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COMMENTS ON THE MORITA EQUIVALENCE

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It is known that the noncommutative Yang—Mills (YM) theory with periodical boundary conditions on a torus at the rational value of the noncommutativity parameter is Morita equivalent to the ordinary YM theory with twisted boundary conditions on the dual torus. We give a simple derivation of this fact. We describe the one-to-one correspondence between these two theories and the corresponding gauge invariant observables. In particular, we show that under the Morita map, the Polyakov loops in the ordinary YM theory go into the open noncommutative Wilson loops discovered by Ishibashi, Iso, Kawai, and Kitazawa.

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1. INTRODUCTION

Noncommutative geometry deals with functions on a deformation of the ordinary space where the coordinates do not commute[1)],

$$[\hat{x}_\mu, \hat{x}_\nu] = 2\pi i\theta_{\mu\nu}, \quad \mu,\nu = 1,\ldots,d. \tag{1.1}$$

The antisymmetric tensor $\theta_{\mu\nu}$ is called the noncommutativity parameter. The deformed flat ($\theta_{\mu\nu} = \text{const}$) and compact space is called the noncommutative (quantum) torus $\mathbf{T}^d_\theta$. Recently, the noncommutative geometry and especially the noncommutative torus were seen to play an important role in the M-theory compactifications [1] and in string theory (see [2] and references therein). The noncommutative geometry also is very useful in compactifications of instanton moduli spaces [3]. The way to deal with the curved quantum spaces is provided by the Kontsevich deformation quantization.

A very intriguing subject from noncommutative geometry is the so-called Morita equivalence [4]. Roughly speaking, it states that certain bundles on different noncommutative tori are dual to each other. From the physical standpoint, this results in the equivalence between certain noncommutative and ordinary gauge theories. In what follows, we try to clarify this statement using a set of simple examples.

2. NOTATION

The algebra $\mathcal{A}_\theta$ of smooth functions on the noncommutative torus is defined using the Moyal star product

$$f * g(\hat{\mathbf{x}}) = \\ = \exp\left(i\pi\theta_{\mu\nu}\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\eta_\nu}\right) f(\xi)g(\eta)\Big|_{\xi=\eta=\hat{\mathbf{x}}}. \tag{2.1}$$

The main property of this product is its associativity. In applications, it is useful to decompose functions on the noncommutative torus into the Fourier components[2)] as

$$f(\hat{\mathbf{x}}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} f_{\mathbf{k}} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}. \tag{2.2}$$

This corresponds to the Weil or symmetric ordering of coordinates. The exponentials

$$\hat{U}_{\mathbf{k}} = e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}$$

can serve as basis elements for the algebra $\mathcal{A}_\theta$.

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1) In what follows, we use the same notation [,] for the ordinary and the star-commutator. To avoid confusion, we supply all noncommutative quantities with the hats.

2) Without loosing the generality, we can consider a torus of size 2π.

A very intriguing phenomenon occurs when the θ-tensor components become rational. We first consider the two-torus $\mathbf{T}^2$,

$$[\hat{x}_\mu, \hat{x}_\nu] = 2\pi i \theta \epsilon_{\mu\nu}, \quad \mu, \nu = 1, 2, \tag{2.3}$$

with the rational noncommutativity parameter $\theta = M/N$, where M and N are coprime integers. Then

$$[\hat{U}_{\mathbf{n}}, \hat{U}_{\mathbf{n}'}] = 2i \sin\left(\pi M \frac{n_2 n_1' - n_1 n_2'}{N}\right) \hat{U}_{\mathbf{n}+\mathbf{n}'} = 2i \sin(\mathbf{n} \times \mathbf{n}')\, \hat{U}_{\mathbf{n}+\mathbf{n}'}, \tag{2.4}$$

where by definition, $\mathbf{n} \times \mathbf{n}' \equiv -\pi \theta_{\mu\nu} n_\mu n'_\nu$. We note that the elements $\hat{U}_{N\mathbf{k}}$ generate the center of $\mathcal{A}_\theta$, that is, we have

$$\left[e^{iN\mathbf{k}\cdot\hat{\mathbf{x}}}, f(\hat{\mathbf{x}})\right] = 0 \tag{2.5}$$

for any $f(\hat{\mathbf{x}})$. This means that the exponentials $\{\hat{U}_{\mathbf{k}},\ \mathbf{k} = 0|_{\mathrm{mod}\, N}\}$ entering the decomposition (2.2) can be treated as if they were ordinary exponentials defined on the ordinary (commutative) space. The other $N^2 - 1$ exponentials obtained from the set $\{\hat{U}_{\mathbf{k}},\ \mathbf{k} \neq 0|_{\mathrm{mod}\, N}\}$ after factorization over the commutative part generate a closed algebra under the star-commutator. This algebra is isomorphic to $SU(N)$, as we see momentarily. Therefore, at the rational value of the noncommutativity parameter, one can identify the algebra of functions on the noncommutative torus with the algebra of matrix-valued functions on the commutative torus.

We conclude this section by giving an explicit matrix representation for the algebra of the noncommutative exponentials (see also [5]). This representation has been well-known for many years [6, 7]. We introduce the clock and shift generators

$$Q = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}, \tag{2.6}$$

where $\omega = e^{2\pi i \theta}$. The matrices P and Q are unitary, traceless and satisfy the relations

$$P^N = Q^N = \mathbf{1}, \quad PQ = \omega\, QP. \tag{2.7}$$

Moreover,

$$\mathrm{Tr}(P^n Q^m) = \begin{cases} N & \text{if} \quad n = 0|_{\mathrm{mod}\, N} \quad \text{and} \quad m = 0|_{\mathrm{mod}\, N}, \\ 0 & \text{if} \quad n \neq 0|_{\mathrm{mod}\, N} \quad \text{or} \quad m \neq 0|_{\mathrm{mod}\, N}. \end{cases} \tag{2.8}$$

It is straightforward to verify that the generators defined as

$$J_{\mathbf{n}} = \omega^{n_1 n_2/2} Q^{n_1} P^{n_2}, \quad \mathbf{n} = (n_1, n_2), \tag{2.9}$$

satisfy commutation relations (2.4),

$$[J_{\mathbf{n}}, J_{\mathbf{n}'}] = 2i \sin(\mathbf{n} \times \mathbf{n}')\, J_{\mathbf{n}+\mathbf{n}'}. \tag{2.10}$$

This identity can be rewritten as the Lie algebra commutation relations

$$[J_{\mathbf{n}}, J_{\mathbf{m}}] = f^{\mathbf{k}}_{\mathbf{nm}} J_{\mathbf{k}} \tag{2.11}$$

with the structure constants

$$f^{\mathbf{k}}_{\mathbf{nm}} = 2i \delta_{\mathbf{n}+\mathbf{m},\mathbf{k}} \sin(\mathbf{n} \times \mathbf{m}). \tag{2.12}$$

The set of unitary unimodular $N \times N$ matrices (2.9) is sufficient to span the $SU(N)$ algebra.

3. THE MORITA EQUIVALENCE

3.1. The two-torus. $U(1)|_{\theta=M/N} \to U(N)$

To define the Morita map, we use an additional decomposition of function (2.2) on the noncommutative two-torus

$$\hat{f} = \sum_{k \in \mathbb{Z}^2} e^{iN\mathbf{k}\cdot\hat{\mathbf{x}}} \sum_{n_1, n_2 = 0}^{N-1} f_{\mathbf{kn}} e^{in_1 \hat{x}_1 + in_2 \hat{x}_2}. \tag{3.1}$$

We then define the corresponding $U(N)$-valued function on the ordinary two-torus as

$$f = \sum_{k \in \mathbb{Z}^2} e^{iN\mathbf{k}\cdot\mathbf{x}} \sum_{n_1, n_2 = 0}^{N-1} f_{\mathbf{kn}}\, e^{i\mathbf{n}\cdot\mathbf{x}} J_{\mathbf{n}}. \tag{3.2}$$

Because of the relation

$$J_{\mathbf{n}} J_{\mathbf{n}'} = e^{i\mathbf{n}\times\mathbf{n}'} J_{\mathbf{n}+\mathbf{n}'}, \tag{3.3}$$

Morita map (3.1), (3.2) takes the star-product to the matrix product. Obviously, a general $U(N)$-valued function cannot be represented in form (3.2). It turns out that this particular form corresponds to the functions with nontrivial boundary conditions. This means that under shifts of their arguments, these functions transform as

$$\begin{aligned} f\left(x_1 + 2\pi\frac{M}{N}, x_2\right) &= \Omega_1\ f(x_1, x_2)\ \Omega_1^\dagger, \\ f\left(x_1, x_2 + 2\pi\frac{M}{N}\right) &= \Omega_2\ f(x_1, x_2)\ \Omega_2^\dagger, \end{aligned} \tag{3.4}$$

where

$$\Omega_1 = (P)^M, \quad \Omega_2 = (Q^\dagger)^M. \tag{3.5}$$

This can be considered as a constant gauge transformation. The size $2\pi M/N$ of the dual torus can be fixed by the requirement for the Morita map to be single-valued[3]. To illustrate this, we consider a torus of the size $2\pi(M/N)n$ (where $n \in \mathbf{N}$; there are no other possibilities if the functions of type (3.2) are required to be gauge-conjugate by a constant matrix when translated along the torus lattice). In this case, obviously, there are functions that cannot be represented in form (3.2). These functions are not conjugated when translated along the vectors $(2\pi M/N, 0)$ and $(0, 2\pi M/N)$.

Therefore, having a set of Fourier coefficients $f_{\mathbf{kn}}$, we can construct a function on the noncommutative torus of the size l and a matrix-valued function with twisted boundary conditions (3.4) on the commutative torus of the size $(M/N)l$ as follows:

$$\left\{ \begin{array}{l} e^{i\mathbf{n}\hat{\mathbf{x}}} \leftrightarrow e^{i\mathbf{n}\cdot\mathbf{x}} J_{\mathbf{n}}, \quad n_1, n_2 < N, \\ e^{iN\mathbf{k}\cdot\hat{\mathbf{x}}} \leftrightarrow e^{iN\mathbf{k}\cdot\mathbf{x}}\mathbf{1}. \end{array} \right. \tag{3.6}$$

3.2. $\mathbf{T^d}$. $U(1)|_\theta \to U(N_1) \times \cdots \times U(N_r)$

The generalization to the d-dimensional case goes by simple modifications of the formulas from the previous subsection. It is always possible to rotate $\theta_{\mu\nu}$ into the canonical skew-diagonal form

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \theta_r & \\ & & & -\theta_r & 0 & \\ & & & & & \mathbf{0}_{d-2r} \end{pmatrix}, \tag{3.7}$$

where r is the rank of $\theta_{\mu\nu}$. The algebra of a higher dimensional noncommutative torus is thereby embedded into a d-fold tensor product of r noncommutative two-torus algebras and the ordinary $(d-2r)$-torus commutative algebra. This immediately leads to other examples of the Morita equivalence, where some of these noncommutative two-tori are mapped to the commutative ones via (3.6). If

$$\theta_i = \frac{M_i}{N_i},$$

the Morita map results in the ordinary Yang–Mills (YM) theory with the gauge group $U(N_1)\times\cdots\times U(N_r)$.

3.3. $\mathbf{T^d}$. $\boldsymbol{U(1)|_\theta \to U(N)}$

The algebra of noncommutative exponentials can also be realized using a set of $SU(N)$-valued matrices Ω_μ, $\mu = 1, \ldots, d$, obeying the relations

$$\Omega_\mu \Omega_\nu = e^{2\pi i\theta_{\mu\nu}} \Omega_\nu \Omega_\mu. \tag{3.8}$$

An explicit construction of these matrices can be found in [8]. We define the generators $J_{\mathbf{n}}$ as

$$J_{\mathbf{n}} = \exp\left(\sum_{\nu<\mu} \theta_{\nu\mu} n_\nu n_\mu\right) \Omega_1^{n_1} \ldots \Omega_d^{n_d}. \tag{3.9}$$

Then

$$[J_{\mathbf{n}}, J_{\mathbf{m}}] = 2i \sin(\mathbf{n}\times\mathbf{m})\, J_{\mathbf{n}+\mathbf{m}}, \tag{3.10}$$

which coincides with the algebra of the noncommutative exponentials. In this case, therefore, the Morita map takes the form

$$\begin{aligned} \hat{f} = \sum_{k\in\mathbb{Z}^d} e^{iN\mathbf{k}\cdot\hat{\mathbf{x}}} \sum_{\mathbf{n}<N^{\otimes d}} f_{\mathbf{kn}} e^{i\mathbf{n}\cdot\hat{\mathbf{x}}} \leftrightarrow \\ \leftrightarrow f = \sum_{k\in\mathbb{Z}^d} e^{iN\mathbf{k}\cdot\mathbf{x}} \sum_{\mathbf{n}<N^{\otimes d}} f_{\mathbf{kn}} e^{i\mathbf{n}\cdot\mathbf{x}} J_{\mathbf{n}}. \end{aligned} \tag{3.11}$$

4. THE NONCOMMUTATIVE YANG—MILLS THEORY VS THE ORDINARY YANG—MILLS THEORY

We now turn to physical applications of the Morita map. One can define a noncommutative version of the YM theory with the action

$$S_{YM} = \frac{1}{4\pi g_{YM}^2} \int d\mathbf{x}\ \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{4.1}$$

by simply replacing the matrix product by the Moyal star-product in all formulas and supplementing all quantities with the hats. Therefore, the noncommutative $U(1)$ YM action is

$$\hat{S} = \frac{1}{4\pi g_{NCYM}^2} \int d\hat{\mathbf{x}}\ \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu}, \tag{4.2}$$

[3] I am indebted to K. Selivanov for this comment.

where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_*.$$

For simplicity, we consider only two-torus in this section. The generalization to the higher-dimensional case is straightforward.

The Morita map takes noncommutative $U(1)$ gauge fields to the $U(N)$ gauge fields with nontrivial boundary conditions. In general, functions on the torus can be gauge-conjugate when shifted by the period of the torus,

$$A_\lambda(\mathbf{x} + \mathbf{l}_\mu) = \Omega_\mu(\mathbf{x}) A_\lambda(\mathbf{x})\, \Omega_\mu^{-1}(\mathbf{x}) + \\ + i\Omega_\mu(\mathbf{x})\, \partial_\lambda \Omega_\mu^{-1}(\mathbf{x}), \quad (4.3)$$

where $\Omega_\mu(\mathbf{x})$ are the elements of the $U(N)$ group that are known as the twist matrices. They must satisfy the consistency conditions

$$\Omega_\mu(\mathbf{x} + \mathbf{l}_\nu)\, \Omega_\nu(\mathbf{x}) = \\ = \exp\left(2\pi i \frac{M}{N} \epsilon_{\mu\nu}\right) \Omega_\nu(\mathbf{x} + \mathbf{l}_\mu)\, \Omega_\mu(\mathbf{x}). \quad (4.4)$$

The integer M entering this formula is the so-called 't Hooft flux. Only three types of possible boundary conditions (solutions of Eqs. (4.4)) are known::

1. twist eaters: $\Omega_\mu = \mathrm{const}$;
2. abelian twists;
3. nonabelian twists.

For more details, see the recent review [9].

The map (3.6) precisely corresponds to the first case. It is not well understood how to realize the Morita map corresponding to other boundary conditions. Roughly speaking, when working in the Fourier basis (2.2), after shifts one can only multiply functions with numbers and cannot add quantities of the form $\Omega_\mu(\mathbf{x})\, \partial_\lambda \Omega_\mu^{-1}(\mathbf{x})$. To do this, one needs another basis for the functions on the noncommutative torus (creation/annihilation operators, noncommutative theta-functions?).

Under the Morita map defined in the previous section, actions go into actions, equations of motions go into equations of motions, and solutions (e.g., instantons) also go into solutions, even at the quantum level. These properties of the Morita map can be encoded in the identity

$$\int d\hat{\mathbf{x}} \hat{A}_\mu * \hat{A}_\nu * \cdots * \hat{A}_\lambda = \\ = \frac{1}{N} \int d\mathbf{x}\ \mathrm{Tr}(A_\mu A_\nu \ldots A_\lambda), \quad (4.5)$$

which is straightforward to prove using the definition

$$\int d\hat{\mathbf{x}} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} = \delta_{\mathbf{k},\mathbf{0}} \quad (4.6)$$

and property (2.8) of the clock and shift generators. In fact, one can insert an arbitrary number of derivatives into the integrals in (4.5) and thus obtain equivalent gauge-invariant quantities in the noncommutative and ordinary gauge theories. Using identity (4.5), we can relate the correlators as

$$\int \mathcal{D}A^\mu_{\mathbf{k},\mathbf{n}} \exp\left(\hat{S}\left[\theta = \frac{M}{N}\right]\right) \hat{\mathcal{O}}_1 \ldots \hat{\mathcal{O}}_l = \\ = \int \mathcal{D}A^\mu_{\mathbf{k},\mathbf{n}} \exp(S_{YM})\Big|_{fxd\ bndry\ conds,\ flux=M} \times \\ \times \mathcal{O}_1 \ldots \mathcal{O}_l, \quad (4.7)$$

where $g^2_{NCYM} = N g^2_{YM}$ and

$$\hat{\mathcal{O}} = \int d\hat{\mathbf{x}} (\hat{F}_{\mu\nu})^{*n}, \\ \mathcal{O} = \frac{1}{N} \int d\mathbf{x}\ \mathrm{Tr}\,(F_{\mu\nu})^n. \quad (4.8)$$

Other important gauge-invariant quantities of the YM theory are the Wilson loops

$$W[C] = \mathrm{Tr}\ P \exp\left(i \oint_C A_\mu(\mathbf{x}) dx_\mu\right) \quad (4.9)$$

corresponding to a closed path C. On the torus, there are paths from different homotopy classes, which can be classified by winding numbers w_μ around the μ-th direction. The corresponding Wilson loops are called the Polyakov loops. The simplest Polyakov loop corresponds to the straight line along the μ-th direction,

$$W_P[\mathbf{x}, \mu] = \\ = \mathrm{Tr}\left[P \exp\left(i \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{l}_\mu} A_\mu(\mathbf{x}) dx_\mu\right) \Omega_\mu e^{ix_\mu}\right], \quad (4.10)$$

where the insertion of twist matrix (3.5) is necessary to guarantee gauge invariance.

Wilson lines were constructed in the noncommutative YM theory by Ishibashi, Iso, Kawai, and Kitazawa [10] (see also [11, 12]). This construction goes as follows. One first introduces an oriented curve C in the auxiliary commutative two-dimensional space parametrized by the functions $\xi(\sigma)$ with $0 \le \sigma \le 1$. One fixes the starting point $\xi_\mu(0) = 0$ and the endpoint $\xi_\mu(1) = v_\mu$. One then assigns to this curve a noncommutative analog of the parallel transport operator

$$\mathcal{U}[\hat{\mathbf{x}}, C] = 1 + \sum_{n=1}^{\infty} i^n \int_0^1 d\sigma_1 \int_{\sigma_1}^1 d\sigma_2 \ldots$$
$$\ldots \int_{\sigma_{n-1}}^1 d\sigma_n \frac{d\xi_{\mu_1}(\sigma_1)}{d\sigma_1} \ldots \frac{d\xi_{\mu_n}(\sigma_n)}{d\sigma_n} \times$$
$$\times A_{\mu_1}(\hat{\mathbf{x}} + \xi(\sigma_1)) * \cdots * A_{\mu_n}(\hat{\mathbf{x}} + \xi(\sigma_n)). \quad (4.11)$$

The series in (4.11) is a noncommutative analog of the P-exponential. The star-gauge invariant quantity is then

$$\hat{\mathcal{O}}[C] = \int d\hat{\mathbf{x}}\, \mathcal{U}[\hat{\mathbf{x}}, C] * S[\hat{\mathbf{x}}, C], \quad (4.12)$$

where $S[\hat{\mathbf{x}}, C] = 1$ if the path C is closed and

$$S[\hat{\mathbf{x}}, C] = \exp\left(i(\theta^{-1})_{\mu\nu} v_\nu \hat{x}_\mu\right) \quad (4.13)$$

if the path is open. Gauge invariance requires that the endpoint coordinates must be equal to

$$v_\mu = 2\pi r_\mu \frac{M}{N}, \quad r_\mu = 0, \ldots, N-1.$$

In the simplest case where C_μ is the straight line along the μ-th direction and $v_\mu = 2\pi M/N$, the function $S[\hat{\mathbf{x}}, C_\mu]$ goes to the twist function $\Omega_\mu e^{ix_\mu}$ under Morita map (3.6). Therefore, identity (4.5) allows us to obtain the following relation between the Polyakov loops in the ordinary YM theory and open noncommutative Wilson loops:

$$\frac{1}{N} \int d\mathbf{x}\, W_P[\mathbf{x}, \mu] = \hat{\mathcal{O}}[C_\mu]. \quad (4.14)$$

5. CONCLUSIONS

In this paper, we have made some comments on the Morita equivalence between noncommutative and ordinary gauge theories. We present a simple prescription whereby gauge fields and correlators of the gauge-invariant observables in the $U(1)$ noncommutative YM theory on a torus at the rational value of the θ-parameter can be identified with those in the ordinary $U(N)$ or $U(N_1) \times \cdots \times U(N_r)$ YM theory with nontrivial boundary conditions on the dual torus. The size of the dual torus is determined by the requirement for the Morita map to be single-valued. We also show that under the Morita map, the Polyakov loops in the ordinary YM theory go into the open noncommutative Wilson loops[4)].

An open question is to generalize the Morita equivalence to boundary conditions of the non-twist-eater type. Another interesting direction is to link three different descriptions of the Morita equivalence: the field theory approach using the Fourier components, the string theory approach using T-duality and the brane language [13, 14], and the mathematical approach via twisted bundles over the noncommutative torus [4, 15].

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