

FREE-FIELD REPRESENTATION OF PERMUTATION BRANES IN GEPNER MODELS

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We consider a free-field realization of Gepner models based on the free-field realization of $N = 2$ superconformal minimal models. Using this realization, we analyze the A/B-type boundary conditions starting from the ansatz with the left-moving and right-moving free-field degrees of freedom glued at the boundary by an arbitrary constant matrix. We show that the only boundary conditions consistent with the singular vector structure of unitary minimal model representations are given by permutation matrices, thereby yielding an explicit free-field construction of the permutation branes of Recknagel.

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1. INTRODUCTION

Investigation of D-branes on Calabi–Yau manifolds at string scales is an interesting and important problem. There is a significant progress in this direction achieved mainly due to the intensive study of D-branes at Gepner points of the Calabi–Yau moduli space initiated in [1].

Because Gepner models are defined by a purely algebraic construction [2, 3], it is natural that the symmetry-preserving boundary states (D-branes) in these models can also be described by algebraic objects [1–6]. The question of their geometric interpretation then appears to be nontrivial and interesting. Considerable progress in the understanding of the geometry of D-branes in the Gepner models has been achieved recently in [7–16]. The main idea developed in these papers is to relate the intersection index of boundary states [17] to the bilinear form of the K-theory classes of bundles on a large-volume Calabi–Yau manifold and use this relation to associate the K-theory classes to the boundary states.

The natural question that arises here but is hard to answer is whether one can find a direct conformal field theory description of the geometry of D-branes in Gepner models instead of interpolation of large-volume topological data of bundles to the Gepner point. In trying to find the direct description (as well as to develop the main ingredients for the integral representation for the boundary correlation functions), the free-field construction of D-branes in Gepner models has been developed in [18]. It was shown there that the free-field representations of the open-string spectrum between the Recknagel–Schomerus boundary states can be described in terms of representations of the chiral de Rham complex [19] on a Landau–Ginzburg orbifold. The chiral de Rham complex is a string generalization of the usual de Rham complex and is a sheaf of vertex algebras [19–21]. Hence, it is a geometric object, and this property has been used in [18] to geometrically interprete the boundary states in Gepner models (constructed in purely algebraic terms) as fractional branes on Landau–Ginzburg orbifolds. This suggests that the chiral de Rham complex might be a natural and efficient object for the description of D-brane geometry at string scales.

With this in mind, we try in this paper to extend the free-field representation in [18, 22] to the case of permutation branes [6]. Our aim is to analyze and represent the free-field construction of permutation branes, while the important question of the study and comparison of the free-field geometry of D-branes to the results in [23, 24] is left for the future.

In Sec. 2, we briefly review the free-field construction of irreducible representations in $N = 2$ minimal

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models developed in [25]. In Sec. 3, we schematically consider the free-field realization of Gepner models. In Sec. 4, we investigate A- and B-type gluing conditions in terms of free fields. We start from the ansatz where the left-moving and right-moving free-field degrees of freedom are glued at the boundary by an arbitrary constant matrix and analyze the A- and B-type boundary conditions in terms of free fields. Section 5 is the main part of the paper. We analyze the consistency of the boundary conditions with the singular vector structure of minimal models (the butterfly resolution) and show that only permutation matrices survive, thereby giving the free-field representation of permutation Ishibashi states. In Sec. 6, we use the Recknagel solution [6] of Cardy's constraints as the well as the orbifold construction to obtain a free-field realization of permutation branes in Gepner models.

2. FREE-FIELD REALIZATION OF IRREDUCIBLE REPRESENTATIONS IN THE $N = 2$ MINIMAL MODELS

In this section, we briefly discuss the free-field construction in [25] of irreducible modules in $N = 2$ superconformal minimal models. The free-field approach to $N = 2$ minimal models was also considered in [26, 27].

2.1. Free-field representations of the $N = 2$ super-Virasoro algebra

We introduce (in the left-moving sector) free bosonic fields $X(z)$ and $X^*(z)$ and free fermionic fields $\psi(z)$ and $\psi^*(z)$ with the operator product expansions given by

$$\begin{aligned} &X^*(z_1)X(z_2) = \ln(z_{12}) + \text{reg}, \\ &\psi^*(z_1)\psi(z_2) = z_{12}^{-1} + \text{reg}, \end{aligned} \tag{1}$$

where $z_{12} = z_1 - z_2$ and reg denotes regular terms as $z_1 \to z_2$. Then for an arbitrary number, μ, the currents of the $N = 2$ super-Virasoro algebra are given by

$$\begin{aligned} &G^+(z) = \psi^*(z)\partial X(z) - \frac{1}{\mu}\partial\psi^*(z), \\ &G^-(z) = \psi(z)\partial X^*(z) - \partial\psi(z), \\ &J(z) = \psi^*(z)\psi(z) + \frac{1}{\mu}\partial X^*(z) - \partial X(z), \\ &T(z) = \partial X(z)\partial X^*(z) + \frac{1}{2}(\partial\psi^*(z)\psi(z) - \\ &\quad - \psi^*(z)\partial\psi(z)) - \frac{1}{2}(\partial^2 X(z) + \frac{1}{\mu}\partial^2 X^*(z)), \end{aligned} \tag{2}$$

and the central charge is

$$c = 3\left(1 - \frac{2}{\mu}\right). \tag{3}$$

As usual, the fermions $\psi(z)$, $\psi^*(z)$, and $G^{\pm}(z)$ in the Neveu–Schwartz (NS) and Ramond (R) sectors are expanded in half-integer (integer) modes. The bosons $X(z)$, $X^*(z)$, $J(z)$, and $T(z)$ are expanded in integer modes in both sectors.

In the NS sector, the $N = 2$ Virasoro superalgebra acts naturally in the Fock module F_{p,p^*} generated by the fermionic operators $\psi^*[r]$, $\psi[r]$, $r < 1/2$, and bosonic operators $X^*[n]$, $X[n]$, $n < 0$, from the vacuum state $|p, p^*\rangle$ such that

$$\begin{aligned} &\psi[r]|p,p^*\rangle = \psi^*[r]|p,p^*\rangle = 0, \quad r \ge \frac{1}{2}, \\ &X[n]|p,p^*\rangle = X^*[n]|p,p^*\rangle = 0, \quad n \ge 1, \\ &X[0]|p,p^*\rangle = p|p,p^*\rangle, \quad X^*[0]|p,p^*\rangle = p^*|p,p^*\rangle. \end{aligned} \tag{4}$$

The state $|p, p^*\rangle$ is primary with respect to the $N = 2$ Virasoro algebra,

$$\begin{aligned} &G^{\pm}[r]|p,p^*\rangle = 0, \quad r > 0, \\ &J[n]|p,p^*\rangle = L[n]|p,p^*\rangle = 0, \quad n > 0, \\ &J[0]|p,p^*\rangle = \frac{j}{\mu}|p,p^*\rangle = 0, \\ &L[0]|p,p^*\rangle = \frac{h(h+2) - j^2}{4\mu}|p,p^*\rangle = 0, \end{aligned} \tag{5}$$

where

$$j = p^* - \mu p, \quad h = p^* + \mu p.$$

The character $f_{p,p^*}(q, u)$ of the Fock module F_{p,p^*} is given by

$$\begin{aligned} f_{p,p^*}(q,u) &\equiv \operatorname{Tr}_{F_{p,p^*}}(q^{L[0]-(c/24)}u^{J[0]}) = \\ &= q^{[h(h+2)-j^2]/4\mu - c/24}u^{j/\mu}\frac{\Theta(q,u)}{\eta(q)^3}, \end{aligned} \tag{6}$$

where the Jacoby theta-function

$$\Theta(q,u) = q^{1/8}\sum_{m\in Z} q^{(1/2)m^2}u^{-m} \tag{7}$$

and the Dedekind eta-function

$$\eta(q) = q^{1/24}\prod_{m=1}(1 - q^m) \tag{8}$$

are used.

The $N = 2$ Virasoro algebra has the following set of automorphisms, known as the spectral flow [28]:

$$G^{\pm}[r] \to G^{\pm}_t[r] \equiv G^{\pm}[r \pm t],$$
$$L[n] \to L_t[n] \equiv L[n] + tJ[n] + t^2 \frac{c}{6}\delta_{n,0}, \qquad (9)$$
$$J[n] \to J_t[n] \equiv J[n] + t\frac{c}{3}\delta_{n,0},$$

where $t \in Z$.

The spectral flow action on the free fields can be easily described if we bosonize the fermions ψ^* and ψ as

$$\psi(z) = \exp(-y(z)), \quad \psi^*(z) = \exp(+y(z)) \qquad (10)$$

and introduce the spectral flow vertex operator

$$U^t(z) = \exp\left(-t\left(y + \frac{1}{\mu}X^* - X\right)(z)\right). \qquad (11)$$

It gives the action of the spectral flow on the free-field modes,

$$\psi[r] \to \psi[r-t], \quad \psi^*[r] \to \psi^*[r+t],$$
$$X^*[n] \to X^*[n] + t\delta_{n,0}, \quad X[n] \to X[n] - \frac{t}{\mu}\delta_{n,0}. \qquad (12)$$

The action of the spectral flow on the vertex operator $V_{(p,p^*)}(z)$ is given by the normal ordered product of the vertex $U^t(z)$ and $V_{p,p^*}(z)$. It follows from (12) that the spectral flow generates twisted sectors.

2.2. Irreducible $N = 2$ super-Virasoro representations and the butterfly resolution

The $N = 2$ minimal models are characterized by the condition that μ is integer and $\mu \geq 2$. In the NS sector, the irreducible highest-weight modules, constituting the (left-moving) space of states of the minimal model, are unitary and are labeled by two integers h, j, where $h = 0, \dots, \mu - 2$ and $j = -h, -h+2, \dots, h$. The highest-weight vector $|h,j\rangle$ of the module satisfies the conditions

$$G^{\pm}[r]|h,j\rangle = 0, \quad r > 0,$$
$$J[n]|h,j\rangle = L[n]|h,j\rangle = 0, \quad n > 0,$$
$$J[0]|h,j\rangle = \frac{j}{\mu}|h,j\rangle, \qquad (13)$$
$$L[0]|h,j\rangle = \frac{h(h+2) - j^2}{4\mu}|h,j\rangle.$$

The Fock modules are highly reducible representations of the $N = 2$ Virasoro algebra and hence contain an infinite number of singular vectors. To describe the singular vector structure, we introduce, following [25], the pair of fermionic screening currents $S^{\pm}(z)$ and the screening charges $Q^{\pm}$ as

$$S^+(z) = \psi^* \exp(X^*)(z),$$
$$S^-(z) = \psi \exp(\mu X)(z), \qquad (14)$$
$$Q^{\pm} = \oint dz S^{\pm}(z).$$

The screening charges commute with the generators of $N = 2$ super-Virasoro algebra (2). But they do not act within each Fock module. Instead, they map between different Fock modules. The space where the screening charges act naturally is the direct sum of Fock modules

$$F_{\pi} = \bigoplus_{(p,p^*)\in\pi} F_{p,p^*}, \qquad (15)$$

where π is the momentum lattice:

$$\pi = \left\{(p,p^*)|p = \frac{n}{\mu}, p^* = m, n, m \in Z\right\}. \qquad (16)$$

Application of the screening charge to an arbitrary vector $|p,p^*\rangle \in F_{\pi}$ gives a singular vector in another Fock module.

The screening charges are nilpotent and mutually anticommute,

$$(Q^+)^2 = (Q^-)^2 = \{Q^+, Q^-\} = 0. \qquad (17)$$

Due to important properties (17), we can combine the charges $Q^{\pm}$ into a BRST operator acting in F_{π} and build a BRST complex of Fock modules $F_{p,p^*} \in F_{\pi}$. This complex, which has been constructed in [25], describes the structure of the $N = 2$ Virasoro singular vectors and the corresponding submodules, and the cohomology of the complex gives the irreducible module $M_{h,j}$.

We first consider the free-field construction for the chiral module $M_{h,j=h}$. In this case, the complex (which is known as the butterfly resolution [25]) can be represented by the diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \\
 & & \uparrow & & \uparrow & & & & \\
\ldots & \leftarrow & F_{1,h+\mu} & \leftarrow & F_{0,h+\mu} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
\ldots & \leftarrow & F_{1,h} & \leftarrow & F_{0,h} & & & & \\
 & & & & & \nwarrow & & & \\
 & & & & & & F_{-1,h-\mu} & \leftarrow & F_{-2,h-\mu} \leftarrow \ldots \\
 & & & & & & \uparrow & & \uparrow \\
 & & & & & & F_{-1,h-2\mu} & \leftarrow & F_{-2,h-2\mu} \leftarrow \ldots \\
 & & & & & & \uparrow & & \uparrow \\
 & & & & & & \vdots & & \vdots
\end{array}
\qquad (18)
$$

We let C_h denote this resolution and let Γ denote the set where the momenta of the Fock spaces of the resolution take values. The horizontal arrows in this diagram are given by the action of Q^+ and the vertical arrows are given by the action of Q^-. The diagonal arrow in the middle of the butterfly resolution is given by the action of Q^+Q^- (which equals $-Q^-Q^+$ due to (17)). The ghost-number operator g of the complex is defined for an arbitrary vector $|v_{n,m}\rangle \in F_{n,m\mu+h}$ by

$$
\begin{aligned}
&g|v_{n,m}\rangle = (n+m)|v_{n,m}\rangle, \text{ if } n,m \geq 0,\\
&g|v_{n,m}\rangle = (n+m+1)|v_{n,m}\rangle, \text{ if } n,m < 0.
\end{aligned}
\qquad (19)
$$

The main statement in [25] is that complex (18) is exact except at the $F_{0,h}$ module, where the cohomology is given by the chiral module $M_{h,j=h}$.

The butterfly resolution allows writing the character

$$\chi_h(q,u) \equiv \mathrm{Tr}_{M_{h,h}}(q^{L[0]-c/24}u^{J[0]})$$

of the module $M_{h,h}$ as the Euler characteristic of the complex:

$$
\begin{aligned}
&\chi_h(q,u) = \chi_h^{(l)}(q,u) - \chi_h^{(r)}(q,u),\\
&\chi_h^{(l)}(q,u) = \sum_{n,m\geq 0} (-1)^{n+m} f_{n,h+m\mu}(q,u),\\
&\chi_h^{(r)}(q,u) = \sum_{n,m>0} (-1)^{n+m} f_{-n,h-m\mu}(q,u),
\end{aligned}
\qquad (20)
$$

where $\chi_h^{(l)}(q,u)$ and $\chi_h^{(r)}(q,u)$ are the characters of the left and right wings of the resolution.

To obtain the resolutions for other (antichiral and nonchiral) modules, we can use the observation in [25] that all irreducible modules can be obtained from the chiral modules $M_{h,j=h}$, $h = 0,\ldots,\mu-2$, by the spectral flow action $U^{-t}, t = h, h-1, \ldots, 1$. Equivalently, we can restrict the set of chiral modules to the range $h = 0,\ldots,[\mu/2]-1$ and extend the spectral flow action by $t = \mu-1,\ldots,1$ (when μ is even and $h = [\mu/2]-1$, the spectral flow orbit becomes shorter: $t = [\mu/2]-1,\ldots,1$) [29]. Thus, the set of irreducible modules can also be labeled by the set

$$\{(h,t)|h = 0,\ldots,[\mu/2]-1, \quad t = \mu-1,\ldots,0\},$$

except in the case where μ is even and the spectral flow orbit becomes shorter. It turns out that all the resolutions can also be obtained by the spectral flow action.

In view of this discussion, it is more convenient to change the notation for irreducible modules. In what follows, we let $M_{h,t}$ denote the irreducible modules, indicating the spectral flow parameter by t.

As with the modules and resolutions, the characters can also be obtained by the spectral flow action [25]:

$$\chi_{h,t}(q,u) = q^{ct^2/6}u^{ct/3}\chi_h(q,uq^t). \qquad (21)$$

There are the following important automorphism properties of the irreducible modules and characters [25, 29]:

$$
\begin{aligned}
&M_{h,t} \equiv M_{\mu-h-2,t-h-1},\\
&\chi_{h,t}(q,u) = \chi_{\mu-h-2,t-h-1}(q,u),
\end{aligned}
\qquad (22)
$$

$$M_{h,t} \equiv M_{h,t+\mu}, \quad \chi_{h,t+\mu}(q,u) = \chi_{h,t}(q,u), \qquad (23)$$

where μ is odd and

$$
\begin{aligned}
&M_{h,t} \equiv M_{h,t+\mu}, \quad \chi_{h,t+\mu}(q,u) = \chi_{h,t}(q,u),\\
&\qquad\qquad h \neq [\mu/2]-1,\\
&M_{h,t} \equiv M_{h,t+[\mu/2]},\\
&\chi_{h,t+[\mu/2]}(q,u) = \chi_{h,t}(q,u), \quad h = [\mu/2]-1,
\end{aligned}
\qquad (24)
$$

where μ is even.

We note that the butterfly resolution is not periodic under the spectral flow, unlike the characters. It is also not invariant under automorphism (22). Instead, the periodicity and invariance are recovered at the level of cohomology. Therefore, the $U^{\pm\mu}$ spectral flow and automorphism (22) are quasi-isomorphisms of complexes.

The modules, resolutions, and characters in the R sector are generated from the modules, resolutions, and characters in the NS sector by the spectral flow operator $U^{-1/2}$.

3. FREE-FIELD REALIZATION OF THE GEPNER MODEL

3.1. Free-field realization of the product of minimal models

It is easy to generalize the free-field representation in Sec. 1 to the case of tensor product of r $N=2$ minimal models characterized by an r-dimensional vector

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$$

with integer $\mu_i \geq 2$.

Let E be a real r-dimensional vector space and let E^* be the dual space to E. We write $\langle , \rangle$ for the natural scalar product in the direct sum $E \oplus E^*$. In the subspaces E and E^*, we fix the sets of basic vectors $\mathfrak{R}$ and $\mathfrak{R}^*$ as

$$\begin{aligned} &\mathfrak{R} = \{\mathbf{s}_i, i = 1, \ldots, r\}, \\ &\mathfrak{R}^* = \{\mu_i \mathbf{s}_i^*, i = 1, \ldots, r\}, \\ &\langle \mathbf{s}_i, \mathbf{s}_j^* \rangle = \delta_{i,j}. \end{aligned} \tag{25}$$

Given the sets $\mathfrak{R}$ and $\mathfrak{R}^*$, we introduce (in the left-moving sector) the free bosonic fields $X_i(z), X_i^*(z)$ and free fermionic fields $\psi_i(z), \psi_i^*(z)$, $i = 1, \ldots, r$, with singular operator product expansions given by Eq. (1), and the following fermionic screening currents and their charges:

$$\begin{aligned} &S_i^+(z) = \mathbf{s}_i \psi^* \exp(\mathbf{s}_i X^*)(z), \\ &S_i^-(z) = \mathbf{s}_i^* \psi \exp(\mu_i \mathbf{s}_i^* X)(z), \\ &Q_i^{\pm} = \oint dz S_i^{\pm}(z). \end{aligned} \tag{26}$$

For each $i = 1, \ldots, r$, we use formulas (2) to define the $N=2$ Virasoro superalgebra with central charge $c_i = 3(1 - 2/\mu_i)$ as

$$\begin{aligned} &G_i^+ = \mathbf{s}_i \psi^* \mathbf{s}_i^* \partial X - \frac{1}{\mu_i} \mathbf{s}_i \partial \psi^*, \\ &G_i^- = \mathbf{s}_i^* \psi \mathbf{s}_i \partial X^* - \mathbf{s}_i^* \partial \psi, \\ &J_i = \mathbf{s}_i \psi^* \mathbf{s}_i^* \psi + \frac{1}{\mu_i} \mathbf{s}_i \partial X^* - \mathbf{s}_i^* \partial X, \\ &T_i(z) = \frac{1}{2}(\mathbf{s}_i \partial \psi^* \mathbf{s}_i^* \psi - \mathbf{s}_i \psi^* \mathbf{s}_i^* \partial \psi) + \\ &\quad + \mathbf{s}_i \partial X^* \mathbf{s}_i^* \partial X - \frac{1}{2}\left(\mathbf{s}_i^* \partial^2 X + \frac{1}{\mu_i} \mathbf{s}_i \partial^2 X^*\right), \end{aligned} \tag{27}$$

and the vertex operators

$$V_{(p_i, p_i^*)} = \exp(p_i^* \mathbf{s}_i^* X + p_i \mathbf{s}_i X^*), \tag{28}$$

which are the conformal fields whose conformal dimensions and charges are labeled by integers

$$h_i = p_i^* + \mu_i p_i, \quad j_i = p_i^* - \mu_i p_i.$$

The vertex operators are naturally associated with the lattice

$$\Pi = P \oplus P^* \in E \oplus E^*,$$

where $P \in E$, $P^* \in E^*$ such that P is generated by $(1/\mu_i)\mathbf{s}_i$ and P^* is generated by the basis $\mathbf{s}_i^*$, $i = 1, \ldots, r$. For an arbitrary vector $(\mathbf{p}, \mathbf{p}^*) \in \Pi$, we introduce the Fock vacuum state $|\mathbf{p}, \mathbf{p}^*\rangle$ in the NS sector by formulas similar to (4) and let $F_{\mathbf{p},\mathbf{p}^*}$ denote the Fock module generated from $|\mathbf{p}, \mathbf{p}^*\rangle$ by the creation operators of the fields $X_i(z)$, $X_i^*(z)$, $\psi_i(z)$, and $\psi_i^*(z)$.

Let F_Π be the direct sum of Fock modules associated with the lattice Π. As an obvious generalization of the results in Sec. 1, for each vector

$$\mathbf{h} = \sum_i h_i \mathbf{s}_i^* \in P^*,$$

where $h_i = 0, 1, \ldots, \mu_i - 2$, we form the butterfly resolution $C_{\mathbf{h}}^{\star}$ as the product $\otimes_{i=1}^r C_{h_i}^{\star}$ of butterfly resolutions of the minimal models. The corresponding ghost-number operator g is given by the sum of the ghost-number operators of each resolution. The differential ∂ acting on the ghost-number-N subspace of the resolution is given by the sum of differentials of each complex $C_{h_i}^{\star}$. It is obvious that the complex $C_{\mathbf{h}}^{\star}$ is exact except at the $F_{0,\mathbf{h}}$ module, where the cohomology is given by the product

$$M_{\mathbf{h},0} = \otimes_{i=1}^r M_{h_i,0}$$

of the chiral modules of each minimal model. Hence, we can represent the character

$$\chi_{\mathbf{h},0}(q,u) \equiv \mathrm{Tr}_{M_{\mathbf{h},0}}(q^{L[0]-c/24} u^{J[0]}) \tag{29}$$

of $M_{\mathbf{h},0}$ as the product of characters

$$\chi_{\mathbf{h},0}(q,u) = \prod_i \chi_{h_i,0}(q,u).$$

According to the discussion at the end of Sec. 1, we obtain the resolution and character for the product of an arbitrary irreducible module of minimal models by acting on $C^\star_{\mathbf{h}}$ by the spectral flow operators

$$U^{-\mathbf{t}} = \prod_i U_i^{-t_i}$$

of the minimal models. Hence, we can label the resolutions, modules, and characters by the pairs of vectors $(\mathbf{h},\mathbf{t})$ from the set

$$\tilde{\Delta} = \{(\mathbf{h},\mathbf{t})|h_i = 0,\dots,[\mu_i/2]-1,\\ t_i = 0,\dots,\mu_i-1, \quad i = 1,\dots,r\}.$$

Equivalently, we can use the set

$$\tilde{\Delta}' = \{(\mathbf{h},\mathbf{t})|h_i = 0,\dots,\mu_i-2,\\ t_i = 0,\dots,h_i,\ i = 1,\dots,r\}.$$

It is also clear that the R-sector resolutions, modules, and characters are generated from the NS sector ones by the spectral flow

$$U^{-\mathbf{v}/2} = \prod_{i=1}^r U_i^{-1/2},$$

where $\mathbf{v} = (1,\dots,1)$.

The same free-field realization can be used in the right-moving sector. Therefore, the sets of screening vectors $\bar{\mathfrak{R}}$ and $\bar{\mathfrak{R}}^*$ have to be fixed in the right-moving sector. This can be done in many ways, the only restriction is that the corresponding cohomology space has to be isomorphic to the space of states of the product of the minimal models in the right-moving sector. Therefore, $\bar{\mathfrak{R}}$ and $\bar{\mathfrak{R}}^*$ are defined modulo the $O(r,r)$ transformations that leave the matrix of scalar products $\langle \mathbf{s}_i, \mathbf{s}_j^*\rangle$ unchanged. In what follows, we set

$$\bar{\mathfrak{R}} = \mathfrak{R}, \quad \bar{\mathfrak{R}}^* = \mathfrak{R}^*. \tag{30}$$

Hence, we can use the same complex to describe the irreducible modules in the right-moving sector.

3.2. Free-field realization and Calabi–Yau extension

It is well known that a product of minimal models cannot be applied straightforwardly to describe the string theory on a complex dimension D Calabi–Yau manifold. First, one has to require that

$$\sum_i c_i = 3D.$$

Second, the so-called simple-current orbifold $CY_{\boldsymbol{\mu}}$ [3, 30, 31] of the product of minimal models has to be constructed. The currents of the $N=2$ Virasoro superalgebra of this model are given by the sum of currents of each minimal model:

$$G^\pm(z) = \sum_i G_i^\pm,\\ J(z) = \sum_i J_i, \quad T(z) = \sum_i T_i. \tag{31}$$

The left-moving (as well as the right-moving) sector of $CY_{\boldsymbol{\mu}}$ is given by projecting the space of states on the subspace of integer $J[0]$-charges and organizing the projected space into the orbits $[\mathbf{h},\mathbf{t}]$ under the spectral flow operator

$$U^{\mathbf{v}} = \prod_{i=1}^r U_i,$$

see [31].

The partition function in the NS sector of the $CY_{\boldsymbol{\mu}}$ sigma-model is the diagonal modular invariant of the spectral flow orbit characters restricted to the subset of integer $J[0]$-charges. From the standpoint of the $N=2$ Virasoro superalgebra representations, there is no difference as to which of the sets $\tilde{\Delta}$ or $\tilde{\Delta}'$ we use to parameterize the orbit characters (although their free-field realizations are different). In what follows, we combine these two sets into the extended set

$$\Delta = \{(\mathbf{h},\mathbf{t})|h_i = 0,\dots,\mu_i-2,\\ t_i = 0,\dots,\mu_i-1,\ i = 1,\dots,r\}$$

and take this extension into account by a corresponding multiplier («field identification») [2].

The orbit characters (with the restriction to the integer-charge subspace) can be written in the explicit form such that the structure of the simple-current extension becomes clear [3, 31]:

$$\mathrm{ch}_{\mathbf{h},\mathbf{t}}(q,u) = \frac{1}{\kappa^2}\sum_{n,m=0}^{\kappa-1} \mathrm{Tr}_{M_{\mathbf{h},\mathbf{t}}}(U^{n\mathbf{v}} q^{(L[0]-c/24)} u^{J[0]} \times \\ \times \exp(i2\pi m J[0]) U^{-n\mathbf{v}}) = \\ = \frac{1}{\kappa^2}\sum_{n,m=0}^{\kappa-1} \chi_{\mathbf{h},\mathbf{t}+n\mathbf{v}}(\tau, \upsilon+m), \tag{32}$$

where

$$q = \exp(i2\pi\tau), \quad u = \exp(i2\pi\upsilon), \quad \kappa = lcm\{\mu_i\}.$$

The partition function of the $CY_{\boldsymbol{\mu}}$ model is given by

$$Z_{CY}(q,\bar{q}) = \frac{1}{2^r} \sum_{[\mathbf{h},\mathbf{t}]\in\Delta_{CY}} \kappa|\,\mathrm{ch}_{[\mathbf{h},\mathbf{t}]}(q)|^2, \qquad (33)$$

where Δ_{CY} denotes the subset of Δ restricted to the space of integer $J[0]$ charges and $[\mathbf{h},\mathbf{t}]$ denotes the spectral flow orbit of the point $(\mathbf{h},\mathbf{t})$. The factor $1/2^r$ corresponds to the extended set Δ of irreducible modules and κ is the length of the orbit $[\mathbf{h},\mathbf{t}]$. In the general case, the orbits with different lengths could appear, but we do not consider these cases in order to avoid the problem of the fixed-point resolution [5, 30, 31].

3.3. Free-field realization of Gepner models

The Gepner models [2] of Calabi–Yau superstring compactification are given by the (generalized) Gliozzi–Scherk–Olive projection [2, 3] applied to the product of the space of states of the $CY_{\boldsymbol{\mu}}$ model and the space of states of external fermions and bosons describing space–time degrees of freedom of the string. In the framework of the simple-current extension formalism, Gepner's construction has been further developed in [30–32].

We introduce the so-called supersymmetrized (Green–Schwartz) characters [2, 3]

$$\mathrm{Ch}_{[\mathbf{h},\mathbf{t}]}(q,u) = \frac{1}{4\kappa^2} \sum_{n,m=0}^{2\kappa-1} \mathrm{Tr}_{(M_{\mathbf{h},\mathbf{t}}\otimes\Phi)}(U_{tot}^{m/2} \times \\ \times \exp(i\pi n J_{tot}[0]) q^{(L_{tot}-c_{tot}/24)} u^{J_{tot}[0]} U_{tot}^{-m/2}), \qquad (34)$$

where the trace is calculated in the product of the $M_{\mathbf{h},\mathbf{t}}$ and the Fock module Φ generated by the external (space–time) fermions and bosons in the NS sector, $J_{tot}[0]$ and $L_{tot}[0]$ are zero modes of the total $U(1)$ current and stress-energy tensor, which includes the contributions from space–time degrees of freedom,

$$c_{tot} = c + \frac{3}{2}(8-2D) = 12$$

is the total central charge, and U_{tot} is the total spectral flow operator acting in the product $M_{\mathbf{h},\mathbf{t}} \otimes \Phi$.

The modular-invariant Gepner model partition function is given by [2, 3, 31]

$$Z_{Gep}(q,\bar{q}) = \frac{1}{2^r}(\mathrm{Im}\,\tau)^{-(4-D/2)} \times \\ \times \sum_{[\mathbf{h},\mathbf{t}]\in\Delta_{CY}} \kappa|\,\mathrm{ch}_{[\mathbf{h},\mathbf{t}]}(q)|^2. \qquad (35)$$

4. THE ISHIBASHI STATES IN FOCK MODULES

The boundary states to be constructed in what follows can be regarded as bilinear forms on the space of states of the model. It is understood in what follows that the right-moving sector of the model is realized by the free fields $\bar{X}_i(\bar{z}), \bar{X}_i^*(\bar{z}), \bar{\psi}_i(\bar{z}), \bar{\psi}_i^*(\bar{z})$, $i = 1, \ldots, r$, and the right-moving $N = 2$ super-Virasoro algebra is given by the formulas similar to (2).

There are two types of boundary states preserving the $N = 2$ super-Virasoro algebra [33], usually called the B-type

$$(L[n] - \bar{L}[-n])|B\rangle\rangle = (J[n] + \bar{J}[-n])|B\rangle\rangle = 0, \\ (G^+[r] + i\eta\bar{G}^+[-r])|B\rangle\rangle = \\ = (G^-[r] + i\eta\bar{G}^-[-r])|B\rangle\rangle = 0, \qquad (36)$$

and A-type states

$$(L[n] - \bar{L}[-n])|A\rangle\rangle = \\ = (J[n] - \bar{J}[-n])|A\rangle\rangle = 0, \\ (G^+[r] + i\eta\bar{G}^-[-r])|A\rangle\rangle = \\ = (G^-[r] + i\eta\bar{G}^+[-r])|A\rangle\rangle = 0, \qquad (37)$$

where $\eta = \pm 1$.

In the tensor product of the left-moving Fock module $F_{\mathbf{p},\mathbf{p}^*}$ and right-moving Fock module $\bar{F}_{\bar{\mathbf{p}},\bar{\mathbf{p}}^*}$, we construct the simplest states satisfying conditions (36) and (37). We call these states the Fock-space Ishibashi [34] states.

4.1. B-type Ishibashi states in the Fock module

In the NS sector, we consider the following ansatz for fermions:

$$(\psi_i^*[r] - i\eta\Omega_{ij}\bar{\psi}_j^*[-r])|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^*,\eta,B\rangle\rangle = 0, \\ (\psi_i[r] - i\eta\Omega_{ij}^*\bar{\psi}_j[-r])|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^*,\eta,B\rangle\rangle = 0, \qquad (38)$$

where Ω_{ij}, Ω_{ij}^* are arbitrary nondegenerate matrices. Substituting these relations in (36) and using (27) and (31), we find

$$\Omega_{ik}\Omega_{in}^* = \delta_{kn}, \\ \Omega_{ij}d_i = d_j, \quad \Omega_{ij}^*d_i^* = d_j^* \\ \bar{p}_k = -\Omega_{jk}p_j - d_k, \quad \bar{p}_k^* = -\Omega_{jk}^*p_j^* - d_k^*, \\ (\Omega_{jk}X_j[n] + \bar{X}_k[-n] + \\ + d_k\delta_{n,0})|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^*,\eta,B\rangle\rangle = 0, \\ (\Omega_{jk}^*X_j^*[n] + \bar{X}_k^*[-n] + \\ + d_k^*\delta_{n,0})|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^*,\eta,B\rangle\rangle = 0, \qquad (39)$$

where $d_k = 1/\mu_k$, $d_k^* = 1$, and we combine these coefficients into the r-dimensional vectors

$$\mathbf{d} = (d_1, \ldots, d_r), \quad \mathbf{d}^* = (d_1^*, \ldots, d_r^*).$$

It is helpful to rewrite the boundary conditions in toric coordinates on the target space:

$$\begin{aligned} \theta_i[n] &= \frac{i}{\sqrt{2\mu_i}}(X_i^*[n] - \mu_i X_i[n]), \\ R_i[n] &= \frac{1}{\sqrt{2\mu_i}}(X_i^*[n] + \mu_i X_i[n]), \\ \gamma_i[s] &= \frac{i}{\sqrt{2\mu_i}}(\psi_i^*[s] - \mu_i \psi_i[s]), \\ \sigma_i[s] &= \frac{1}{\sqrt{2\mu_i}}(\psi_i^*[s] + \mu_i \psi_i[s]). \end{aligned} \tag{40}$$

Then Eqs. (38) and (39) become

$$\begin{aligned} &\left(\sigma_i[s] - \frac{i\eta}{2}\left(\sqrt{\frac{\mu_j}{\mu_i}}\Omega_{ij} + \sqrt{\frac{\mu_i}{\mu_j}}\Omega_{ij}^*\right)\bar{\sigma}_j[-s] - \right. \\ &\left. - \frac{\eta}{2}\left(\sqrt{\frac{\mu_j}{\mu_i}}\Omega_{ij} - \sqrt{\frac{\mu_i}{\mu_j}}\Omega_{ij}^*\right)\bar{\gamma}_j[-s]\right)|B\rangle\rangle = 0, \\ &\left(\gamma_i[s] + \frac{\eta}{2}\left(\sqrt{\frac{\mu_j}{\mu_i}}\Omega_{ij} - \sqrt{\frac{\mu_i}{\mu_j}}\Omega_{ij}^*\right)\bar{\sigma}_j[-s] - \right. \\ &\left. - \frac{i\eta}{2}\left(\sqrt{\frac{\mu_j}{\mu_i}}\Omega_{ij} + \sqrt{\frac{\mu_i}{\mu_j}}\Omega_{ij}^*\right)\bar{\gamma}_j[-s]\right)|B\rangle\rangle = 0, \\ &\left(\bar{R}_j[-n] + \frac{1}{2}\left(\sqrt{\frac{\mu_i}{\mu_j}}\Omega_{ij}^* + \sqrt{\frac{\mu_j}{\mu_i}}\Omega_{ij}\right)R_i[n] - \right. \\ &- \frac{i}{2}\left(\sqrt{\frac{\mu_i}{\mu_j}}\Omega_{ij}^* - \sqrt{\frac{\mu_j}{\mu_i}}\Omega_{ij}\right)\theta_i[n] + \\ &\left. \qquad + \sqrt{\frac{2}{\mu_j}}\delta_{n,0}\right)|B\rangle\rangle = 0, \\ &\left(\bar{\theta}_j[-n] + \frac{i}{2}\left(\sqrt{\frac{\mu_i}{\mu_j}}\Omega_{ij}^* - \sqrt{\frac{\mu_j}{\mu_i}}\Omega_{ij}\right)R_i[n] + \right. \\ &\left. + \frac{1}{2}\left(\sqrt{\frac{\mu_i}{\mu_j}}\Omega_{ij}^* + \sqrt{\frac{\mu_j}{\mu_i}}\Omega_{ij}\right)\theta_i[n]\right)|B\rangle\rangle = 0. \end{aligned} \tag{41}$$

Because the toric coordinates (θ_i, R_i) are real, we must impose the reality constraint

$$\Omega_{ij}^* = \frac{\mu_j}{\mu_i}\bar{\Omega}_{ij}. \tag{42}$$

The linear Fock-space B-type Ishibashi state in the NS sector is given by the standard expression [35, 36]

$$\begin{aligned} |\mathbf{p}, \mathbf{p}^*, \Omega, \eta, B\rangle\rangle = \prod_{n=1} \exp\left(-\frac{1}{n}(X_i^*[-n]\Omega_{ik}^*\bar{X}_k[-n] + \right. \\ \left. + X_i[-n]\Omega_{ik}\bar{X}_k^*[-n])\right) \times \\ \times \prod_{r=1/2} \exp(i\eta(\psi_i^*[-r]\Omega_{ik}^*\bar{\psi}_k[-r] + \\ + \psi_i[-r]\Omega_{ik}\bar{\psi}_k^*[-r])) \times \\ \times |\mathbf{p}, \mathbf{p}^*, -\Omega^T\mathbf{p} - \mathbf{d}, -(\Omega^*)^T\mathbf{p}^* - \mathbf{d}^*\rangle. \end{aligned} \tag{43}$$

4.2. *A*-type Ishibashi states in the Fock module

The A-type Ishibashi states in the Fock module can be found similarly. The linear ansatz for the fermions has the form

$$\begin{aligned} (\psi_i^*[r] - i\eta\Upsilon_{ij}\bar{\psi}_j[-r])|\mathbf{p}, \mathbf{p}^*, \bar{\mathbf{p}}, \bar{\mathbf{p}}^*, \eta, A\rangle\rangle = 0, \\ (\psi_i[r] - i\eta\Upsilon_{ij}^*\bar{\psi}_j^*[-r])|\mathbf{p}, \mathbf{p}^*, \bar{\mathbf{p}}, \bar{\mathbf{p}}^*, \eta, A\rangle\rangle = 0, \end{aligned} \tag{44}$$

where Υ_{ij} and Υ_{ij}^* are arbitrary nondegenerate matrices. Substituting these relations in (37) and using (2), we find

$$\begin{aligned} &\Upsilon_{ik}\Upsilon_{in}^* = \delta_{kn}, \\ &\Upsilon_{ij}d_i = d_j^*, \quad \Upsilon_{ij}^*d_j^* = d_i, \\ &\bar{p}_k = -\Upsilon_{jk}^*p_j^* - d_k, \quad \bar{p}_k^* = -\Upsilon_{jk}p_j - d_k^*, \\ &(\Upsilon_{jk}X_j[n] + \bar{X}_k^*[-n] + d_k^*\delta_{n,0}) \times \\ &\qquad \times |\mathbf{p}, \mathbf{p}^*, \bar{\mathbf{p}}, \bar{\mathbf{p}}^*, \eta, A\rangle\rangle = 0, \\ &(\Upsilon_{jk}^*X_j^*[n] + \bar{X}_k[-n] + d_k\delta_{n,0}) \times \\ &\qquad \times |\mathbf{p}, \mathbf{p}^*, \bar{\mathbf{p}}, \bar{\mathbf{p}}^*, \eta, A\rangle\rangle = 0. \end{aligned} \tag{45}$$

In toric coordinates (40), the conditions become

$$\begin{aligned} &\left(\sigma_i[s] - \frac{i\eta}{2}\left(\frac{\Upsilon_{ij}}{\sqrt{\mu_j\mu_i}} + \sqrt{\mu_j\mu_i}\Upsilon_{ij}^*\right)\bar{\sigma}_j[-s] + \right. \\ &\left. + \frac{\eta}{2}\left(\frac{\Upsilon_{ij}}{\sqrt{\mu_i\mu_j}} - \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*\right)\bar{\gamma}_j[-s]\right)|A\rangle\rangle = 0, \\ &\left(\gamma_i[s] + \frac{\eta}{2}\left(\frac{\Upsilon_{ij}}{\sqrt{\mu_i\mu_j}} - \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*\right)\bar{\sigma}_j[-s] + \right. \\ &\left. + \frac{i\eta}{2}\left(\frac{\Upsilon_{ij}}{\sqrt{\mu_j\mu_i}} + \sqrt{\mu_j\mu_i}\Upsilon_{ij}^*\right)\bar{\gamma}_j[-s]\right)|A\rangle\rangle = 0, \\ &\left(\bar{R}_j[-n] + \frac{1}{2}\left(\frac{\Upsilon_{ij}}{\sqrt{\mu_j\mu_i}} + \sqrt{\mu_j\mu_i}\Upsilon_{ij}^*\right)R_i[n] + \right. \\ &+ \frac{i}{2}\left(\frac{\Upsilon_{ij}}{\sqrt{\mu_i\mu_j}} - \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*\right)\theta_i[n] + \\ &\left. \qquad + 2d_j^*\delta_{n,0}\right)|A\rangle\rangle = 0, \\ &\left(\bar{\theta}_j[-n] + \frac{i}{2}\left(\frac{\Upsilon_{ij}}{\sqrt{\mu_i\mu_j}} - \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*\right)R_i[n] - \right. \\ &\left. - \frac{1}{2}\left(\frac{\Upsilon_{ij}}{\sqrt{\mu_j\mu_i}} + \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*\right)\theta_i[n]\right)|A\rangle\rangle = 0. \end{aligned} \tag{46}$$

The reality constraint takes the form

$$\Upsilon_{ij}^* = \frac{1}{\mu_i\mu_j}\bar{\Upsilon}_{ij}. \tag{47}$$

The linear A-type Ishibashi state in the NS sector is given similarly to the B-type one,

$$
\begin{aligned}
&|\mathbf{p},\mathbf{p}^*,\Upsilon,\eta,A\rangle\rangle = \\
&\quad = \prod_{n=1} \exp\left(-\frac{1}{n}(X_i[-n]\Upsilon_{ik}\bar{X}_k[-n]+ \right.\\
&\qquad \left. + X^*_i[-n]\Upsilon^*_{ik}\bar{X}^*_k[-n])\right) \times \\
&\quad \times \prod_{r=1/2} \exp(i\eta(\psi_i[-r]\Upsilon_{ik}\bar{\psi}_k[-r] + \\
&\qquad + \psi^*_i[-r]\Upsilon^*_{ik}\bar{\psi}^*_k[-r])) \times \\
&\quad \times |\mathbf{p},\mathbf{p}^*,-(\Upsilon^*)^T\mathbf{p}^* - \mathbf{d}, -\Upsilon^T\mathbf{p}-\mathbf{d}^*\rangle. \qquad (48)
\end{aligned}
$$

5. PERMUTATION ISHIBASHI STATES IN THE PRODUCT OF MINIMAL MODELS

5.1. *B*-type permutation Ishibashi states

The free-field realizations of the irreducible modules described in Secs. 1 and 2 and the constructions in (43) and (48) suggest that the Ishibashi states in the product of minimal models can also be represented by the free fields. We consider the following superposition of B-type Fock-modules Ishibashi states (43):

$$
\begin{aligned}
&|I_{\mathbf{h}},\Omega,\eta,B\rangle\rangle = \delta(\Omega\mathbf{h}-\mathbf{h}) \times \\
&\qquad \times \sum_{(\mathbf{p},\mathbf{p}^*)\in\Gamma_{\mathbf{h}}} c_{\mathbf{p},\mathbf{p}^*}|\mathbf{p},\mathbf{p}^*,\Omega,\eta,B\rangle\rangle, \qquad (49)
\end{aligned}
$$

where the coefficients $c_{\mathbf{p},\mathbf{p}^*}$ are arbitrary and the summation is performed over the momenta of the butterfly resolution $C^\star_{\mathbf{h}}$. Because the partition function is diagonal, the delta-function $\delta(\Omega\mathbf{h}-\mathbf{h})$ has been inserted. It is clear that this state satisfies relations (36).

Before the Gliozzi–Scherk–Olive projection, the closed-string states of the model that can interact with Ishibashi state (49) come from the product of the left-moving and right-moving Fock modules

$$F_{\mathbf{p},\mathbf{p}^*} \otimes \bar{F}_{-\Omega^T\mathbf{p}-\mathbf{d},-(\Omega^*)^T\mathbf{p}^*-\mathbf{d}^*},$$

where $(\mathbf{p},\mathbf{p}^*) \in \Gamma_{\mathbf{h}}$. The left-moving modules in superposition (49) constitute the butterfly resolution $C^\star_{\mathbf{h}}$, whose cohomology is given by the module $M_{\mathbf{h}}$. What about the Fock modules in the right-moving sector? To have a nontrivial interaction with the states in the model, the right-moving Fock modules must also form the product of resolutions of minimal models (18). But this contradicts the relations between left-moving and right-moving momenta in (39). This contradiction may be resolved if we allow the right-moving Fock modules to form the product of resolutions each of which is dual to (18). The dual resolution $\tilde{C}^\star_h$ to the minimal-model resolution (18) is given by the diagram

$$
\begin{array}{cccccccccc}
 & & \vdots & & \vdots & & & & & \\
 & & \downarrow & & \downarrow & & & & & \\
\ldots & \to & \bar{F}_{-1-\frac{1}{\mu},-1-h-\mu} & \to & \bar{F}_{-\frac{1}{\mu},-1-h-\mu} & & & & & \\
 & & \downarrow & & \downarrow & & & & & \\
\ldots & \to & \bar{F}_{-1-\frac{1}{\mu},-1-h} & \to & \bar{F}_{-\frac{1}{\mu},-1-h} & & & & & \\
 & & & & & \searrow & & & & \\
 & & & & & & \bar{F}_{1-\frac{1}{\mu},-1-h+\mu} & \to & \bar{F}_{2-\frac{1}{\mu},-1-h+\mu} & \to \ldots \\
 & & & & & & \downarrow & & \downarrow & \\
 & & & & & & \bar{F}_{1-\frac{1}{\mu},-1-h+2\mu} & \to & \bar{F}_{2-\frac{1}{\mu},-1-h+2\mu} & \to \ldots \\
 & & & & & & \downarrow & & \downarrow & \\
 & & & & & & \vdots & & \vdots &
\end{array}
\qquad (50)
$$

(here, h is an integer taking values from 0 to $\mu-2$). The arrows in this diagram are given by the same operators as in diagram (18).

Hence, the right-moving Fock modules have to form the dual resolution

$$\tilde{C}^\star_{\mathbf{h}} = \bigotimes_{i=1}^{r} \tilde{C}^\star_{h_i}$$

and the matrices Ω^T and $(\Omega^*)^T$ have to map the set of left-moving momenta $\Gamma_{\mathbf{h}}$ onto a set of momenta $\bar{\Gamma}_{\mathbf{h}}$ that has to be isomorphic to $\Gamma_{\mathbf{h}}$. Therefore, we conclude that Ω^T has to be an element of the direct product of the permutation groups $\mathfrak{N}_{r_i}$ on r_i-elements

$$\Omega \in \mathfrak{N}_{r_1\ldots r_N} = \mathfrak{N}_{r_1} \otimes \mathfrak{N}_{r_2} \ldots \otimes \mathfrak{N}_{r_N}, \qquad (51)$$

which are determined by the sets $r_1, \ldots, r_N$ of coinciding elements in the vector $\boldsymbol{\mu}$. In other words, it is implied here that

$$\mu_1 = \ldots = \mu_{r_1}, \quad \mu_{r_1+1} = \ldots = \mu_{r_1+r_2}, \ldots$$

In view of (42), we also have

$$\Omega^*_{ij} = \Omega_{ij}. \qquad (52)$$

Therefore, relations (41) take the form

$$\begin{aligned} &(\sigma_i[s] - i\eta\Omega_{ij}\bar{\sigma}_j[-s])|B\rangle\rangle = 0, \\ &(\gamma_i[s] - i\eta\Omega_{ij}\bar{\gamma}_j[-s])|B\rangle\rangle = 0, \\ &\left(\bar{R}_j[-n] + \Omega_{ij}R_i[n] + \sqrt{\frac{2}{\mu_j}}\delta_{n,0}\right)|B\rangle\rangle = 0, \\ &(\bar{\theta}_j[-n] + \Omega_{ij}\theta_i[n])|B\rangle\rangle = 0. \end{aligned} \qquad (53)$$

Hence, the ith minimal model in the right-moving sector interacts with the $\Omega^{-1}(i)$th minimal model in the left-moving sector.

With the matrix Ω fixed by (51), we can define the coefficients $c_{\mathbf{p},\mathbf{p}^*}$ from the BRST-invariance condition. It is a straightforward generalization of the condition found for the $N=2$ minimal models in [22]. To formulate this condition, we must describe the total space of states of the model in terms of free fields.

For this, we first form the product of complexes $C^\star_{\mathbf{h}} \otimes \tilde{C}^\star_{\mathbf{h}}$ to build the complex

$$\ldots \to \mathbf{C}^{-2}_{\mathbf{h}} \to \mathbf{C}^{-1}_{\mathbf{h}} \to \mathbf{C}^{0}_{\mathbf{h}} \to \mathbf{C}^{+1}_{\mathbf{h}} \to \ldots, \qquad (54)$$

which is graded by the sum of the ghost numbers $g+\bar{g}$ and, for an arbitrary ghost number I, the space $\mathbf{C}^I_{\mathbf{h}}$ is given by the sum of products of the Fock modules from the resolution $C^\star_{\mathbf{h}}$ and $\tilde{C}^\star_{\mathbf{h}}$ such that $g+\bar{g}=I$. The differential δ of the complex is defined by the differentials ∂ and $\bar{\partial}$ of the complexes $C^\star_{\mathbf{h}}$ and $\tilde{C}^\star_{\mathbf{h}}$,

$$\delta|v_g \otimes \bar{v}_{\bar{g}}\rangle = |\partial v_g \otimes \bar{v}_{\bar{g}}\rangle + (-1)^g|v_g \otimes \bar{\partial}\bar{v}_{\bar{g}}\rangle, \qquad (55)$$

where $|v_g\rangle$ is an arbitrary vector from the complex $C^\star_{\mathbf{h}}$ with ghost number g, while $|\bar{v}_{\bar{g}}\rangle$ is an arbitrary vector from the complex $\tilde{C}^\star_{\mathbf{h}}$ with ghost number $\bar{g}$ and $g+\bar{g}=I$. The cohomology of complex (54) is nonzero only at grade 0 and is given by the product of irreducible modules

$$M_{\mathbf{h}} \otimes \bar{M}_{\mathbf{h},\mathbf{t}=2\mathbf{h}},$$

where $\bar{M}_{\mathbf{h},\mathbf{t}=2\mathbf{h}}$ is the product of antichiral modules of minimal models.

The Ishibashi state that we seek can be considered a linear functional on the Hilbert space of the product of models; it then has to be an element of the homology group. Therefore, the BRST-invariance condition for the state can be formulated as follows.

We define the action of the differential δ on the state $|I_{\mathbf{h}}, \Omega, \eta, B\rangle\rangle$ as

$$\begin{aligned} &\langle\langle \delta^*(I_{\mathbf{h}}, \Omega, \eta, B)|v_g \otimes \bar{v}_{\bar{g}}\rangle \equiv \\ &\equiv \langle\langle I_{\mathbf{h}}, \Omega, \eta, B|\delta_{g+\bar{g}}|v_g \otimes \bar{v}_{\bar{g}}\rangle, \end{aligned} \qquad (56)$$

where $v_g \otimes \bar{v}_{\bar{g}}$ is an arbitrary element from $\mathbf{C}^{g+\bar{g}}_{\mathbf{h}}$. Then the BRST-invariance condition means that

$$\delta^*|I_{\mathbf{h}}, \Omega, \eta, B\rangle\rangle = 0. \qquad (57)$$

As a straightforward generalization of Theorem 2 in [22], we find that superposition (49) satisfies BRST-invariance condition (57) if the coefficients $c_{\mathbf{p},\mathbf{p}^*}$ take values ± 1 in accordance with the expression

$$c_{\mathbf{p},\mathbf{p}^*} = \sqrt{2}\cos\left((2g_{\mathbf{p},\mathbf{p}^*}+1)\frac{\pi}{4}\right)c_{0,\mathbf{h}}, \qquad (58)$$

where $g_{\mathbf{p},\mathbf{p}^*}$ is the ghost number.

Thus, superposition (49) respects the singular vector structure of the product of minimal N=2 Virasoro algebra representations and gives an explicit construction of permutation Ishibashi states. We also note that the BRST-condition does not fix the phase of the overall coefficient $c_{0,\mathbf{h}}$.

We now consider the closed-string transition amplitude between a pair of permutation Ishibashi states with the permutations Ω' and Ω. It is given by

$$\begin{aligned} &\langle\langle I_{\mathbf{h}'}, \Omega', \eta, B|(-1)^{g(\Omega',\Omega)} \times \\ &\times q^{L[0]-c/24}u^{J[0]}|I_{\mathbf{h}}, \Omega, \eta, B\rangle\rangle = \\ &= \delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}'-\mathbf{h}')\delta(\Omega\mathbf{h}-\mathbf{h}) \times \\ &\times \sum_{(\mathbf{p},\mathbf{p}^*)\in\Gamma_{\mathbf{h}}} (-1)^{g(\Omega',\Omega)}|c_{\mathbf{p},\mathbf{p}^*}|^2 \times \\ &\times \delta(\Omega'\Omega^{-1}\mathbf{p}-\mathbf{p})\delta(\Omega'\Omega^{-1}\mathbf{p}^*-\mathbf{p}^*) \times \\ &\times \langle\langle \mathbf{p}, \mathbf{p}^*, \Omega', \eta|(-1)^{g(\Omega',\Omega)}q^{L[0]-c/24}u^{J[0]} \times \\ &\times |\mathbf{p}, \mathbf{p}^*, \Omega, \eta, B\rangle\rangle. \end{aligned} \qquad (59)$$

Due to the insertion of $(-1)^{g(\Omega',\Omega)}$, the amplitude is calculated according to the ghost number of the intermediate closed-string states, and the ghost number operator $g(\Omega', \Omega)$ depends on the permutation matrices. To simplify the calculation, we set the number N of permutation groups equal to 1 (and hence $\mu_1 = \ldots = \mu_r = \mu$). Due to the factor

$$\delta(\Omega'\Omega^{-1}\mathbf{p}-\mathbf{p})\delta(\Omega'\Omega^{-1}\mathbf{p}^*-\mathbf{p}^*),$$

the summation is restricted to the subspace of $\Gamma_{\mathbf{h}}$ that is invariant with respect to the permutation $\Omega'\Omega^{-1}$. This allows us to write

$$\langle\langle \mathbf{p},\mathbf{p}^*,\Omega',\eta|(-1)^{g(\Omega',\Omega)}q^{L[0]-c/24}u^{J[0]}|\mathbf{p},\mathbf{p}^*,\Omega,\eta,B\rangle\rangle = \\ = q^{(1/2)(|\Xi|_1(2p_1^*p_1+p_1+p_1^*/\mu)+\ldots|\Xi|_{\nu(\Xi)}(2p^*_{\nu(\Xi)}p_{\nu(\Xi)}+p_{\nu(\Xi)}+p^*_{\nu(\Xi)}/\mu)-c/24}\times \\ \times u^{(|\Xi|_1(p_1^*/\mu-p_1)+\ldots+|\Xi|_{\nu(\Xi)}(p^*_{\nu(\Xi)}/\mu-p_{\nu(\Xi)})}\text{(oscillator contribution)}, \quad (60)$$

where $|\Xi|_i$ is the length of ith cycle of the permutation $\Xi\equiv\Omega'\Omega^{-1}$ and $\nu(\Xi)$ is the number of cycles of the permutation.

The oscillator contribution can be conveniently calculated for the bosons and fermions separately. The bosonic contribution can be calculated as follows. First, from (43), we have

$$\prod_{a,c}\langle \mathbf{p},\mathbf{p}^*,-(\Omega')^T\mathbf{p}-\mathbf{d},-(\Omega')^T\mathbf{p}^*-\mathbf{d}^*|\times \\ \times\prod_{n=1}\exp\left(-\frac{1}{n}X^*_a[n]\bar{X}_{\Omega'(a)}[n]\right)\times \\ \times\prod_{m=1}\exp\left(-\frac{q^m}{m}X_c[-m]\bar{X}^*_{\Omega(c)}[-m]\right)\times \\ \times|\mathbf{p},\mathbf{p}^*,-\Omega^T\mathbf{p}-\mathbf{d},-\Omega^T\mathbf{p}^*-\mathbf{d}^*\rangle = \\ =\prod_{n=1}\sum_{k_1,\ldots,k_r=0}\sum_{l_1,\ldots,l_r=0}\frac{1}{k_1!\ldots k_r!l_1!\ldots l_r!}\times \\ \times\frac{q^{n(l_1+\ldots+l_r)}}{n^{k_1+\ldots+l_r}}\times \\ \times\langle \mathbf{p},\mathbf{p}^*,-(\Omega')^T\mathbf{p}-\mathbf{d},-(\Omega')^T\mathbf{p}^*-\mathbf{d}^*|\times \\ \times(X_1^*[n])^{k_1}(\bar{X}_{\Omega'(1)}[n])^{k_1}\ldots\times \\ \times(X_r^*[n])^{k_r}(\bar{X}_{\Omega'(r)}[n])^{k_r}\times \\ \times(X_1[-n])^{l_1}(\bar{X}^*_{\Omega(1)}[-n])^{l_1}\ldots\times \\ \times(X_r[-n])^{l_r}(\bar{X}^*_{\Omega(r)}[-n])^{l_r}\times \\ \times|\mathbf{p},\mathbf{p}^*,-\Omega^T\mathbf{p}-\mathbf{d},-\Omega^T\mathbf{p}^*-\mathbf{d}^*\rangle = \\ =\prod_{n=1}(1-q^{n|\Xi|_1})^{-1}\ldots(1-q^{n|\Xi|_{\nu(\Xi)}})^{-1}. \quad (61)$$

Similarly,

$$\prod_{a,c}\langle \mathbf{p},\mathbf{p}^*,-(\Omega')^T\mathbf{p}-\mathbf{d},-(\Omega')^T\mathbf{p}^*-\mathbf{d}^*|\times \\ \times\prod_{n=1}\exp\left(-\frac{1}{n}X_a[n]\bar{X}^*_{\Omega'(a)}[n]\right)\times \\ \times\prod_{m=1}\exp\left(-\frac{q^m}{m}(X^*_c[-m]\bar{X}_{\Omega(c)}[-m])\right)\times \\ \times|\mathbf{p},\mathbf{p}^*,-\Omega^T\mathbf{p}-\mathbf{d},-\Omega^T\mathbf{p}^*-\mathbf{d}^*\rangle = \\ =\prod_{n=1}(1-q^{n|\Xi|_1})^{-1}\ldots(1-q^{n|\Xi|_{\nu(\Xi)}})^{-1}. \quad (62)$$

The first part of the fermionic contribution is given by

$$\prod_{b,d}\langle \mathbf{p},\mathbf{p}^*,-(\Omega')^T\mathbf{p}-\mathbf{d},-(\Omega')^T\mathbf{p}^*-\mathbf{d}^*|\times \\ \times\prod_{r=1/2}(1-i\eta\psi^*_b[r]\bar{\psi}_{\Omega^{-1}(b)}[r])\times \\ \times\prod_{s=1/2}(1+i\eta u^{-1}q^s\psi_d[-s]\bar{\psi}^*_{\Omega(d)}[-s])\times \\ \times|\mathbf{p},\mathbf{p}^*,-\Omega^T\mathbf{p}-\mathbf{d},-\Omega^T\mathbf{p}^*-\mathbf{d}^*\rangle = \\ =\prod_{s=1/2}\sum_{k=0}^{r}\mathrm{Tr}(\Xi)|_{\wedge^k V}u^{-k}q^{ks}, \quad (63)$$

where $\mathrm{Tr}(\Xi)|_{\wedge^k V}$ denotes the trace of the matrix $\Xi=\Omega'\Omega^{-1}$ acting by permuting components in the space $\wedge^k V$ for the r-dimensional real vector space V. We see that the last expression can be rewritten similarly to (61),

$$\prod_{b,d}\langle \mathbf{p},\mathbf{p}^*,-(\Omega')^T\mathbf{p}-\mathbf{d},-(\Omega')^T\mathbf{p}^*-\mathbf{d}^*|\times \\ \times\prod_{r=1/2}(1-i\eta(\psi^*_b[r]\bar{\psi}_{(\Omega')^{-1}(b)}[r])\times \\ \times\prod_{s=1/2}(1+i\eta u^{-1}q^s\psi_d[-s]\bar{\psi}^*_{\Omega(d)}[-s])\times \\ \times|\mathbf{p},\mathbf{p}^*,-\Omega^T\mathbf{p}-\mathbf{d},-\Omega^T\mathbf{p}^*-\mathbf{d}^*\rangle = \\ =\prod_{s=1/2}(1-(-1)^{|\Xi|_1}u^{-|\Xi|_1}q^{s|\Xi|_1})\ldots\times \\ \times(1-(-1)^{|\Xi|_{\nu(\Xi)}}u^{-|\Xi|_{\nu(\Xi)}}q^{s|\Xi|_{\nu(\Xi)}}). \quad (64)$$

Analogously,

$$\prod_{a,c}\langle \mathbf{p},\mathbf{p}^*,-(\Omega')^T\mathbf{p}-\mathbf{d},-(\Omega')^T\mathbf{p}^*-\mathbf{d}^*|\times \\ \times\prod_{r=1/2}(1-i\eta\psi^*_a[r]\bar{\psi}_{(\Omega')^{-1}(a)}[r])\times \\ \times\prod_{s=1/2}(1+i\eta u^{-1}q^s\psi_c[-s]\bar{\psi}^*_{\Omega(c)}[-s])\times \\ \times|\mathbf{p},\mathbf{p}^*,-\Omega^T\mathbf{p}-\mathbf{d},-\Omega^T\mathbf{p}^*-\mathbf{d}^*\rangle = \\ =\prod_{s=1/2}(1-(-1)^{|\Xi|_1}u^{|\Xi|_1}q^{s|\Xi|_1})\ldots\times \\ \times(1-(-1)^{|\Xi|_{\nu(\Xi)}}u^{|\Xi|_{\nu(\Xi)}}q^{s|\Xi|_{\nu(\Xi)}}). \quad (65)$$

Collecting the results, we obtain

$$\begin{aligned}&\langle\langle I_{\mathbf{h}'},\Omega',\eta,B|(-1)^{g(\Omega',\Omega)}q^{L[0]-c/24}u^{J[0]}|I_{\mathbf{h}},\Omega,\eta,B\rangle\rangle=\\&=\delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{h}-\mathbf{h})|c_{0,\mathbf{h}}|^2\times\\&\times\sum_{(\mathbf{p},\mathbf{p}^*)\in\Gamma_{\mathbf{h}}}(-1)^{g(\Omega',\Omega)}\delta(\Xi\mathbf{p}-\mathbf{p})\delta(\Xi\mathbf{p}^*-\mathbf{p}^*)\times\\&\times q^{(1/2)(\sum_{i=1}^{\nu(\Xi)}|\Xi|_i(2p_i^*p_i+p_i+p_i^*/\mu))-c/24}u^{\sum_{i=1}^{\nu(\Xi)}|\Xi|_i(p_i^*/\mu-p_i)}\times\\&\times\prod_{i=1}^{\nu(\Xi)}\prod_{n=1}(1-(-1)^{|\Xi|_i}u^{-|\Xi|_i}q^{(n-1/2)|\Xi|_i})\times\\&\times(1-(-1)^{|\Xi|_i}u^{|\Xi|_i}q^{(n-1/2)|\Xi|_i})(1-q^{n|\Xi|_i})^{-2}.\end{aligned}\tag{66}$$

The transition amplitude between

$$|I_{\mathbf{h}'},\Omega',-\eta,B\rangle\rangle$$

and

$$|I_{\mathbf{h}},\Omega,\eta,B\rangle\rangle$$

is given by a similar expression. Indeed, the change $\eta\to-\eta$ affects only the fermionic contribution in (63)–(65) and therefore

$$\begin{aligned}&\langle\langle I_{\mathbf{h}'},\Omega',-\eta,B|(-1)^{g(\Omega',\Omega)}q^{L[0]-c/24}u^{J[0]}|I_{\mathbf{h}},\Omega,\eta,B\rangle\rangle=\delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{h}-\mathbf{h})|c_{0,\mathbf{h}}|^2\times\\&\times\sum_{(\mathbf{p},\mathbf{p}^*)\in\Gamma_{\mathbf{h}}}(-1)^{g(\Omega',\Omega)}\delta(\Xi\mathbf{p}-\mathbf{p})\delta(\Xi\mathbf{p}^*-\mathbf{p}^*)q^{(1/2)(\sum_{i=1}^{\nu(\Xi)}|\Xi|_i(2p_i^*p_i+p_i+p_i^*/\mu))-c/24}u^{\sum_{i=1}^{\nu(\Xi)}|\Xi|_i(p_i^*/\mu-p_i)}\times\\&\times\prod_{i=1}^{\nu(\Xi)}\prod_{n=1}(1-u^{-|\Xi|_i}q^{(n-1/2)|\Xi|_i})(1-u^{|\Xi|_i}q^{(n-1/2)|\Xi|_i})(1-q^{n|\Xi|_i})^{-2}.\end{aligned}\tag{67}$$

We now fix the dependence of the ghost-number operator on the permutation matrices. Taking representation (20) into account, we find that the amplitude is given by the product of minimal-model characters if

$$g(\Omega',\Omega)=\sum_{i=1}^{\nu(\Xi)}g_i.\tag{68}$$

Thus, the ghost number receives the contribution g_i from the ith invariant subspace of $\Gamma_{\mathbf{h}}$. In other words, we consider the space of intermediate closed-string states as the product of the $\nu(\Xi)$ minimal-model butterfly resolutions (18). Hence, the amplitude is given by the product of minimal-model characters as

$$\begin{aligned}&\langle\langle I_{\mathbf{h}'},\Omega',\eta,B|(-1)^{g(\Omega',\Omega)}q^{L[0]-c/24}u^{J[0]}\times\\&\times|I_{\mathbf{h}},\Omega,\eta,B\rangle\rangle=\\&=\delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{h}-\mathbf{h})|c_{0,\mathbf{h}}|^2\times\\&\times\prod_{i=1}^{\nu(\Xi)}\exp\left(-i\pi(1-|\Xi|_i)\frac{h_i}{\mu}\right)\times\\&\times\chi_{h_i}\left(\tau,\upsilon+\frac{1-|\Xi|_i}{2}\right),\end{aligned}\tag{69}$$

where f is the fermion-number operator and we use the relation

$$\begin{aligned}&\mathrm{Tr}_{M_{h_i}}((-1)^{(1-|\Xi|_i)f}q^{(L_i[0]-c_i/24)}u^{J_i[0]})=\\&=\exp\left(-i\pi(1-|\Xi|_i)\frac{h_i}{\mu}\right)\times\\&\times\chi_{h_i}\left(\tau,\upsilon+\frac{1-|\Xi|_i}{2}\right).\end{aligned}\tag{70}$$

Analogously,

$$\begin{aligned}&\langle\langle I_{\mathbf{h}'},\Omega',-\eta,B|(-1)^{g(\Omega',\Omega)}q^{L[0]-c/24}u^{J[0]}\times\\&\times|I_{\mathbf{h}},\Omega,\eta,B\rangle\rangle=\\&=\delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{h}-\mathbf{h})|c_{0,\mathbf{h}}|^2\times\\&\times\prod_{i=1}^{\nu(\Xi)}\exp\left(-i\pi\frac{h_i}{\mu}\right)\chi_{h_i}\left(\tau,\upsilon+\frac{1}{2}\right).\end{aligned}\tag{71}$$

It was mentioned in Sec. 2 that the irreducible representations are generated by the spectral flow action. Hence, for an arbitrary module $M_{\mathbf{h},\mathbf{t}}$, $(\mathbf{h},\mathbf{t})\in\Delta$, the Ishibashi state is generated by the action of the spectral flow operators on Ishibashi state (49). It is easy to verify that the state

$$|I_{\mathbf{h},\mathbf{t}},\Omega,\eta,B\rangle\rangle=\prod_i U_i^{t_i}\bar{U}_i^{-t_i}|I_{\mathbf{h}},\Omega,\eta,B\rangle\rangle\tag{72}$$

satisfies the B-type boundary conditions if

$$\Omega\mathbf{t}-\mathbf{t}=0.\tag{73}$$

However, we must take into account that the right-moving space of states of the model is governed by the dual butterfly resolutions (twisted by the right-moving spectral flow operators). A representative of the chiral primary field from the dual resolution is

$$U^h G^+\left[\frac{1}{2}-h\right]\dots G^+\left[-\frac{1}{2}\right]\left|-\frac{1}{\mu},-1-h\right\rangle \sim \left|-\frac{1+h}{\mu},-1\right\rangle. \qquad (74)$$

Thus, the highest-weight vectors of the model are given by the products of the minimal-model states as

$$\left|p_i=\frac{t_i}{\mu_i}\,,\quad p_i^*=h_i-t_i,\quad \bar{p}_i=-\frac{1+h_i-t_i-l}{\mu_i},\quad \bar{p}_i^*=-1-t_i-l\right\rangle. \qquad (75)$$

Therefore, Ishibashi states (72) nontrivially overlap with states (75) if, in addition to (73), we have

$$\begin{aligned} &h_{\Omega^{-1}(i)}=h_i,\\ &h_i-2t_i-l=0 \bmod \mu_i. \end{aligned} \qquad (76)$$

It is easy to see from (12) that this state satisfies boundary conditions (38), (39). Hence, (36) is fulfilled. The state is also BRST-closed because the spectral flow commutes with screening charges.

The transition amplitude between such states is a spectral-flow twist of amplitudes (69) and (71),

$$\begin{aligned} &\langle\langle I_{\mathbf{h}',\mathbf{t}'},\Omega',\eta,B|(-1)^{g(\Omega',\Omega)}q^{L[0]-c/24}u^{J[0]}\times\\ &\times|I_{\mathbf{h},\mathbf{t}},\Omega,\eta,B\rangle\rangle=\\ &=\delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{h}-\mathbf{h})\delta^{(\boldsymbol{\mu})}(\mathbf{t}-\mathbf{t}')\times\\ &\times|c_{0,\mathbf{h}}|^2\prod_{i=1}^{\nu(\Xi)}\exp\left(-i\pi(1-|\Xi|_i)\frac{h_i-2t_i}{\mu}\right)\times\\ &\times\chi_{h_i,t_i}\left(\tau,\upsilon+\frac{1-|\Xi|_i}{2}\right), \end{aligned} \qquad (77)$$

$$\begin{aligned} &\langle\langle I_{\mathbf{h}',\mathbf{t}'},\Omega',-\eta,B|(-1)^{g(\Omega',\Omega)}q^{L[0]-c/24}u^{J[0]}\times\\ &\times|I_{\mathbf{h},\mathbf{t}},\Omega,\eta,B\rangle\rangle=\\ &=\delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{h}-\mathbf{h})\delta^{(\boldsymbol{\mu})}(\mathbf{t}-\mathbf{t}')\times\\ &\times|c_{0,\mathbf{h}}|^2\prod_{i=1}^{\nu(\Xi)}\exp\left(-i\pi\frac{h_i-2t_i}{\mu}\right)\times\\ &\times\chi_{h_i,t_i}\left(\tau,\upsilon+\frac{1}{2}\right). \end{aligned} \qquad (78)$$

5.2. *A*-type permutation Ishibashi states

We consider the free-field representation for A-type Ishibashi states. It is obvious that A-type Ishibashi states are given by superpositions like (49).

Similarly to the B-type case, we can conclude that the matrix Υ^T is proportional to the element of the permutation group $\mathfrak{N}_{r_1\dots r_N}$. More precisely,

$$\begin{aligned} &\Upsilon=\mu_1\Omega_1\otimes\dots\otimes\mu_N\Omega_N,\\ &\Upsilon^*=\frac{1}{\mu_1}\Omega_1\otimes\dots\otimes\frac{1}{\mu_N}\Omega_N, \end{aligned} \qquad (79)$$

where $\Omega_i\in\mathfrak{N}_{r_i}$, $i=1,\dots,N$.

Boundary conditions (46) take the form that is mirror to (53),

$$\begin{aligned} &(\sigma_i[s]-i\eta\Omega_{ij}\bar{\sigma}_j[-s])|A\rangle\rangle=0,\\ &(\gamma_i[s]+i\eta\Omega_{ij}\bar{\gamma}_j[-s])|A\rangle\rangle=0,\\ &\left(\bar{R}_j[-n]+\Omega_{ij}R_i[n]+\sqrt{\frac{2}{\mu_j}}\delta_{n,0}\right)|A\rangle\rangle=0,\\ &(\bar{\theta}_j[-n]-\Omega_{ij}\theta_i[n])|A\rangle\rangle=0. \end{aligned} \qquad (80)$$

The BRST-condition for A-type states is slightly different from that in the B-type case. The reason is that in accordance with (44) and (45), the application of one of the left-moving BRST charges, e.g., Q_i^+, to an A-type state gives the right-moving BRST charge $\bar{Q}^-_{G^{-1}(i)}$ multiplied by μ_i, as opposed to the B-type case. In fact, we are free to arbitrarily rescale the right-moving BRST charges because this does not change the cohomology of the complex in the right-moving sector and the cohomology of the total complex (54). Hence, we define the right-moving BRST charges such that this effect is canceled,

$$\begin{aligned} &\bar{S}_i^+(\bar{z})=\frac{i\eta}{\mu_i}\mathbf{s}_i\bar{\psi}^*\exp(\mathbf{s}_i\bar{X}^*)(\bar{z}),\\ &\bar{S}_i^-(\bar{z})=i\eta\mu_i\mathbf{s}_i^*\bar{\psi}\exp(\mu_i\mathbf{s}_i^*\bar{X})(\bar{z}),\\ &\bar{Q}_i^\pm=\oint d\bar{z}\bar{S}_i^\pm(\bar{z}). \end{aligned} \qquad (81)$$

As a result, the BRST-invariant A-type Ishibashi state $|I_{\mathbf{h}},\Omega,\eta,A\rangle\rangle$ is given by a formula similar to (49) and (58) with the restriction $\delta(\Omega\mathbf{h}-\mathbf{h})$, and, similarly to the B-type case, the phase of the coefficient $c_{0,\mathbf{h}}$ is also arbitrary.

The A-type version of transition amplitude (69) can be calculated similarly to the B-type case, with the result given by

$$\langle\langle I_{\mathbf{h}'}, \Omega', \eta, A|(-1)^{g(\Omega',\Omega)} q^{L[0]-c/24} u^{J[0]} \times \times |I_{\mathbf{h}}, \Omega, \eta, A\rangle\rangle = = \delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{h}-\mathbf{h})|c_{0,\mathbf{h}}|^2 \times \times \prod_{i=1}^{\nu(\Xi)} \exp\left(-i\pi(1-|\Xi|_i)\frac{h_i}{\mu}\right) \times \times \chi_{h_i}\left(\tau, \upsilon + \frac{1-|\Xi|_i}{2}\right), \quad (82)$$

where $\Xi = \Omega'\Omega^{-1}$ and we have set the number N of permutation groups equal to 1 for simplicity.

For an arbitrary module $M_{\mathbf{h},\mathbf{t}}$, $(\mathbf{h},\mathbf{t}) \in \Delta$, the A-type Ishibashi state is generated by the action of spectral flow operators. It is easy to verify that the state

$$|I_{\mathbf{h},\mathbf{t}}, \Omega, \eta, A\rangle\rangle = \prod_i U_i^{t_i}\bar{U}_i^{t_i}|I_{\mathbf{h}}, \Omega, \eta, A\rangle\rangle \quad (83)$$

satisfies the A-type boundary condition if the spectral flow parameter $\mathbf{t}$ satisfies (73). Although the right-moving space of states of the model is governed by the dual butterfly resolutions (twisted by right-moving spectral flow operators), the only restrictions on $\mathbf{h}$ and $\mathbf{t}$ are

$$\Omega\mathbf{h} = \mathbf{h}, \quad \Omega\mathbf{t} = \mathbf{t}. \quad (84)$$

The corresponding transition amplitude is given similarly to the B-type case as

$$\langle\langle I_{\mathbf{h}',\mathbf{t}'}, \Omega', \eta, A|(-1)^{g(\Omega',\Omega)} q^{L[0]-c/24} u^{J[0]} \times \times |I_{\mathbf{h},\mathbf{t}}, \Omega, \eta, A\rangle\rangle = = \delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{h}-\mathbf{h})\delta^{\boldsymbol{\mu}}(\mathbf{t}-\mathbf{t}')|c_{0,\mathbf{h}}|^2 \times \times \prod_{i=1}^{\nu(\Xi)} \exp\left(-i\pi\,(1-|\Xi|_i)\,\frac{h_i-2t_i}{\mu}\right) \times \times \chi_{h_i,t_i}\left(\tau, \upsilon + \frac{1-|\Xi|_i}{2}\right), \quad (85)$$

$$\langle\langle I_{\mathbf{h}',\mathbf{t}'}, \Omega', -\eta, A|(-1)^{g(\Omega',\Omega)} q^{L[0]-c/24} u^{J[0]} \times \times |I_{\mathbf{h},\mathbf{t}}, \Omega, \eta, A\rangle\rangle = = \delta(\mathbf{h}-\mathbf{h}')\delta(\Omega'\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{h}-\mathbf{h})\delta^{\boldsymbol{\mu}}(\mathbf{t}-\mathbf{t}')|c_{0,\mathbf{h}}|^2 \times \times \prod_{i=1}^{\nu(\Xi)} \exp\left(-i\pi\frac{h_i-2t_i}{\mu}\right)\chi_{h_i,t_i}\left(\tau, \upsilon + \frac{1}{2}\right). \quad (86)$$

Thus, expressions (77) and (85) reproduce the corresponding results in [6] correctly (with the correct fermionic contribution). This allows us to use the solution of Cardy's constraint found for permutation branes in [6] to construct the free-field representation of permutation branes.

6. FREE-FIELD REPRESENTATION OF PERMUTATION BRANES IN THE GEPNER MODEL

6.1. A-type boundary states in the Calabi–Yau extension

It has already been noted that a product of minimal models cannot be applied straightforwardly to describe string theory on a Calabi–Yau manifold in the bulk. Instead, the so-called simple-current orbifold, whose partition function is a diagonal modular-invariant partition function with respect to orbit characters (32), must be introduced. The extension of this technique to conformal field theory with a boundary has been developed in [1, 5, 6, 8, 31].

As we have seen, the BRST invariance fixes the free-field permutation Ishibashi states up to an arbitrary constant $c_{\mathbf{h},\mathbf{t}}$. Hence, our problem is to apply the (simple current) orbifold construction and Cardy's constraint to the superposition of free-field permutation Ishibashi states to fix the coefficients $c_{\mathbf{h},\mathbf{t}}$. Fortunately, Cardy's constraint for the permutation branes has been found in [6]. It therefore suffices only to quote the solution.

Thus, the free-field realization of permutation A-type branes can be given as follows. We start from the spectral-flow-invariant permutation boundary states

$$|[\boldsymbol{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, A\rangle\rangle = \frac{\alpha}{\kappa^2}\sum_{(\mathbf{h},\mathbf{t})\in\tilde{\Delta}} \delta(\Omega\mathbf{h}-\mathbf{h})\delta(\Omega\mathbf{t}-\mathbf{t}) \times \times W^{\mathbf{h},\mathbf{t}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\Omega} \sum_{m,n=0}^{\kappa-1} \exp\left(i2\pi nJ[0]\right) \times \times U^{m\mathbf{v}}\bar{U}^{m\mathbf{v}}|I_{\mathbf{h},\mathbf{t}}, \Omega, \eta, A\rangle\rangle \quad (87)$$

(α is the normalization constant). They are labeled by the spectral-flow orbit classes $[\boldsymbol{\Lambda}, \boldsymbol{\lambda}]$ of the vectors $(\boldsymbol{\Lambda}, \boldsymbol{\lambda}) \in \Delta$. The coefficients $W^{\mathbf{h},\mathbf{t}}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\Omega}$ that solve Cardy's constraint are given by [6]

$$
\begin{gathered}
W^{\mathbf{h},\mathbf{t}}_{\mathbf{\Lambda},\boldsymbol{\lambda},\Omega} = \\
= \prod_{a=1}^{\nu(\Omega)} S_{(\Lambda_a,\lambda_a)(h_a,t_a)} (S_{(0,0),(h_a,t_a)})^{-1/2|\Omega|_a}, \\
S_{(\Lambda_a,\lambda_a)(h_a,t_a)} = \\
= S_{\Lambda_a,h_a} \exp(i\pi \frac{(h_a-2t_a)(\Lambda_a-2\lambda_a)}{\mu}), \\
S_{\Lambda_a,h_a} = \frac{\sqrt{2}}{\mu} \sin\left(\pi \frac{(h_a+1)(\Lambda_a+1)}{\mu}\right).
\end{gathered}
\tag{88}
$$

The summation over n implements the $J[0]$-projection, while the summation over m introduces spectral-flow-twisted sectors. This state depends only on the spectral-flow orbit class. Moreover, the restriction of an integer $J[0]$-charge of the orbits $[\mathbf{\Lambda}, \boldsymbol{\lambda}]$ is necessary for the self-consistency of expression (87).

We now apply the inner automorphism group of the Gepner model to construct additional boundary states. Namely, we use the operator

$$
\exp\left(-i2\pi \sum_i \phi_i J_i[0]\right) \in U(1)^r
$$

to generate new boundary states. We consider the properties of the state

$$
\begin{gathered}
|[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, A\rangle\rangle_{\boldsymbol{\phi}} \equiv \\
\equiv \exp\left(-i2\pi \sum_i \phi_i J_i[0]\right) |[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, A\rangle\rangle.
\end{gathered}
\tag{89}
$$

It satisfies the conditions similar to (37) except the relations for fermionic fields,

$$
\begin{gathered}
\left(G^{\pm}[r] + i\eta \sum_i \exp(\pm i2\pi\phi_{\Omega(i)}) \bar{G}^{\mp}_{\Omega(i)}[-r]\right) \times \\
\times |[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, A\rangle\rangle_{\boldsymbol{\phi}} = 0, \\
\left(\psi^*_i[r] - i\eta\mu_i \exp(i2\pi\phi_{\Omega(i)}) \bar{\psi}_{\Omega(i)}[-r]\right) \times \\
\times |[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, A\rangle\rangle_{\boldsymbol{\phi}} = 0, \\
\left(\psi_i[r] - i\frac{\eta}{\mu_i} \exp(-i2\pi\phi_{\Omega(i)}) \bar{\psi}^*_{\Omega(i)}[-r]\right) \times \\
\times |[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, A\rangle\rangle_{\boldsymbol{\phi}} = 0.
\end{gathered}
\tag{90}
$$

This state is not invariant under the diagonal $N = 2$ Virasoro algebra unless

$$
\phi_i \in Z, \quad i = 1, \ldots, r. \tag{91}
$$

Hence, the group $U(1)^r$ reduces to Z^r. It is worth noting that the case where all the ϕ_i are half-integer can be ignored because of the cancelation achieved by the $\eta \to -\eta$ redefinition. It is easy to see directly that the states thus obtained are given by

$$
\begin{gathered}
|[\mathbf{\Lambda}, \boldsymbol{\lambda}]\Omega, \eta, A\rangle\rangle_{\boldsymbol{\phi}} = \frac{\alpha}{\kappa^2} \sum_{(\mathbf{h},\mathbf{t})\in\Delta_\Omega} W^{\mathbf{h},\mathbf{t}}_{[\Lambda,\lambda]} \times \\
\times \sum_{m,n=0}^{\kappa-1} \exp(i2\pi n J[0]) \times \\
\times \exp\left(-i2\pi m \sum_i \phi_i \frac{c_i}{3}\right) U^{m\mathbf{v}} \bar{U}^{m\mathbf{v}} \times \\
\times \exp\left(-i2\pi \sum_i \phi_i \frac{h_i - 2t_i}{\mu_i}\right) |I_{\mathbf{h},\mathbf{t}}, \Omega, \eta, A\rangle\rangle = \\
= \frac{\alpha}{\kappa^2} \sum_{(\mathbf{h},\mathbf{t})\in\Delta_\Omega} \prod_{e=1}^{\nu(\Omega)} S_{\Lambda_e,h_e} (S_{0,h_e})^{-|\Omega|_e/2} \times \\
\times \exp\left(i\pi \frac{(\Lambda_e - 2\lambda_e)(h_e - 2t_e)}{\mu}\right) \times \\
\times \exp\left(-i\pi \frac{(h_e - 2t_e)}{\mu} \sum_{a=1}^{|\Omega|_e} 2\phi_{e+a}\right) \times \\
\times \exp\left(i4\pi m \sum_{a=1}^{|\Omega|_e} \frac{\phi_{e+a}}{\mu}\right) \times \\
\times \sum_{m,n=0}^{\kappa-1} \exp(i2\pi n J[0]) U^{m\mathbf{v}} \bar{U}^{m\mathbf{v}} |I_{\mathbf{h},\mathbf{t}}, \Omega, \eta, A\rangle\rangle.
\end{gathered}
\tag{92}
$$

Hence, the boundary states can be parameterized by

$$
\begin{gathered}
|[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, A\rangle\rangle = \\
= \exp\left(-i2\pi \sum_i \lambda_i J_i[0]\right) |[\mathbf{\Lambda}, 0], \Omega, \eta, A\rangle\rangle,
\end{gathered}
\tag{93}
$$

such that different boundary states are labeled by different values of

$$
|\lambda|_e = \sum_{a=1}^{|\Omega|_e} \lambda_{e+a}, \quad e = 1, \ldots, \nu(\Omega),
$$

and spectral-flow-invariant boundary states are recovered when

$$
\frac{2}{\mu} \sum_{e=1}^{\nu(\Omega)} |\lambda|_e \in Z. \tag{94}
$$

6.2. *B*-type boundary states in the Calabi–Yau extension

We let Δ_Ω denote the subset of Δ satisfying (73) and (76). Then for an arbitrary pair of vectors $(\mathbf{\Lambda}, \boldsymbol{\lambda}) \in \Delta_{CY}$, the free-field realization of a spectral-flow-invariant B-type boundary state is given by

$$|[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, B\rangle\rangle = \frac{\alpha}{\kappa^2} \sum_{(\mathbf{h},\mathbf{t})\in\Delta_\Omega} W^{\mathbf{h},\mathbf{t}}_{[\mathbf{\Lambda},\boldsymbol{\lambda}],\Omega} \times \\ \times \sum_{m,n=0}^{\kappa-1} \exp\left(i2\pi n J[0]\right) \times \\ \times U^{m\mathbf{v}} \bar{U}^{-m\mathbf{v}} |I_{\mathbf{h},\mathbf{t}}, \Omega, \eta, B\rangle\rangle, \quad (95)$$

where the coefficients $W^{\mathbf{h},\mathbf{t}}_{\mathbf{\Lambda},\boldsymbol{\lambda},\Omega}$ are given by (88). It can be verified that this state depends only on the spectral-flow orbit class $[\mathbf{\Lambda}, \boldsymbol{\lambda}]$ of the vectors $(\mathbf{\Lambda}, \boldsymbol{\lambda})$. It is also obvious that $[\mathbf{\Lambda}, \boldsymbol{\lambda}]$ has to be restricted to the set of $J[0]$-integer charges by the reasons similar to those given in the A-type case.

The other boundary states are generated by the inner automorphism group of the Gepner model, similarly to the A-type case. Namely, the state

$$|[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, B\rangle\rangle_{\boldsymbol{\phi}} \equiv \\ \equiv \exp\left(-i2\pi \sum_i \phi_i J_i[0]\right) |[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, B\rangle\rangle \quad (96)$$

satisfies the conditions similar to (90) and is not invariant under the diagonal $N = 2$ Virasoro algebra unless

$$\phi_i \in Z, \quad i = 1, \ldots, r. \quad (97)$$

Hence, the group $U(1)^r$ reduces to Z^r and we can parameterize the boundary states by

$$|[\mathbf{\Lambda}, \boldsymbol{\lambda}], \Omega, \eta, B\rangle\rangle = \\ = \exp\left(-i2\pi \sum_i \lambda_i J_i[0]\right) |[\mathbf{\Lambda}, 0], \Omega, \eta, B\rangle\rangle, \quad (98)$$

such that different boundary states are labeled by different values of

$$|\lambda|_e = \sum_{a=1}^{|\Omega|_e} \lambda_{e+a}, \quad e = 1, \ldots, \nu(\Omega).$$

In conclusion of this section, we make the following remarks. First, our free-field construction allows interpreting A/B-type gluing conditions (37) and (36) geometrically. Indeed, in terms of the free fields, the B-type gluing conditions, for example, are given by (53). Thus ± 1 eigenvalues of the permutation matrix Ω can be interpreted as labeling the Neumann and Dirichlet boundary conditions, while complex eigenvalues realize mixed boundary conditions [33]. This result seems to contradict the calculation of D-brane charges performed in Refs. [23, 24]. It has been found there that $D0$-branes correspond to transposition matrices permuting only one pair of minimal models. It follows from (53) that in this case, we have only one Dirichlet condition and the corresponding free-field boundary state gives a codimension-one D-brane. We do not know at the moment how to resolve or explain the contradiction. Perhaps, a more profound geometric investigation of the open-string spectrum in terms of the chiral de Rham complex has to be performed, but this requires additional investigation.

Second, we note that the free-field representations of permutation boundary states are determined modulo BRST-exact states satisfying A- or B-type boundary conditions. We interpret this ambiguity in the free-field representation as a result of adding brane–antibrane pairs annihilating under the tachyon condensation process [37]. Strictly speaking, the BRST-exact state ambiguity is not the usual brane–antibrane-pair ambiguity and has to be considered in a generalized sense, because BRST-exact states also contain states with opposite charges in the NS sector. In this context, the free-field representations of boundary states can be regarded as superpositions of branes flowing under the (generalized) tachyon condensation process to nontrivial boundary states in Gepner models. It is also important to note that automorphisms (22) give different free-field representations of boundary states because the corresponding butterfly resolutions are not invariant under these automorphisms. However, their cohomology are invariant. Hence, these different representations have to be identified and the free-field boundary state construction is to be considered in the sense of derived categories [38].

6.3. Free-field representation of permutation boundary states in Gepner models

It is completely clear from (34) and (35) how to incorporate the space–time degrees of freedom in our construction in order to obtain a free-field construction of permutation branes in the Calabi–Yau extension to the case of Gepner models. This is straightforward (see, e.g., [1, 7, 18, 31]), and we do not give the details here.

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