2 + d\Omega^2/d\ln R}{k_x^2 v_A^2 + \kappa^2} + \frac{k_x^2 \rho_s^2}{c_\parallel}\right) = 0. \quad (6.5)$$

6.2. Rotational Alfvén waves

We consider the approximation of an infinite parallel conductivity

$$k_x^2 \rho_s^2 / c_\parallel = 0. \quad (6.6)$$

Equation (6.5) then becomes

$$\omega^2 = \frac{k_z^2 v_A^2}{1 + \kappa^2 / k_x^2 v_A^2}\left(1 + \frac{1}{k_x^2 v_A^2}\frac{d\Omega^2}{d\ln R}\right). \quad (6.7)$$

For a weak plasma rotation,

$$d\Omega^2 / d\ln R \ll k_x^2 v_A^2, \quad (6.8)$$

it follows from Eq. (6.7) that

$$\omega^2 = k_z^2 v_A^2 (1 - 4\Omega^2 / k_x^2 v_A^2). \quad (6.9)$$

The oscillation branches described by (6.9) can be called the rotational Alfvén waves.

6.3. Kinetic Alfvén waves in a rotating plasma

Let

$$\beta > M_e / M. \quad (6.10)$$

Then we can use the approximation

$$c_\parallel \to 1. \quad (6.11)$$

In this case, Eq. (6.5) is transformed to

$$\omega^2 = k_z^2 v_A^2 \left(\frac{k_x^2 v_A^2 + d\Omega^2/d\ln R}{k_x^2 v_A^2 + \kappa^2} + k_x^2 \rho_s^2\right). \quad (6.12)$$

In the case of a weak plasma rotation with condition (6.8) satisfied, Eq. (6.12) reduces to (cf. (6.9))

$$\omega^2 = k_z^2 v_A^2 \left(1 - 4\Omega^2 / k_x^2 v_A^2 + k_x^2 \rho_s^2\right). \quad (6.13)$$

With the term involving Ω^2 neglected, this dispersion relation describes the kinetic Alfvén waves [62]. It can be seen that the rotational dispersion exceeds the Larmor dispersion for

$$k_x^2 \lesssim \Omega / c d_e. \quad (6.14)$$

Instead of the positive dispersion, we then have Alfvén waves with the negative dispersion.

6.4. Inertial Alfvén waves in a rotating plasma

We now take

$$\beta < M_e / M. \quad (6.15)$$

Then Eq. (6.4) is transformed to

$$c_\parallel = -k_z^2 v_{Te}^2 / 2\omega^2. \quad (6.16)$$

Substituting Eq. (6.16) in Eq. (6.5) yields

$$\omega^2 = \frac{k_z^2 v_A^2}{1 + c^2 k_x^2 / \omega_{pe}^2}\, \frac{k_x^2 v_A^2 + d\Omega^2/d\ln R}{k_x^2 v_A^2 + \kappa^2}, \quad (6.17)$$

where $\omega_{pe}^2 = 4\pi n_0 e^2 / M_e$ is the squared electron plasma frequency. With the rotation neglected, this dispersion relation describes the inertial Alfvén waves [63].

For weak rotation, $\Omega^2 \ll k_x^2 v_A^2$, and weak electron inertia, $c^2 k_x^2 \ll \omega_{pe}^2$, Eq. (6.17) yields (cf. Eqs. (6.9) and (6.13))

$$\omega^2 = k_z^2 v_A^2 \left(1 - \frac{4\Omega^2}{k_x^2 v_A^2} - \frac{c^2 k_x^2}{\omega_{pe}^2}\right). \quad (6.18)$$

It follows that rotation leads to dispersion of the same sign as the electron inertia, i. e., they are both negative. The rotational dispersion exceeds the inertial dispersion for

$$k_x^2 < \Omega \omega_{pe} / c v_A. \quad (6.19)$$

The sign of the dispersion then remains unchanged.

7. DISCUSSION

We have collected and analyzed the results of the one-fluid MHD theory of MRI in an ideal plasma. We have shown that this instability can occur for an arbitrary β. In general, such a theory has the goal to predict regularities of MRI in a collisionless plasma. To verify these predictions, it is necessary to develop the kinetic theory of MRI. The simplest version of this theory has been formulated in the present paper (see also [64–66]). Then we have shown that the one-fluid and kinetic instability criteria are identical and are given by Eq. (2.38). At the same time, the one-fluid and kinetic growth rates of MRI turn out to be different, cf. (2.40) and (3.17). Roughly speaking,

$$\gamma^{kin}/\gamma^{MHD} \approx \beta^{-1/2}. \tag{7.1}$$

This means that for $\beta > 1$, the kinetic growth rate is small compared with the MHD one, while for $\beta < 1$ the situation is the opposite. This difference is due to the imaginary term in kinetic dispersion relation (3.13). Physically, this term describes the gyrorelaxation effect discovered in [67] and [68]. In the case of a collision-dominated plasma, this effect is described in terms of the parallel viscosity [53, 69].

We have discussed the mechanism of MRI and have explained that it is intrinsically related to the appearance of the parallel perturbed electric field (see Eqs. (4.5) and (4.10)).

To incorporate the notion of MRI into the general theory of plasma instabilities, we have developed the electrodynamic theory generalizing the known dispersion relation for a homogeneous plasma by including the rotation effects. Such a generalized dispersion relation is given by Eq. (5.40). In the approximation of an infinite parallel conductivity, it reduces to Eq. (5.24). According to the electrodynamic theory presented, plasma rotation leads to two modifications of the permittivity tensor entering the dispersion relation. The first is the appearance of the Velikhov effect in the element ε_{22} and the second is the appearance of the nondiagonal components ε_{12} and ε_{21}, see Eq. (5.27).

We have noted that the rotation effects are additive, implying that the permittivity tensor can be represented as a sum of nonrotational and rotational parts (see Eq. (5.25)). It is remarkable that the rotational part of the permittivity tensor has a universal structure independent of the detailed plasma properties (see Eq. (5.27)).

We have taken the effect of a finite electron temperature on MRI into account. In this regard, it is reasonable to note that the one-fluid MHD approach developed in [3, 61] is valid only for cold electrons. Therefore, one of the goals for future studies on MRI is a generalization of the Balbus–Hawley and Kim–Ostriker dispersion relations (see Subsecs. 2.1.3 and 2.1.4) to the case of a finite electron temperature.

Using that the rotational part of the permittivity tensor is invariant, we have developed a heuristic kinetic electrodynamic theory of MRI with both the finite electron temperature effects and the effects of a finite ion Larmor radius taken into account. In the scope of the present paper, we restricted ourselves to using this theory for studying the rotation effects on Alfvén modes. As a result, we have shown that in addition to the kinetic and inertial Alfvén waves, one more mode, the rotational Alfvén waves, can be realized in a rotating plasma. At the same time, according to our analysis, the rotation effects can essentially modify both kinetic and inertial Alfvén waves, transforming them into rotational Alfvén waves for not too small rotation frequencies. The kinetic and inertial Alfvén waves have been studied in Refs. [63, 70–72] as a possible reason of zonal flow generation. It is evident from our analysis that the same role can be played by the rotational Alfvén waves.

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