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PHOTON PROPAGATION IN A SUPERCRITICAL MAGNETIC FIELD

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We show that for the asymptotically strong (super-Schwinger) magnetic field B exceeding the critical value $B_{cr} = m^2c^3/eh = 4.4 \cdot 10^{13}$ Gs, the vacuum polarization effects become important not only in the γ-range, but also for softer electromagnetic quanta, including X-rays and optic photons, and for electromagnetic waves of the radio frequencies. This is a consequence of the linearly growing term $\asymp B/B_{cr}$ present in the vacuum polarization in the asymptotically strong magnetic field. The results may be essential in studying reflection, refraction, and splitting of X-rays, light and radio waves by magnetic fields of magnetars, and in considering emission of such waves by charged particles.

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1. INTRODUCTION

Although it is long since the refracting and birefringing properties of a strong magnetic field in the vacuum have been realized, their only essential consequences considered in a realistic astrophysical context remain the photon splitting effect [1] and the effect of photon capture [2–8]. Both effects are currently discussed mostly in application to electromagnetic radiation in the γ-range. They depend crucially on the deviation of the photon dispersion curve from its customary shape in the empty vacuum, $k_0^2 = |\mathbf{k}|^2$, where k_0 is the photon energy and $\mathbf{k}$ is its momentum. For the magnetic fields B below the Schwinger critical value,

$$B \leq B_{cr} = m^2c^3/eh = 4.4 \cdot 10^{13} \text{ Gs},$$

where m and e are the electron mass and charge, the only essential source of this deviation is the singular behavior of the polarization operator $\Pi_{\mu\nu}(k)$ near the creation thresholds of mutually independent electron and positron on Landau levels n, n' by a photon (the cyclotron resonance) [2–4] or an even stronger singular behavior of $\Pi_{\mu\nu}$ near the points of a mutually bound e^+e^--pair (the positronium atom) formation [5–7, 9]. To reach (at least the lower of) these positions, the photon must belong to the γ-ray range, with its energy above or of the order 1 MeV. For this reason, the effect of photon capture, with its transformation into an electron–positron pair, derived from the singular behavior of $\Pi_{\mu\nu}(k)$, applies mostly to the γ-quanta, as long as their propagation in a pulsar magnetosphere of traditional pulsars is concerned. It was estimated that the fields about $B = 0.1B_{cr}$ are sufficient to provide this effect [4] and to protect the positronium atom into which the captured γ-quantum is transformed against ionization by the accelerating electric field in the polar gap and by the thermal photons [5–9].

Also the Adler effect [1] of photon splitting $\gamma \to \gamma\gamma$ in such fields is usually discussed for γ-quanta [10–13]. There are two reasons why, again, the γ-range is important. The first is that the photon splitting becomes possible in the magnetic field because the deviation of the dispersion curve from the $k_0^2 = \mathbf{k}^2$ law opens a kinematical aperture for this process — the wider, the stronger the deviation (and the deviation is strong near the thresholds). In addition, there is a strong birefringence for the photons in the γ-range, because only one eigenvalue $\kappa_2(k)$ of the tensor $\Pi_{\mu\nu}$ is singular near the lowest ($n = n' = 0$) threshold, while the other two eigenvalues $\kappa_{1,3}(k)$ remain finite, until the next thresholds ($n = 0$, $n' = 1$ or $n = 1$, $n' = 0$) are reached. This implies that the photons of only one polarization mode are essentially affected by the medium. This birefringence leads to polarization selection rules in the photon

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splitting process, which are well pronounced. The second reason is dynamical. The matrix elements of the photon splitting are subject to the same resonant behavior near the thresholds as the polarization operator. The aforesaid explains why mainly the γ-range is first to be affected by the magnetized vacuum.

The situation changes considerably in passing to super-Schwinger magnetic fields $B \gg B_{cr}$, expected to exist in soft γ-ray repeaters and anomalous X-ray pulsars (see, e.g., Ref. [14]). In this asymptotic range, a linearly growing term proportional to B/B_{cr} appears in one of the eigenvalues, κ_2, of the polarization operator [15, 16], thus providing an extra large contribution (additional to the cyclotron resonance) to the refraction of the vacuum.

In Sec. 3, we study the consequences of this phenomenon for the photon propagation, basing on the first three leading contributions to the asymptotic expansion of the polarization operator eigenvalues for large B, obtained within the one-loop approximation. One of these consequences is a frequency-independent, but direction-sensitive, large refraction index for propagation nonparallel to the magnetic field in one (out of three) polarization modes in the kinematical domain far from the threshold. The corresponding strong polarization- and direction-sensitive refraction occurs for electromagnetic radiation of any frequency range, including X-ray, optic, and radio range.

This study is preceded by Sec. 2, where exact results concerning the electromagnetic radiation propagation in the magnetized vacuum are described. These follow only from the general properties of the relativistic, gauge, and charge invariance [17] and the Onsager theorem [18]. The results in Sec. 2 are valid irrespective of any approximation and the field strength, unless the opposite is explicitly indicated.

In the Appendix, the asymptotic expansion used in Sec. 3 is derived.

2. EXACT FACTS ABOUT ELECTROMAGNETIC EIGENMODES IN AN EXTERNAL MAGNETIC FIELD

There are three propagating eigenmodes corresponding to the vacuum excitations with photon quantum numbers in an external magnetic field B. The dispersion law, i.e., the dependence of the energy k_0 of the quantum (or the frequency in the wave) on its momentum $\mathbf{k}$, is given for each mode by a solution of the equation

$$k^2 = \kappa_i(k_0^2 - k_\parallel^2, k_\perp^2), \quad i = 1,2,3, \tag{1}$$

where $k_\parallel$ and $k_\perp$ are the respective momentum components parallel and perpendicular to the magnetic field $\mathbf{B}$ and k^2 is the photon 4-momentum squared,

$$k^2 = k_\perp^2 + k_\parallel^2 - k_0^2.$$

The κ_i in the right-hand sides in Eqs. (1) are eigenvalues of the polarization operator [2, 3, 17].

A general consequence of the relativistic covariance is that the eigenvalues depend on the two combinations of the momentum specified in (1). This implies that solutions of dispersion equations (1) have the general structure

$$k_0^2 = k_\parallel^2 + f_i(k_\perp^2), \quad i = 1,2,3, \tag{2}$$

and that the direction of the group velocity $\mathbf{v} = \partial k_0/\partial \mathbf{k}$ in each mode does not coincide (for $k_\perp \neq 0$) with that of the phase velocity $\mathbf{k}/k_0$. To see this, we calculate the components of the respective group velocities $v_\perp$ and $v_\parallel$ across and along the magnetic field $\mathbf{B}$ on solutions (2) of each dispersion equation (1),

$$\begin{aligned} v_\perp &\equiv \frac{\partial k_0}{\partial k_\perp} = \frac{k_\perp}{k_0}\frac{\partial k_0^2}{\partial k_\perp^2} = \\ &= \frac{k_\perp}{k_0}\frac{1 - \partial\kappa_i/\partial k_\perp^2}{1 + \partial\kappa_i/\partial(k_0^2 - k_\parallel^2)} = \frac{k_\perp}{k_0}\frac{df_i(k_\perp^2)}{dk_\perp^2}, \\ v_\parallel &\equiv \frac{\partial k_0}{\partial k_\parallel} = \frac{k_\parallel}{k_0}. \end{aligned} \tag{3}$$

It follows from (3) that the angle θ between the direction $\mathbf{v}$ of the electromagnetic energy propagation and the external magnetic field satisfies the relation

$$\frac{v_\perp}{v_\parallel} \equiv \operatorname{tg}\theta = \left(1 - \frac{\partial\kappa_i}{\partial k_\perp^2}\right)\left(1 + \frac{\partial\kappa_i}{\partial(k_0^2 - k_\parallel^2)}\right)^{-1} \operatorname{tg}\vartheta, \tag{4}$$

where ϑ is the angle between the photon momentum (phase velocity) and the external field, $\operatorname{tg}\vartheta \equiv k_\perp/k_\parallel$. The following statement holds: if the phase velocity $\mathbf{k}/k_0$ exceeds the velocity of light c, i.e., if $k_\perp^2 + k_\parallel^2 > k_0^2$ (or $f_i(k_\perp^2) < k_\perp^2$ in (2)), but the group velocity (3) does not, $v_\perp^2 + v_\parallel^2 \le 1$ (or $d^2 f_i(k_\perp^2)/(dk_\perp^2)^2 < 0$), then $\operatorname{tg}\theta < \operatorname{tg}\vartheta$. The conditions of this statement are fulfilled for the dispersion laws found within approximation-dependent calculations of the κ_i. For the super-Schwinger fields, treated within the one-loop approximation, this fact follows explicitly from equations in Sec. 3 below. Therefore, the photon tends to deviate closer to the magnetic field line.

It follows from the gauge invariance that

$$\kappa_i(0,0) = 0, \quad i = 1,2,3. \tag{5}$$

This property implies that for each mode, there always exists a dispersion curve with $f_i(0) = 0$, which passes through the origin in the $(k_0^2 - k_\parallel^2, k_\perp^2)$ plane. But only two of these three solutions may simultaneously correspond to physical massless particles, the photons. The third solution is a nonphysical degree of freedom, characteristic of gauge theories: in a magnetic field, a photon has two degrees of freedom, the same as in the empty vacuum. Which of the modes becomes nonphysical depends on the propagation direction and on the specific form of the function $f_i(k_\perp^2)$ in (2). We discuss this point for the super-Schwinger field limit in the next section. Massive branches of solutions of (1), with $f_i(0) > 0$, may also exist, despite (5). For them, the number of physical degrees of freedom is three, and hence all the three equations (1) can have physical solutions simultaneously (see, e.g., the positronium branches found in [7, 19, 20])

The refraction index n_i in mode i is

$$n_i \equiv \frac{|\mathbf{k}|}{k_0} = \left(1+\frac{\kappa_i}{k_0^2}\right)^{1/2} = \left(1+\frac{k_\perp^2 - f_i(k_\perp^2)}{k_0^2}\right)^{1/2}. \quad (6)$$

Unlike κ_i, the refraction index n_i is not a Lorentz scalar and may depend on two energy-momentum variables, after it is reduced to dispersion law (2). Gauge invariance property (5) implies that the refraction index (6) for parallel propagation, $k_\perp = 0$, is exactly equal to unity for the massless ($f_i(0) = 0$) branches in every mode,

$$n_i^\parallel = 1. \quad (7)$$

The electromagnetic wave propagating strictly along the external constant and homogeneous magnetic field propagates with the velocity of light c in the vacuum, the phase and group velocities coinciding in this case.

If, within a certain approximation, the eigenvalue κ_i is a linear function of its arguments with condition (5) satisfied, refraction index (6) for the corresponding dispersion law depends on a single combination of the photon energy and momentum, which is the propagation direction ϑ. This happens in a nonresonant situation, for instance, as described in the next section.

The polarizations of the modes are described in an approximation-independent way [3, 17] by the relations

$$\mathbf{e}^{(1)} = -\frac{\mathbf{k}_\perp}{k_\perp} k_0, \quad \mathbf{h}^{(1)} = \left(\frac{\mathbf{k}_\perp}{k_\perp} \times \mathbf{k}_\parallel\right), \quad (8)$$

$$\mathbf{e}_\perp^{(2)} = \mathbf{k}_\perp k_\parallel, \quad \mathbf{e}_\parallel^{(2)} = \frac{\mathbf{k}_\parallel}{k_\parallel}(k_\parallel^2 - k_0^2), \quad \mathbf{h}^{(2)} = -k_0 \left(\mathbf{k}_\perp \times \frac{\mathbf{k}_\parallel}{k_\parallel}\right), \quad (9)$$

$$\mathbf{e}^{(3)} = -k_0 \left(\frac{\mathbf{k}_\perp}{k_\perp} \times \frac{\mathbf{k}_\parallel}{k_\parallel}\right), \quad \mathbf{h}_\perp^{(3)} = -\frac{\mathbf{k}_\perp}{k_\perp} k_\parallel, \quad \mathbf{h}_\parallel{}^{(3)} = \frac{\mathbf{k}_\parallel}{k_\parallel} k_\perp, \quad (10)$$

where $\mathbf{e}^{(i)}$ and $\mathbf{h}^{(i)}$ are the electric and magnetic fields in the wave belonging to mode the number $i = 1, 2, 3$, the cross denotes the vector product, and boldface letters with the subscripts «$\parallel$» and «$\perp$» denote vectors along the directions parallel and perpendicular to the external magnetic field respectively. In the mode 1 wave, the electric field $\mathbf{e}$ is parallel to $\mathbf{k}_\perp$, in mode 2 it lies in the plane containing the vectors $\mathbf{k}$ and $\mathbf{B}$, and in mode 3 it is orthogonal to this plane, which means that mode 3 is always transversely polarized.

We note that the normalizations in Eqs. (8), (9), and (10) are different, and we can therefore judge about vanishing of some components compared to others within one equation, but not between different equations.

Concerning the direction of propagation, two cases are essentially different. If $k_\perp = 0$, we speak about longitudinal propagation. Otherwise, there exists a Lorentz boost along the external (constant and homogeneous) magnetic field, which does not change the value of the magnetic field and does not introduce an extra electric field, but nullifies $k_\parallel$. Hence, the general case of nonparallel propagation $k_\perp \neq 0$, $k_\parallel \neq 0$ is reduced to purely transversal propagation, $k_\parallel = 0$ (in the corresponding reference frame). One should keep in mind, however, that the above transformation changes the photon energy k_0 and should be treated with caution when one considers a field with curved force lines.

For transversal propagation, $\mathbf{k} \perp \mathbf{B}$ ($k_\parallel = 0$), modes 2 and 3 are transversely polarized ($\mathbf{e}^{(2),(3)} \perp \mathbf{k}$) in two mutually orthogonal planes, $\mathbf{e}^{(2)} \perp \mathbf{e}^{(3)}$, while mode 1 is longitudinally polarized ($\mathbf{e}^{(1)} \parallel \mathbf{k}$) with no magnetic field in it, $\mathbf{h}^{(1)} = 0$. It is expected not to correspond to a photon (depending on the dispersion law).

On the contrary, for longitudinal propagation, $\mathbf{k} \parallel \mathbf{B}$, ($k_\perp = 0$), modes 1 and 3 are transversely polarized ($\mathbf{e}^{(1,3)} \perp \mathbf{B}$) and their electric field vectors lie in mutually orthogonal planes, $\mathbf{e}^{(1)} \perp \mathbf{e}^{(3)}$, as they always do, while mode 2 is longitudinally polarized ($\mathbf{e}^{(2)} \parallel \mathbf{B}$) and does not contain a magnetic field, $\mathbf{h}^{(2)} = 0$. Mode 2 is then expected not to correspond to a photon, whereas mode 1 is a physical electromagnetic wave, which matches the electromagnetic wave of mode 3: together, they may form a circularly polarized transversal wave because of the degeneracy property

$$\kappa_1((k_0^2 - k_\parallel^2), 0) = \kappa_3((k_0^2 - k_\parallel^2), 0). \quad (11)$$

This relation reflects the cylindrical symmetry of the problem of a photon propagating along the external magnetic field.

Another remark of almost general character is in order. One might expect the possibility of the Cherenkov radiation by a charged particle moving in an optically dense medium formed by the magnetized vacuum. This effect (with the Cherenkov photons softer than $k_0 = 2m$) does not occur in known situations, however. We consider emission of a photon by an electron in a magnetic field, not accompanied by a change of its Landau quantum number, $n = n'$ (otherwise, that would be the cyclotron, and not Cherenkov, radiation). According to the kinematical analysis of the energy and momentum conservation in [21] (and to the study [21] of analyticity regions of the one-loop photon polarization operator in the electron-positron plasma in a magnetic field, calculated in [18]), the Cherenkov photon with $k_0 < 2m$ can only belong to the right lower sector

$$k_0^2 - k_\parallel^2 \leq 0, \quad k_\perp^2 \geq 0 \tag{12}$$

in the $(k_0^2 - k_\parallel^2, k_\perp^2)$ plane. The substantial reason for this is the degeneration of the electron energy with respect to the center-of-orbit position in the transversal plane. No dynamical calculations, hitherto known, provide penetration of photon dispersion curves into this sector. The only exception is the nonphysical situation due to exponentially strong external fields, to be mentioned in Sec. 3.2 below. We conclude that no Cherenkov emission of a photon softer than $k_0 = 2m$ is possible under standard conditions.

3. PHOTON DISPERSION IN A SUPER-SCHWINGER MAGNETIC FIELD

3.1. Asymptotic expansion of polarization tensor eigenvalues

In the asymptotic region of supercritical magnetic fields $B \gg B_{cr}$ and a restricted energy of longitudinal motion,

$$k_0^2 - k_\parallel^2 \ll (B/B_{cr})m^2,$$

the three eigenvalues $\varkappa_{1,2,3}(k)$ of the polarization operator (if it is calculated within the one-loop approximation as in [17, 22]) have the following behavior, derived from equations of Ref. [3] (see the Appendix),

$$\varkappa_1(k_0^2 - k_\parallel^2, k_\perp^2) = \frac{\alpha k^2}{3\pi}\left(\ln\frac{B}{B_{cr}} - C - 1.21\right), \tag{13}$$

$$\varkappa_2(k_0^2 - k_\parallel^2, k_\perp^2) = \frac{\alpha B m^2 (k_0^2 - k_\parallel^2)}{\pi B_{cr}} \times$$
$$\times \exp\left(-\frac{k_\perp^2}{2m^2}\frac{B_{cr}}{B}\right)\int_{-1}^{1}\frac{(1-\eta^2)d\eta}{4m^2-(k_0^2-k_\parallel^2)(1-\eta^2)}, \tag{14}$$

$$\varkappa_3(k_0^2 - k_\parallel^2, k_\perp^2) = \frac{\alpha k^2}{3\pi}\left(\ln\frac{B}{B_{cr}} - C\right) -$$
$$- \frac{\alpha}{3\pi}\left(0.21 k_\perp^2 - 1.21(k_0^2 - k_\parallel^2)\right). \tag{15}$$

Here, $\alpha = 1/137$ is the fine structure constant and $C = 0.577$ is the Euler constant. Equations (13) and (15) are accurate up to terms decreasing with B as $(B_{cr}/B)\ln(B/B_{cr})$ and faster. Equation (14) is accurate up to terms logarithmically growing with B. In $\varkappa_{1,3}$, we also took the limit

$$k_\perp^2 \ll (B/B_{cr})m^2,$$

which is not the case for $\varkappa_2$, where the factor $\exp(-k_\perp^2 B_{cr}/2m^2 B)$ is kept different from unity, because it is important near the cyclotron resonance, as explained in Sec. 3.2 below. The integral in (14) can readily be calculated, but we do not need its explicit form here.

The parts growing with B in $\varkappa_{1,2,3}$ were written in [16], their derivation from equations of Ref. [3] is traced in detail in [19, 20]. The linearly growing term in Eq. (14) was obtained in [15] in a different way using a two-dimensional (one time, one space) diagram technique developed to serve the asymptotic magnetic field regime. The logarithmic terms in the expressions above do not dominate over the constant terms unless exponentially large magnetic fields are included into consideration[1)]. The derivation of all terms in Eqs. (13), (14), and (15), including those that do not grow with B, is given in the Appendix using a straightforward method different from the one applied earlier in [19, 20]. The asymptotic expressions used in [13] do not coincide with ours, except for the term linear in B.

The limiting expressions (13), (14), and (15) do satisfy the exact properties (11) and (5).

In this paper, we only deal with the transparency region, $k_0^2 - k_\parallel^2 \leq 4m^2$ (i.e., with the kinematical domain

[1)] That would be unreasonable not only because such fields are hardly expected to exist in nature, but mainly because their consideration is beyond the scope of quantum electrodynamics: the logarithmically growing terms in (13) and (15) are associated with the absence of asymptotic freedom in QED (*cf.* analogous asymptotic behavior [23] in the Euler–Heisenberg effective Lagrangian).

where $\kappa_{1,2,3}$ are real), because we are interested in photons with $k_0 < 2m$, or even $k_0 \ll 2m$, which never reach the free pair creation threshold $k_0^2 - k_\parallel^2 = 4m^2$. The eigenvalue κ_2 in (14) has a singular branching point in the complex plane of the variable $(k_0^2 - k_\parallel^2)$ near the lowest pair creation threshold $(k_0^2 - k_\parallel^2)_{thr} = 4m^2$. Thresholds of creation of e^+e^--pairs with the electron and the positron on excited Landau levels $n, n' \neq 0$,

$$(k_0^2 - k_\parallel^2)_{thr}^{n,n'} = \\ = m^2 \left[\left(1 + n\frac{B}{B_{cr}}\right)^{1/2} + \left(1 + n'\frac{B}{B_{cr}}\right)^{1/2} \right]^2, \quad (16)$$

are shifted in the asymptotic regime to the infinitely remote region. Therefore, the eigenvalues $\kappa_{1,3}$, which are responsible for photons of such polarizations that can only create e^+e^--pairs with at least one charged particle in an excited Landau state, do not contain imaginary parts or singular branching points in this regime. On the other hand, the eigenvalue κ_2 has only one singular branching point, corresponding to the possibility of creation of the electron and positron in the lowest Landau states by the photon polarized as in mode 2. The singular threshold behavior of (14) near the point

$$k_0^2 - k_\parallel^2 = 4m^2 - \epsilon, \quad \epsilon > 0, \quad \epsilon \to 0$$

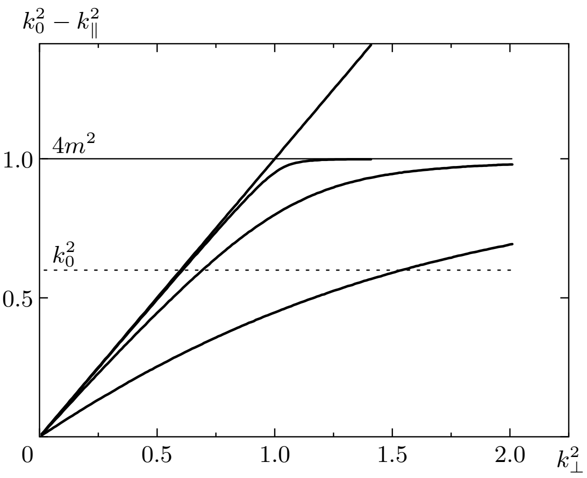
is

$$\kappa_2(k) \sim \frac{2\alpha B m^3}{B_{cr}} \exp\left(-\frac{k_\perp^2}{2m^2}\frac{B_{cr}}{B}\right) \times \\ \times \left(4m^2 - k_0^2 + k_\parallel^2\right)^{-1/2}. \quad (17)$$

As could be expected, this is the same as the behavior of the exact one-loop expression for $\kappa_2(k)$ [3] near this threshold, before the limiting transition to large fields has been performed.

3.2. Propagation of eigenmodes in the super-Schwinger field limit

If Eq. (13) for κ_1 is taken as the right-hand side of Eq. (1), the latter has only one solution, which is the trivial dispersion law $k^2 = 0$. With the relation $k^2 = 0$ satisfied, however, the 4-potential corresponding to the electromagnetic field of mode 1 becomes proportional to the photon 4-momentum vector k_μ, unless $k_\parallel = 0$ (see [3, 19, 20]). Therefore, for nonparallel propagation, mode 1 corresponds to only the gauge degree of freedom discussed in Sec. 2, with no real electromagnetic field associated with it.



A family of dispersion curves for mode 2 (solutions of Eq. (1) with Eq. (14) taken for the right-hand side) below the threshold $k_0^2 - k_\parallel^2 = 4m^2$. The values of the external magnetic field corresponding to the curves are (from left to right) $B = 10B_{cr}, 100B_{cr}, 1000B_{cr}$. The straight line is the light cone dispersion curve for $B = 0$. The dashed horizontal line marks the maximum to which the photon with the energy k_0 may proceed if $k_0 < 2m$. The variables along the axes are plotted in the units of $4m^2 \approx 1$ MeV^2

Solutions of Eq. (1) for the second mode $i = 2$ with Eq. (14) taken for κ_2 are plotted in the Figure for three values of the field B using MATHCAD code. These solutions are dominated by cyclotron resonance (17), which causes a strong deviation of the dispersion curves in the Figure from the shape $k^2 = 0$ (the light cone). As $k_\perp^2 \to \infty$ near the threshold on the dispersion curves, the quantity $k_\perp^2 B_{cr}/m^2 B$ must be kept different from zero even in the large-field limit under consideration.

Behavior of the dispersion curves of mode 2 near the threshold for super-Schwinger magnetic fields $B \gg B_{cr}$ is the same as for the «moderate» fields $B \leq B_{cr}$, and therefore it also presents the photon capture effect for photons harder than $2m$, known for such fields [4]: if we calculate (4) near the threshold $k_0^2 - k_\parallel^2 = 4m^2$ using Eq. (17) as κ_2 to obtain

$$\operatorname{tg}\theta = \frac{k_\perp}{k_\parallel}\frac{B_{cr}}{Bm^2}(4m^2 - k_0^2 + k_\parallel^2), \quad (18)$$

we conclude that the angle θ between the external magnetic field and the direction of the wave packet propagation in mode 2 tends to zero, the faster, the stronger the field. If the photon energy k_0 is slightly less than $2m$, the photon may be close to the threshold when its $k_\parallel$ disappears. At this upper point, the wave packet stops, because the group velocity length

$$v_\perp^2 + v_\parallel^2 = v_\parallel^2(1 + \mathrm{tg}^2\,\theta),$$

equal to $k_\parallel^2/k_0^2$ according to the second line in (3) and (18), disappears together with $k_\parallel$.

Applied to the conventional pattern of a pulsar magnetosphere, this effect acts as follows [4]. A curvature γ-quantum emitted tangentially to the magnetic force line, i.e., placed initially at the origin in the Figure, then evolves along its dispersion curve as it propagates in the dipole magnetic field with its force line curved, because the components $k_\parallel$ and $k_\perp$ are changing. The maximum value of the ordinate $k_0^2 - k_\parallel^2$ occurs at $k_\parallel = 0$, and it is the photon energy squared, k_0^2. If the latter is greater than $4m^2$, the photon may reach the horizontal asymptote in the Figure. Here, its group velocity $dk_0/dk_\perp$ across the magnetic field disappears, $dk_0/dk_\perp \to 0$, and hence it propagates along the magnetic field and does not cross the threshold, because the other branch of the dispersion curve, which passes above the threshold, is separated from the initial branch by a gap. A mixed state — photon-pair — is actually formed [4], analogous to the polariton known in condensed matter physics. The massless part of its spectrum is presented by the dispersion curves in the Figure. The photon gradually turns into the e^+e^- pair and exists mostly in that form when it is finally propagating along the magnetic force lines. This capturing effect is important for the formation of radiation of pulsars with the fields $B > 0.1B_{cr}$, because it prevents the screening of the accelerating electric field in the polar gap (if the binding of the electron–positron pair into a positronium atom is taken into account [5–9]). It may also be essential for magnetars with their fields approximately 10^{14}–10^{15} Gs.

The new features introduced by super-Schwinger fields are that the dispersion curves for mode 2 in the Figure already step aside from the light cone far from the resonance region. This means that although the photons softer than $2m = 1$ MeV cannot proceed to the values of the ordinate in the Figure higher than their energy squared (corresponding to $k_\parallel = 0$), they can still reach the region where the transversal group velocity $dk_0/dk_\perp$ becomes much less than unity and are therefore captured to the trajectory almost parallel to the magnetic field. This is how the capture effect extends to the photon energies below the border $k_0 = 2m$. The cyclotron singularity at the pair-creation threshold in such fields is so strong that even low-energy photons that are unable to create a pair are sensitive to it, provided that they belong to mode 2!

In addition to extension of the photon capture effect to softer photons, the inclusion of super-Schwinger fields into consideration has another impact. It leads to a large direction-dependent refraction of mode 2 electromagnetic waves of low frequency. To see this, we consider the limit

$$k_0^2 - k_\parallel^2 \ll 4m^2 \tag{19}$$

in Eq. (14), which reduces to neglecting $k_0^2 - k_\parallel^2$ in the integrand in (14). Then (14) becomes

$$\kappa_2(k) = \frac{\alpha}{3\pi}(k_0^2 - k_\parallel^2)\frac{B}{B_{cr}}\exp\left(-\frac{k_\perp^2}{2m^2}\frac{B_{cr}}{B}\right). \tag{20}$$

The exponential factor in (20) cannot be essential within region (19). Dispersion equation (1) for mode 2 ($i = 2$) then has solutions expressing the photon energy k_0 as a function of its transversal and longitudinal momentum,

$$k_0^2 = k_\parallel^2 + k_\perp^2\left(1 + \frac{\alpha}{3\pi}\frac{B}{B_{cr}}\right)^{-1}. \tag{21}$$

Equation (21) analytically presents the straight line parts of the dispersion curves in the Figure adjacent to the origin for various values of B. The components $v_{\perp,\parallel}$ of the group velocity, Eq. (3), calculated from (21) are

$$v_\perp = \frac{k_\perp}{k_0}\left(1 + \frac{\alpha}{3\pi}\frac{B}{B_{cr}}\right)^{-1}, \quad v_\parallel = \frac{k_\parallel}{k_0}. \tag{22}$$

The modulus of the group velocity squared is now given by

$$v_\perp^2 + v_\parallel^2 = \frac{1}{1 + \frac{\alpha}{3\pi}\frac{B}{B_{cr}}} + \frac{\frac{\alpha}{3\pi}\frac{B}{B_{cr}}\cos^2\vartheta}{1 + \frac{\alpha}{3\pi}\frac{B}{B_{cr}}\cos^2\vartheta}, \tag{23}$$

where ϑ is the angle between the photon momentum and the field, $\mathrm{tg}\,\vartheta = k_\perp/k_\parallel$. Equation (23) has the maximum value of unity for the parallel propagation, $\vartheta = 0$, in accordance with the general statement in Sec. 2, and is minimum for perpendicular propagation, $\vartheta = \pi/2$.

Expression (4) for the angle θ between the direction of the electromagnetic energy propagation and the external magnetic field in the super-Schwinger limit for mode 2 becomes

$$\frac{v_\perp}{v_\parallel} = \mathrm{tg}\,\theta = \frac{k_\perp}{k_\parallel}\left(1 + \frac{\alpha}{3\pi}\frac{B}{B_{cr}}\right)^{-1} = \mathrm{tg}\,\vartheta\left(1 + \frac{\alpha}{3\pi}\frac{B}{B_{cr}}\right)^{-1}. \tag{24}$$

Because $\operatorname{tg}\theta < \operatorname{tg}\vartheta$, the photon emitted tangentially to curved force lines bends towards these lines. This is also related to low-frequency radiation.

The refraction index (6) in mode 2 for $k_0^2 - k_\parallel^2 \ll \ll 4m^2$ and $B \gg B_{cr}$ is given by

$$n_2 = \left(\frac{1 + \dfrac{\alpha}{3\pi} \dfrac{B}{B_{cr}}}{1 + \dfrac{\alpha}{3\pi} \dfrac{B}{B_{cr}} \cos^2 \vartheta} \right)^{1/2}. \qquad (25)$$

The refraction index obtained depends on the direction of the photon momentum, characterized by the angle ϑ, but does not depend on its energy. In other words, there is no frequency dispersion in a wide range from slow radio waves up to soft γ-rays with $k_0 \ll 2m$. This is a consequence of the fact that only linear parts in momenta squared were actually left in κ_2 (correspondingly, $f(k_\perp^2)$ in (2) is proportional to $k_\perp^2$ according to (21)).

Refraction index (25) reaches its maximum for transversal propagation ($k_\parallel = 0$, $\vartheta = \pi/2$),

$$n_2^\perp = \left(1 + \frac{\alpha}{3\pi} \frac{B}{B_{cr}}\right)^{1/2} = \left(1 + 7.7 \cdot 10^{-4} \frac{B}{B_{cr}}\right)^{1/2}. \qquad (26)$$

For $B \sim 10 B_{cr}$, the deviation of refraction index (26) from unity exceeds that value for gases at atmospheric pressure in the optic range by an order of magnitude; for $B \sim 1000 B_{cr}$, it reaches the value characteristic of transparent liquids and glass; the refraction index (26) becomes equal to that of diamond ($n = 2.4$) for $B = 27 \cdot 10^{16}$ Gs.

Contrary to the case of mode 2 just considered, the polarization tensor eigenvalue κ_3 in (15) contains neither the contribution linearly growing with the external field nor the resonance. For mode 3, dispersion equation (1) with its right-hand side given as (15) has the solution

$$k_0^2 = k_\parallel^2 + k_\perp^2 \frac{Z - \alpha/3\pi}{Z}, \qquad (27)$$

where

$$Z = 1 - \frac{\alpha}{3\pi} \left(\ln \frac{B}{B_{cr}} - C - 1.21 \right). \qquad (28)$$

The known absence of the asymptotic freedom in QED manifests itself in the negative sign in front of the logarithm in (28). This results in pathological consequences for the fields as large as $B_{cr} \exp(3\pi/\alpha)$. In this domain, the coefficient of $k_\perp^2$ in (27) first becomes less than zero and then greater than unity as the field grows. The corresponding dispersion laws are nonphysical because they lead to the group velocity greater than unity. In the negative slope case in (27),

$$e > \frac{B}{B_{cr}} \exp(-0.21 - C - 3\pi/\alpha) > 1,$$

the dispersion curve enters the sector (12) acceptable for the Cherenkov radiation. But this is the Cherenkov emission of tachyons! It is also odd that in the latter case, electromagnetic waves can only propagate inside the cone

$$0 < \operatorname{tg}\vartheta < -1 + \frac{\alpha}{3\pi Z}$$

with its axis along the external field, irrespective of the way they are produced. This domain of exponentially large external fields is not of our interest in this paper.

For the fields that are not exponentially large, with the logarithmic terms of the order of unity, one should treat all the terms marked by the coefficient $\alpha/3\pi$ in (27) as small. Then, finally, the dispersion law for mode 3 becomes

$$k_0^2 = k_\parallel^2 + k_\perp^2 \left(1 - \frac{\alpha}{3\pi}\right). \qquad (29)$$

Notably, the field-containing logarithmic terms have cancelled here. Therefore, dispersion law (29) of mode 3 is saturated in the sense that unlike Eq. (21) for mode 2, it has reached the universal form, independent of the external field in the super-Schwinger limit. The refraction index of mode 3 corresponding to (29) is

$$n_3 = 1 + \frac{\alpha}{6\pi} \sin^2 \vartheta. \qquad (30)$$

Again, similarly to (26), the maximum refraction in mode 3 is achieved at perpendicular propagation, $\vartheta = \pi/2$:

$$n_3^\perp = 1 + 3.8 \cdot 10^{-4}. \qquad (31)$$

This refraction index is of the order of that of gaseous ammonia and cannot be made larger by increasing the external field any further.

4. CONCLUSION

We have found that in the asymptotic case of external magnetic fields B that can be orders of magnitude larger than the Schwinger value 4.4×10^{13} Gs, the refractive capacity of the magnetized vacuum grows unlimitedly with this field for electromagnetic radiation belonging to polarization mode 2, but comes to saturation at a moderate level of corrections of the order of $\alpha/3\pi$ for mode 3. For the «parallel energy» of the photon not close to the cyclotron resonance, $k_0^2 - k_\parallel^2 \ll 4m^2$, the refraction effects for mode 2 essentially exceed the above small corrections, typical of the nonasymptotic domain, already for $B \sim 10B_{cr}$. In the range of photon frequencies/energies extending from zero to soft γ-rays, a regime is established for which the dispersive properties of the magnetized vacuum are independent of the photon frequency/energy in each mode, but do depend on the direction of its propagation. Apart from the fact that the refraction index in mode 2 for the propagation nonparallel to the external field grows numerically with the field, it is remarkable that the angle between the group velocity and the direction of the photon momentum also grows, the wave packet being attracted by the force line of the external field. The effect of γ-quantum capture by a strong magnetic field, known to exist due to resonance phenomena associated with free and bound pair creation, is thus extended to lower energy ranges. Therefore, not only hard γ-rays, but also X-rays, light and radio-waves undergo strong dispersive influence of the magnetized vacuum when the magnetic fields are of the order of magnitude of those estimated to exist in magnetars. In view of this, the electromagnetic energy canalization phenomena may become important not only within the traditional context described in Sec. 3.2 above, but also in application to the scattering of electromagnetic waves falling onto the magnetic field from outside [2]. These may be, for instance, the X-rays emitted from the accretion disk or from the pulsar surface outside the region where the magnetic field enters it. The problem of the bending of electromagnetic radiation by the dipole magnetic field of a neutron star was recently addressed in [24], and the competition of this process with the effects of gravity was considered[2)]. We insist, however, that such effects cannot be adequately treated disregarding the refraction index dependence on the direction of propagation and using the quadratic-in-the-field expressions for the polarization operator, only valid in the low-field limit, as is the case in Ref. [24].

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2) The author is indebted to H. Mosquera Cuesta who attracted his attention to that work.

APPENDIX

In this Appendix, the asymptotic expansion presented in Sec. 3.1 is derived from expressions in Ref. [3, 19, 20].

The three eigenvalues κ_i, $i = 1,2,3$, of the photon polarization operator in the one-loop approximation, calculated using the exact electron propagator in an external magnetic field, can be expressed as linear combinations of the three functions Σ_i,

$$\begin{aligned} \kappa_1 &= -\frac{1}{2}(z_1+z_2)\Sigma_1, \\ \kappa_2 &= -\frac{1}{2}(z_1\Sigma_2+z_2\Sigma_1), \\ \kappa_3 &= -\frac{1}{2}(z_2\Sigma_3+z_1\Sigma_1), \end{aligned} \tag{A.1}$$

where the new notation

$$z_1 = k_\parallel^2 - k_0^2, \quad z_2 = k_\perp^2 \tag{A.2}$$

is introduced for the momentum variables, with $k^2 = z_1 + z_2$. Here, Σ_i are dimensionless functions of the three ratios B_{cr}/B, z_2B_{cr}/m^2B, and z_1B_{cr}/m^2B, given by

$$\Sigma_i = \Sigma_i^{(1)} + \Sigma_i^{(2)}, \tag{A.3}$$

$$\Sigma_i^{(1)}\left(\frac{B_{cr}}{B}\right) = \frac{2\alpha}{\pi}\int_0^\infty dt\exp\left(-\frac{tB_{cr}}{B}\right)\times \\ \times\int_{-1}^1 d\eta\left[\frac{\sigma_i(t,\eta)}{\operatorname{sh}t} - \lim_{t\to 0}\frac{\sigma_i(t,\eta)}{\operatorname{sh}t}\right], \tag{A.4}$$

$$\Sigma_i^{(2)}\left(\frac{B_{cr}}{B},\frac{z_2B_{cr}}{m^2B},\frac{z_1B_{cr}}{m^2B}\right) = \\ = \frac{2\alpha}{\pi}\int_0^\infty dt\exp\left(-\frac{tB_{cr}}{B}\right)\int_{-1}^1 d\eta\frac{\sigma_i(t,\eta)}{\operatorname{sh}t}\times \\ \times\left[\exp\left(-z_2\frac{M(t,\eta)}{eB} - z_1\frac{1-\eta^2}{4eB}t\right) - 1\right], \tag{A.5}$$

where

$$M(t,\eta) = \frac{\operatorname{ch} t - \operatorname{ch} t\eta}{2\operatorname{sh} t} \tag{A.6}$$

and

$$\sigma_1(t,\eta) = \frac{1-\eta}{2}\,\frac{\operatorname{sh}(1+\eta)t}{2\operatorname{sh} t}, \tag{A.7}$$

$$\sigma_2(t,\eta) = \frac{1-\eta^2}{4}\operatorname{ch} t, \tag{A.8}$$

$$\sigma_3(t,\eta) = \frac{\operatorname{ch} t - \operatorname{ch}\eta t}{2\operatorname{sh}^2 t}. \tag{A.9}$$

The notation «lim» in (A.4) stands for the asymptotic limit

$$\lim_{t\to 0}\frac{\sigma_i(t,\eta)}{\operatorname{sh} t} = \frac{1-\eta^2}{4t}, \quad i = 1,2,3. \tag{A.10}$$

The fact that Σ_i are independent of the fourth possible dimensionless variable z_1/z_2 seems to be an approximation-independent manifestation of analyticity properties due to dispersion relations of the Kramers–Kronig nature.

We first consider $\Sigma_i^{(1)}$. It is independent of the photon energy and momentum. With the notation

$$g_i(t) = \int_{-1}^{1}\sigma_i(t,\eta)\,d\eta, \tag{A.11}$$

Eq. (A.4) can be represented as

$$\Sigma_i^{(1)} = \frac{2\alpha}{\pi}\int_0^\infty \exp\left(-\frac{tB_{cr}}{B}\right)\left(\frac{g_i(t)}{\operatorname{sh} t} - \frac{1}{3t}\right)dt. \tag{A.12}$$

The integrals in (A.11) are explicitly calculated to give

$$g_1(t) = \frac{1}{4t\operatorname{sh} t}\left(\frac{\operatorname{sh} 2t}{t} - 2\right), \quad g_2(t) = \frac{\operatorname{ch} t}{3},$$
$$g_3(t) = \frac{1}{\operatorname{sh}^2 t}\left(\operatorname{ch} t - \frac{\operatorname{sh} t}{t}\right). \tag{A.13}$$

Our goal is now to find the asymptotic behavior of (A.12) as

$$\frac{B}{B_{cr}} \to \infty. \tag{A.14}$$

The integrands in (A.12) do not contain singularities at $t = 0$, but would cause divergence at $t\to\infty$ if we just set the limiting value

$$\exp(-tB_{cr}/B) = 1.$$

We must therefore divide the integration domain into two parts. In addition,

$$\frac{3g_2(t)}{\operatorname{sh} t} \to 1$$

as $t\to\infty$, and hence we have to add and subtract this limit beforehand in the integrand of $\Sigma_2^{(1)}$. This is not required in handling the cases of $i = 1,3$, because

$$\frac{g_{1,3}}{\operatorname{sh} t} \to 0$$

sufficiently fast. We thus have

$$\begin{aligned}
\frac{3\pi}{2\alpha}\Sigma_i^{(1)} &= \int_0^\infty \exp\left(-\frac{tB_{cr}}{B}\right)\left(\frac{3g_i(t)}{\operatorname{sh} t} - \frac{1}{t} - \delta_{i2}\right)dt + \\
&\quad + \delta_{i2}\int_0^\infty \exp\left(-\frac{tB_{cr}}{B}\right)dt = \\
&= \int_0^T \exp\left(-\frac{tB_{cr}}{B}\right)\left(\frac{3g_i(t)}{\operatorname{sh} t} - \frac{1}{t} - \delta_{i2}\right)dt + \frac{B}{B_{cr}}\delta_{i2} + \\
&\quad + \int_T^\infty \exp\left(-\frac{tB_{cr}}{B}\right)\left(\frac{3g_i(t)}{\operatorname{sh} t} - \delta_{i2}\right)dt - \\
&\quad - \int_T^\infty \exp\left(-\frac{tB_{cr}}{B}\right)\frac{dt}{t} = \\
&= \int_0^T \left(\frac{3g_i(t)}{\operatorname{sh} t} - \frac{1}{t}\right)dt - \delta_{i2}T + \int_T^\infty \left(\frac{3g_i(t)}{\operatorname{sh} t} - \delta_{i2}\right)dt + \\
&\quad + \frac{B}{B_{cr}}\delta_{i2} + \ln\left(\frac{B_{cr}}{B}\right) + C + \ln T,
\end{aligned} \tag{A.15}$$

where T is an arbitrary positive number, δ_{i2} is the Kronecker delta, and C is the Euler constant. We have omitted the exponentials in the first two integrals after the second equality sign in (A.15) because the resulting integrals converge. We then used the known asymptotic expansion of the standard exponential-integral function, which is given by (up to terms linearly decreasing with B/B_{cr}) [25]

$$-\int_T^\infty \exp\left(-\frac{tB_{cr}}{B}\right)\frac{dt}{t} = \ln\frac{B_{cr}}{B} + \ln T + C. \tag{A.16}$$

The most slowly decreasing term neglected in

$$3\int_T^\infty dt\exp\left(-tB_{cr}/B\right)\frac{g_i(t)}{\operatorname{sh} t}$$

is

$$\delta_{i1}\frac{3B_{cr}}{4B}\ln\frac{B_{cr}}{B},$$

because

$$\frac{g_1(t)}{\operatorname{sh} t}\sim\frac{1}{4t^2}$$

as $t\to\infty$. Other neglected terms decrease at least as fast as B_{cr}/B, because $3g_i(t)/\operatorname{sh} t-\delta_{i2}$ decreases exponentially, as $\exp(-2t)$, for $i=2,3$ when t is large.

Numerical calculations using MATHCAD code allow evaluating the constants ($dh_i/dT=0$)

$$h_i=\int_0^T dt\left(\frac{3g_i(t)}{\operatorname{sh} t}-\frac{1}{t}\right)+ \\ +\int_T^\infty\left(\frac{3g_i(t)}{\operatorname{sh} t}-\delta_{1,2}\right)dt+\ln T-\delta_{i2}T \quad \text{(A.17)}$$

involved in (A.15) as

$$h_1=1.21,\quad h_2=-0.69,\quad h_3=0.21. \quad \text{(A.18)}$$

Finally, in the asymptotic regime $B/B_{cr}\gg 1$, we have

$$\Sigma_i^{(1)}=\frac{2\alpha}{3\pi}\left(\ln\frac{B_{cr}}{B}+C+h_i+\frac{B}{B_{cr}}\,\delta_{i2}\right) \quad \text{(A.19)}$$

up to terms decreasing at least as fast as integral powers of the ratio B_{cr}/B and to the slower term

$$\frac{\alpha}{2\pi}\,\frac{B_{cr}}{B}\ln\frac{B_{cr}}{B},$$

omitted in $\Sigma_1^{(1)}$.

We now turn to $\Sigma_i^{(2)}$ in Eq. (A.5). This depends on the three arguments as indicated in (A.5). We are interested in the asymptotic domain described by condition (A.14) and

$$\frac{2eB}{z_1}\to\infty. \quad \text{(A.20)}$$

We keep the ratio $2eB/z_2$ finite whenever it makes sense.

The asymptotic expansion of (A.7), (A.8), and (A.9) in powers of $\exp(-t)$ and $\exp(\eta t)$ produces an expansion of (A.5) into a sum of contributions coming from thresholds (16), the singular behavior at the threshold points originating from the divergences of the t-integration in (A.5) near $t=\infty$ (see [3, 19, 20] for the details). The leading terms in the expansion of (A.7), (A.8), and (A.9) at $t\to\infty$ are

$$\left.\left(\frac{\sigma_1(t,\eta)}{\operatorname{sh} t}\right)\right|_{t\to\infty}=\frac{1-\eta}{2}\,\exp(t(\eta-1))), \quad \text{(A.21)}$$

$$\left.\left(\frac{\sigma_2(t,\eta)}{\operatorname{sh} t}\right)\right|_{t\to\infty}=\frac{1-\eta^2}{4}\left(1+2\exp(-2t)\right), \quad \text{(A.22)}$$

$$\left.\left(\frac{\sigma_3(t,\eta)}{\operatorname{sh} t}\right)\right|_{t\to\infty}=2\exp(-2t). \quad \text{(A.23)}$$

Changing the variable as $\tau=t/eB$ and taking into account that $M(\infty,\eta)=1/2$ (see (A.6)), we evaluate (A.5) near the lowest singular thresholds ($n=0$, $n'=1$ or $n'=0$, $n=1$ for $i=1$ in (16), $n=n'=1$ for $i=3$, and $n=n'=0$ and $n=n'=1$ for $i=2$) as

$$\Sigma_i^{(2)}=\frac{2\alpha eB}{\pi}\int_{-1}^1 d\eta\int_0^\infty d\tau\exp(-m^2\tau)\times \\ \times\left.\left(\frac{\sigma_i(eB\tau,\eta)}{\operatorname{sh} eB\tau}\right)\right|_{t\to\infty}\times \\ \times\left[\exp\left(-\frac{z_2}{2eB}-\frac{z_1(1-\eta^2)}{4}\tau\right)-1\right]. \quad \text{(A.24)}$$

After integration over τ we obtain, e.g.,

$$\Sigma_1^{(2)}=\frac{4\alpha eB}{\pi}\int_{-1}^1 d\eta(1-\eta)\times \\ \times\left(\frac{\exp\left(-z_2/2eB\right)}{4m^2+4(1-\eta)eB+z_1(1-\eta^2)}- \right. \\ \left. -\frac{1}{4m^2+4(1-\eta)eB}\right). \quad \text{(A.25)}$$

The pole in the above expression, caused by the integration over t, turns into the inverse square root singularity after the integration over η (cf. the derivation of (17) from (14)). In the limit (A.14), (A.20) when $B\gg m^2$, $B\gg|z_1|$, no singularity remains in this expression (it is shifted to the infinitely remote region) and we are left with

$$\Sigma_1^{(2)}=\frac{2\alpha}{\pi}\left(\exp\left(-\frac{z_2}{2eB}\right)-1\right).$$

The same situation occurs for $\Sigma_3^{(2)}$ and for higher thresholds (also for contributions into $\Sigma_2^{(2)}$ other than those coming from the first term in (A.22)). The result of the calculation analogous to (A.25) is

$$\Sigma_3^{(2)}=\frac{4\alpha}{\pi}\left(\exp\left(-\frac{z_2}{2eB}\right)-1\right).$$

We conclude that in the limit (A.14), (A.20), there are no cyclotron resonances in the eigenvalues $\kappa_{1,\ 3}$ according to (A.1), and that $\Sigma_1^{(2)}$ does not introduce a singular contribution into κ_2. Consequently, there is no reason to keep the ratio eB/z_2 finite as $B \to \infty$, in $\Sigma_{1,3}$, because z_2 may grow infinitely on the dispersion curve only when there is a resonance.

We must therefore consider only the limit when all the three arguments in $\Sigma_{1,3}^{(2)}$ tend to zero. Handling this limit in (A.5) is straightforward:

$$\lim \Sigma_{1,3}^{(2)} = \frac{-2\alpha}{\pi eB} \int_0^\infty dt \times \int_{-1}^{1} d\eta \, \frac{\sigma_{1,3}(t,\eta)}{\operatorname{sh} t} \left(z_2 M(t,\eta) + z_1 \frac{1-\eta^2}{4} t \right). \quad \text{(A.26)}$$

Both integrations here converge, and hence this contribution decreases as z_1/eB and z_2/eB when $B \to \infty$. This is to be neglected within our scope of accuracy.

The situation is different with $\Sigma_2^{(2)}$. The resonance behavior is here present due to the contribution of the leading asymptotic term $(1-\eta^2)/4$ in (A.22). It is responsible for the first threshold at $-z_1 = 4m^2$ (the ground Landau state $n = n' = 0$ in (16)), which remains in its place as $B \to \infty$. We must therefore keep the ratio z_2/eB nonzero in passing to the limit of large fields (because $z_2 \to \infty$ near the singular threshold on the dispersion curve) for the contribution of this term into $\Sigma_2^{(2)}$. The contributions of nonleading terms in expansion (A.22) to $\Sigma_2^{(2)}$ are nonsingular and should be treated along the same lines as $\Sigma_{1,3}$ above. They decrease as z_1/eB, z_2/eB and are to be neglected. Finally, for (A.5) we are left in the limit (A.14), (A.20) with

$$\Sigma_2^{(2)} = \frac{2\alpha}{\pi} \int_0^\infty dt \exp\left(-\frac{tB_{cr}}{B}\right) \int_{-1}^{1} d\eta \frac{1-\eta^2}{4} \times \left[\exp\left(-z_2 \frac{M(t,\eta)}{eB} - z_1 \frac{1-\eta^2}{4eB} t \right) - 1 \right]. \quad \text{(A.27)}$$

Changing the integration variable as $t = eB\tau$ and using the asymptotic form $M(eB\tau,\eta) = 1/2$, $eB\tau \gg 1$, we finally obtain (after the τ integration) the leading contribution to $\Sigma_2^{(2)}$ in the limit (A.14), (A.20),

$$\Sigma_2^{(2)} = \frac{2\alpha eB}{\pi} \int_{-1}^{1} d\eta \, (1-\eta^2) \times \frac{\exp(-z_2/2eB)}{4m^2 + z_1(1-\eta^2)} - \frac{2\alpha eB}{3\pi m^2}. \quad \text{(A.28)}$$

Combining Eqs. (A.28) and (A.19) in accordance with (A.1), and bearing (A.2) and (A.3) in mind, we obtain expressions (13), (14), and (15) for the polarization operator eigenvalues if we also neglect the constant and logarithmic terms in κ_2 coming from $\Sigma_{1,2}^{(1)}$ in (A.19) as compared to the terms growing linearly with B.

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