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THE QUANTUM POISSON—LIE T-DUALITY AND MIRROR SYMMETRY

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Poisson—Lie T-duality in quantum $N = 2$ superconformal Wess—Zumino—Novikov—Witten models is considered. The Poisson—Lie T-duality transformation rules of the super-Kac—Moody algebra currents are found from the conjecture that, as in the classical case, the quantum Poisson—Lie T-duality transformation is given by an automorphism which interchanges the isotropic subalgebras of the underlying Manin triple in one of the chirality sectors of the model. It is shown that quantum Poisson—Lie T-duality acts on the $N = 2$ super-Virasoro algebra generators of the quantum models as a mirror symmetry acts: in one of the chirality sectors it is a trivial transformation while in another chirality sector it changes the sign of the $U(1)$ current and interchanges the spin-3/2 currents. A generalization of Poisson—Lie T-duality for the quantum Kazama—Suzuki models is proposed. It is shown that quantum Poisson—Lie T-duality acts in these models as a mirror symmetry also.

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1. INTRODUCTION

Target-space (T) dualities in superstring theory relate backgrounds with different geometries and are symmetries of the underlying conformal field theory [1, 2].

The mirror symmetry [3] discovered in superstring theory is a special type of T-duality. At the level of conformal field theory it can be formulated as an isomorphism between two theories, amounting to a change of sign of the $U(1)$ generator and an interchange of the spin-3/2 generators of the leftmoving (or rightmoving) $N = 2$ superconformal algebra.

Mirror symmetry has mostly been studied in the context of Calabi—Yau superstring compactification. Important progress has been achieved in this direction in the last few years, based on the ideas of toric geometry [4]. In particular, in Ref. [5] toric geometry mirror pair construction was proposed. Though it seems quite certain that pairs of Calabi—Yau manifolds constructed by these methods are mirror, one needs to show that the proposed pairs correspond to isomorphic conformal field theories, to prove that they are indeed mirror. Progress in this direction was made in [6], but a complete arguments has yet to be carried out. In fact, the only rigorously established example of mirror symmetry, the Greene—Plesser construction [7], is based on the tensor products of the $N = 2$ minimal models [8]. For a review of mirror symmetry and toric geometry methods in Calabi—Yau superstring compactifications see the lectures of Greene [9].

Recently, Strominger, Yau, Zaslow [10] related mirror symmetry in superstring theory to the quantum Abelian T-duality in fibers of toricaly fibrated Calabi—Yau manifolds.

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The Poisson—Lie (PL) T-duality, recently discovered by Klimcik and Severa in their excellent work [11], is a generalization of Abelian and non-Abelian T-dualities [12–14]. This generalized duality is associated with two groups forming a Drinfeld double [15], and the duality transformation exchanges their roles. Many aspects of these ideas have been developed in Refs. [16–26]. In particular, in [26] it was shown that PL T-duality in the classical $N = 2$ superconformal WZNW (SWZNW) and Kazama—Suzuki models is a mirror duality. It is reasonable to expect that PL T-duality in the quantum versions of these models will be a mirror duality also. Moreover, it is tempting to conjecture that PL T-duality is an adequate geometric structure underlying mirror symmetry in superstring theory. Motivated by this we propose a quantization of PL T-duality transformations in the $N = 2$ SWZNW and Kazama—Suzuki models.

Quantum equivalence among PL T-duality related σ-models was studied perturbatively in [27] and [22], and it was shown that PL dualizability is compatible with renormalization at 1 loop. In particular it was shown in [22] that 1-loop beta functions for the coupling and the parameters in the two simplest examples of PL T-duality related models are equivalent. This allows us to suggest that their equivalence extends beyond the classical level with appropriate quantum modification of PL T-duality transformations rules.

In the present note the PL T-duality transformation rules of the fields in quantum $N = 2$ SWZNW models will be found starting from the conjecture that as in the classical case, quantum $N = 2$ SWZNW models are PL self-dual and the PL T-duality transformation is given by an automorphism of the super-Kac-Moody algebra in the rightmoving sector. Then we obtain PL T-duality transformation rules using the Knizhnik—Zamolodchikov equation, Ward identities and a quantum version of the classical formula which relates the generators of rightmoving super-Kac-Moody algebra to its PL T-duality transformed. We show that the generators of the $N = 2$ super-Virasoro algebras transform under PL T-duality like a mirror duality: the $U(1)$ current changes sign and the spin-3/2 currents permute. Thus, the results are in agreement with the conjecture proposed in [28] that mirror symmetry can be related to a gauge symmetry (automorphism) of the self-dual points of the moduli space of the $N = 2$ superconformal field theories (SCFTs) (for the $N = 0$ version of this conjecture see [29]). Then we consider quantum PL T-duality in the Kazama—Suzuki models and propose a natural generalization of the quantum PL T-duality transformation. We show that as in the SWZNW models quantum PL T-duality in the Kazama—Suzuki models is a mirror duality also.

The structure of the paper is as follows: In section 2 we briefly review PL T-duality in the classical $N = 2$ SWZNW model following [26]. In section 3 we describe Manin triple construction of the quantum $N = 2$ SWZNW models on the compact groups and obtain the PL T-duality transformation rules of the quantum fields. We show that PL T-duality transformation is given by an automorphism of the underlying Manin triple which permutes isotropic subalgebras of the triple. Then we obtain transformation rules of the rightmoving $N = 2$ super-Virasoro algebra generators. In section 4 we present the Manin triple construction of the Kazama—Suzuki models. We show that they can be described as $(Manin\ triple)/(Manin\ subtriple)$-cosets. We define quantum PL T-duality transformation in the Kazama—Suzuki models as the subset of the transformations of the numerator triple which stabilizes the denominator subtriple. Then we easily find transformation rules of the rightmoving $N = 2$ super-Virasoro algebra generators of the coset. At the end of the section PL T-duality in the $N = 2$ minimal models considered briefly as an example.

2. POISSON—LIE T-DUALITY AND MIRROR SYMMETRY IN THE CLASSICAL $N = 2$ SUPERCONFORMAL WZNW MODELS

In this section we briefly review PL T-duality in the classical $N = 2$ SWZNW models, following [25, 26].

We parameterize the super world-sheet by introducing the light cone coordinates $z_\pm$ and Grassman coordinates $\Theta_\pm$ (we use the $N = 1$ superfield formalism). The generators of the supersymmetry and covariant derivatives satisfying the standard relations are given by

$$Q_\mp = \frac{\partial}{\partial\Theta_\pm} + i\Theta_\pm\partial_\mp, \quad D_\mp = \frac{\partial}{\partial\Theta_\pm} - i\Theta_\pm\partial_\mp. \tag{1}$$

The superfield of the $N = 2$ SWZNW model

$$G = g + i\Theta_-\psi_+ + i\Theta_+\psi_- + i\Theta_-\Theta_+F \tag{2}$$

takes values in a compact Lie group $\mathbf{G}$ so that its Lie algebra $\mathbf{g}$ is endowed with an ad-invariant nondegenerate inner product $\langle , \rangle$. The action of the model is given by

$$\begin{aligned} S_{SWZ} &= \int d^2x d^2\Theta(\langle G^{-1}D_+G, G^{-1}D_-G\rangle) - \\ &- \int d^2x d^2\Theta dt \left\langle G^{-1}\frac{\partial G}{\partial t}, \{G^{-1}D_-G, G^{-1}D_+G\}\right\rangle \end{aligned} \tag{3}$$

and possesses manifest $N = 1$ superconformal and super-Kac-Moody symmetries [30]:

$$\begin{aligned} &\delta_{a_+}G(z_+, x_-, \Theta_+, \Theta_-) = a_+(z_-, \Theta_+)G(z_+, z_-, \Theta_+, \Theta_-), \\ &\delta_{a_-}G(z_+, z_-, \Theta_+, \Theta_-) = -G(z_+, z_-, \Theta_+, \Theta_-)a_-(z_+, \Theta_-), \end{aligned} \tag{4}$$

$$\begin{aligned} &G^{-1}\delta_{\epsilon_+}G = (G^{-1}\epsilon_+(z_-)Q_+G), \\ &\delta_{\epsilon_-}GG^{-1} = \epsilon_-(z_+)Q_-GG^{-1}, \end{aligned} \tag{5}$$

where $a_\pm$ are $\mathbf{g}$-valued superfields.

An additional ingredient demanded by the $N = 2$ superconformal symmetry is a complex structure J on the finite-dimensional Lie algebra of the model which is skew-symmetric with respect to the inner product $\langle , \rangle$ [31–33]. That is, we should demand that the following equations be satisfied on $\mathbf{g}$:

$$\begin{aligned} &J^2 = -1, \\ &\langle Jx, y\rangle + \langle x, Jy\rangle = 0, \\ &[Jx, Jy] - J[Jx, y] - J[x, Jy] = [x, y] \end{aligned} \tag{6}$$

for any elements x, y in $\mathbf{g}$. It is clear that the corresponding Lie group is a complex manifold with left (or right) invariant complex structure. In the following we shall denote the real Lie group and the real Lie algebra with the complex structure satisfying (6) by the pairs $(\mathbf{G}, J)$ and $(\mathbf{g}, J)$ respectively.

The complex structure J on the Lie algebra defines the second supersymmetry transformation [31]

$$(G^{-1}\delta_{\eta_+}G)^a = \mp \eta_+(z_-)(J_l)^a_b(G^{-1}D_+G)^b,$$
$$(\delta_{\eta_-}GG^{-1})^a = \eta_-(z_+)(J_r)^a_b(D_-GG^{-1})^b, \tag{7}$$

where J_l, J_r are the left invariant and right invariant complex structures on $\mathbf{G}$ which correspond to the complex structure J.

The notion of Manin triple is closely related to a complex structure on a Lie algebra. By definition [15], a Manin triple $(\mathbf{g}, \mathbf{g}_+, \mathbf{g}_-)$ consists of a Lie algebra $\mathbf{g}$ with nondegenerate invariant inner product $\langle , \rangle$ and isotropic Lie subalgebras $\mathbf{g}_\pm$ such that the vector space $\mathbf{g} = \mathbf{g}_+ \oplus \mathbf{g}_-$.

With each pair $(\mathbf{g}, J)$ one can associate the complex Manin triple $(\mathbf{g}^{\mathbb{C}}, \mathbf{g}_+, \mathbf{g}_-)$, where $\mathbf{g}^{\mathbb{C}}$ is the complexification of $\mathbf{g}$ and $\mathbf{g}_\pm$ are $\pm i$ eigenspaces of J. Moreover, it can be proved that there exists a one-to-one correspondence between a complex Manin triple endowed with an anti-linear involution which conjugates isotropic subalgebras $\tau : \mathbf{g}_\pm \to \mathbf{g}_\mp$ and a real Lie algebra endowed with an ad-invariant nondegenerate inner product $\langle , \rangle$ and complex structure J which is skew-symmetric with respect to $\langle , \rangle$ [32]. The conjugation can be used to extract a real form from a complex Manin triple.

Now we have to consider some geometric properties of the $N = 2$ SWZNW models closely related to the existence of complex structures on the groups. We shall follow [25].

Let us fix some compact Lie group with the left invariant complex structure $(\mathbf{G}, J)$ and consider its Lie algebra with the complex structure $(\mathbf{g}, J)$. The complexification $\mathbf{g}^{\mathbb{C}}$ of $\mathbf{g}$ has the Manin triple structure $(\mathbf{g}^{\mathbb{C}}, \mathbf{g}_+, \mathbf{g}_-)$. The Lie group version of this triple is the double Lie group $(\mathbf{G}^{\mathbb{C}}, \mathbf{G}_+, \mathbf{G}_-)$ [34–36], where the exponential subgroups $\mathbf{G}_\pm$ correspond to the Lie algebras $\mathbf{g}_\pm$. The real Lie group $\mathbf{G}$ is extracted from its complexification with the help of conjugation τ (it will be assumed in the following that τ is the hermitian conjugation)

$$\mathbf{G} = \{g \in \mathbf{G}^{\mathbb{C}} | \tau(g) = g^{-1}\}. \tag{8}$$

Each element $g \in \mathbf{G}^{\mathbb{C}}$ from the vicinity $\mathbf{G}_1$ of the unit element from $\mathbf{G}^{\mathbb{C}}$ admits two decompositions:

$$g = g_+g_-^{-1} = \tilde{g}_-\tilde{g}_+^{-1}. \tag{9}$$

Taking into account (8) and (9) we conclude that the element g ($g \in \mathbf{G}_1$) belongs to $\mathbf{G}$ iff

$$\tau(g_\pm) = \tilde{g}_\mp^{-1}. \tag{10}$$

These equations mean that we can parameterize the elements of

$$\mathbf{C}_1 \equiv \mathbf{G}_1 \cap \mathbf{G} \tag{11}$$

by the elements of the complex group $\mathbf{G}_+$ (or $\mathbf{G}_-$), i.e., we can introduce complex coordinates (they are just matrix elements of g_+ (or g_-)) in the strat $\mathbf{C}_1$.

To generalize (9), (10) one has to consider the set W (which we shall assume in the following to be discrete and finite) of classes $\mathbf{G}_+\backslash\mathbf{G}^{\mathbb{C}}/\mathbf{G}_-$ and choose a representative w for each class $[w] \in W$. It gives us the stratification of $\mathbf{G}^{\mathbb{C}}$ [35]:

$$\mathbf{G}^{\mathbb{C}} = \bigcup_{[w]\in W} \mathbf{G}_+w\mathbf{G}_- = \bigcup_{[w]\in W} \mathbf{G}_w. \tag{12}$$

There is a second stratification:

$$\mathbf{G}^{\mathbb{C}} = \bigcup_{[w]\in W} \mathbf{G}_{-}w\mathbf{G}_{+} = \bigcup_{[w]\in W} \mathbf{G}^{w}. \tag{13}$$

We shall assume, in the following, that the representatives w have been chosen to be unitary:

$$\tau(w) = w^{-1}. \tag{14}$$

It allows us to generalize (9) as follows:

$$g = wg_{+}g_{-}^{-1} = w\tilde{g}_{-}\tilde{g}_{+}^{-1}, \tag{15}$$

where

$$g_{+} \in \mathbf{G}_{+}^{w}, \quad \tilde{g}_{-} \in \mathbf{G}_{-}^{w} \tag{16}$$

and

$$\mathbf{G}_{+}^{w} = \mathbf{G}_{+} \cap w^{-1}\mathbf{G}_{+}w, \quad \mathbf{G}_{-}^{w} = \mathbf{G}_{-} \cap w^{-1}\mathbf{G}_{-}w. \tag{17}$$

In order for the element g to belong to the real group $\mathbf{G}$ the elements $g_{\pm}, \tilde{g}_{\pm}$ from (15) must satisfy (10). Thus, the formulas (10), (15) define the mapping

$$\phi_{w}^{+} : \mathbf{G}_{+}^{w} \rightarrow \mathbf{C}_{w} \equiv \mathbf{G}_{w} \cap \mathbf{G}. \tag{18}$$

In a similar way one can define the mapping

$$\phi_{w}^{-} : \mathbf{G}_{-}^{w} \rightarrow \mathbf{C}_{w} \equiv \mathbf{G}_{w} \cap \mathbf{G}. \tag{19}$$

In [25, 26] the following statements were proved.

1) The mappings (18) are holomorphic and define the natural (holomorphic) action of the complex group $\mathbf{G}_{+}$ on $\mathbf{G}$; the set W parameterizes the $\mathbf{G}_{+}$-orbits $\mathbf{C}_{w}$.

2) The $(\mathbf{G}, J)$-SWZNW model admits PL symmetry [11,37], with respect to $\mathbf{G}_{+}$-action, so that we may associate with each extremal surface $G_{+}(z_{+}, z_{-}, \Theta_{+}, \Theta_{-}) \subset \mathbf{G}_{+}$, of the model a mapping («Noether charge») $V_{-}(z_{+}, z_{-}, \Theta_{+}, \Theta_{-})$ from the super world-sheet into the group $\mathbf{G}_{-}$. The pair $(G_{+}(z_{+}, z_{-}, \Theta_{+}, \Theta_{-}), V_{-}(z_{+}, z_{-}, \Theta_{+}, \Theta_{-}))$ can be lifted into the the double $\mathbf{G}^{\mathbb{C}}$:

$$\Phi(z_{+}, z_{-}, \Theta_{+}, \Theta_{-}) = G_{+}(z_{+}, z_{-}, \Theta_{+}, \Theta_{-})V_{-}(z_{+}, z_{-}, \Theta_{+}, \Theta_{-}). \tag{20}$$

Moreover, the surface (20) can be rewritten in the form

$$\Phi(z_{\pm}, \Theta_{\pm}) = G(z_{\pm}, \Theta_{\pm})H_{-}^{-1}(z_{+}, \Theta_{-}). \tag{21}$$

Here $G(z_{\pm}, \Theta_{\pm}) \subset \mathbf{G}$ is a solution of the $\mathbf{G}$-SWZNW model and the superfield H_{-} is given by the solution of the equation

$$H_{-}^{-1}D_{+}H_{-} = 2(I_{+})^{-}, \tag{22}$$

where $(I_{+})^{-}$ is $\mathbf{g}_{-}$-projection of the conservation current $I_{+} = G^{-1}D_{+}G$ of the model.

3) With the appropriate modifications the above statements are true also for the mappings (19) and $\mathbf{G}_{-}$-action on $\mathbf{G}$. Thus, one can represent the surface (20) in the «dual» parameterization [11]

$$\Phi(z_\pm, \Theta_\pm) = \check{G}(z_\pm, \Theta_\pm) H_+^{-1}(z_+, \Theta_-), \tag{23}$$

where $\check{G}(z_\pm, \Theta_\pm)$ is the dual solution of the **G**-SWZNW model and the superfield H_+ is given by the similar equation

$$H_+^{-1} D_+ H_+ = 2(\check{I}_+)^+, \tag{24}$$

where $(\check{I}_+)^+$ is the $\mathbf{g}_+$-projection of the dual conserved current $\check{I}_+ \equiv \check{G}^{-1} D_+ \check{G}$.

4) Under PL T-duality

$$t : G(z_\pm, \Theta_\pm) \to \check{G}(z_\pm, \Theta_\pm) = G(z_\pm, \Theta_\pm) H(z_+, \Theta_-), \tag{25}$$

where

$$H \equiv H_-^{-1} H_+, \tag{26}$$

the conserved rightmoving current I_+ transforms as

$$t : (I_+)^- \to (\check{I}_+)^+, \quad (I_+)^+ \to (\check{I}_+)^-, \tag{27}$$

while the conserved leftmoving current $I_- \equiv D_- G G^{-1}$ transforms identically:

$$t : (I_-)^\pm \to (I_-)^\pm. \tag{28}$$

Moreover, the classical rightmoving $N = 2$ super-Virasoro algebra maps under PL T-duality as follows [26]:

$$t : \Sigma^\pm \to \check{\Sigma}^\mp, \quad T \pm i\partial K \to \check{T} \mp i\partial \check{K}, \tag{29}$$

where $\Sigma^\pm$ are the spin-3/2 currents, T is the stress-energy tensor, and K is the $U(1)$ current, while the leftmoving $N = 2$ super-Virasoro algebra maps identically. Thus, PL T-duality in the classical $N = 2$ SWZNW models is a mirror duality.

3. POISSON—LIE T-DUALITY AND MIRROR SYMMETRY IN THE QUANTUM $N = 2$ SUPERCONFORMAL WZNW MODELS

We start with the Manin triple construction of the $N = 2$ Virasoro algebra generators of the quantum SWZNW model on the group $(\mathbf{G}, J)$ [32, 33, 38].

Let us specify an orthonormal basis

$$\{E^a, E_a, a = 1, ..., d\} \tag{30}$$

in the Manin triple $(\mathbf{g}^{\mathbb{C}}, \mathbf{g}_+, \mathbf{g}_-)$, so that $\{E^a\}$ is a basis in g_+, and $\{E_a\}$ is a basis in g_-. The commutation relations and Jacoby identity in this basis take the form

$$\begin{aligned} &[E^a, E^b] = f^{ab}_c E^c, \\ &[E_a, E_b] = f^c_{ab} E_c, \\ &[E^a, E_b] = f^a_{bc} E^c - f^{ac}_b E_c, \end{aligned} \tag{31}$$

$$
\begin{gathered}
f_d^{ab} f_e^{dc} + f_d^{bc} f_e^{da} + f_d^{ca} f_e^{db} = 0, \\
f_{ab}^d f_{dc}^e + f_{bc}^d f_{da}^e + f_{ca}^d f_{db}^e = 0, \\
f_{mc}^a f_d^{bm} - f_{md}^a f_c^{bm} - f_{mc}^b f_d^{am} + f_{md}^b f_c^{am} = f_{cd}^m f_m^{ab}.
\end{gathered}
\tag{32}
$$

Let us introduce the matrices

$$
\begin{gathered}
B_a^b = f_c f_a^{cb} + f^c f_{ca}^b, \\
A_a^b = f_{ac}^d f_d^{bc}.
\end{gathered}
\tag{33}
$$

Let $j^a(z), j_a(z)$ be the generators of the affine Kac-Moody algebra $\hat{g}^{\mathbb{C}}$, corresponding to the fixed basis $\{E^a, E_a\}$, so that the currents j^a generate the subalgebra $\hat{g}_+$ and the currents j_a generate the subalgebra $\hat{g}_-$ (we shall omit in the following the super-world-sheet indices $\pm$, keeping in mind that we are in the rightmoving sector). The singular operator product expansions (OPEs) between these currents are the following:

$$
\begin{gathered}
j^a(z)j^b(w) = -(z-w)^{-2}\frac{1}{2}k(E^a, E^b) + (z-w)^{-1} f_c^{ab} j^c(w) + \text{reg}, \\
j_a(z)j_b(w) = -(z-w)^{-2}\frac{1}{2}k(E_a, E_b) + (z-w)^{-1} f_{ab}^c j_c(w) + \text{reg}, \\
j^a(z)j_b(w) = -(z-w)^{-2}\frac{1}{2}(q\delta_b^a + k(E^a, E_b)) + (z-w)^{-1}(f_{bc}^a j^c - f_b^{ac} j_c)(w) + \text{reg},
\end{gathered}
\tag{34}
$$

where $k(x, y)$ denotes the Killing form for the vectors x, y of $\mathbf{g}^{\mathbb{C}}$. Let $\psi^a(z), \psi_a(z)$ be free fermion currents which have the following singular OPEs:

$$
\psi^a(z)\psi_b(w) = -(z-w)^{-1}\delta_b^a + \text{reg}. \tag{35}
$$

Then the $N = 2$ Virasoro superalgebra currents and the central charge are given by [31–33, 38]

$$
\begin{gathered}
\Sigma^+ = \frac{2}{\sqrt{q}}\left(\psi^a j_a + \frac{1}{2} f_{ab}^c : \psi^a \psi^b \psi_c :\right), \\
\Sigma^- = \frac{2}{\sqrt{q}}\left(\psi_a j^a + \frac{1}{2} f_c^{ab} : \psi_a \psi_b \psi^c :\right), \\
K = \left(\frac{2B_a^b}{q} - \delta_a^b\right) : \psi^a \psi_b : - \frac{2}{q}(f_c j^c - f^c j_c), \\
T = -\frac{1}{q} : (j^a j_a + j_a j^a) : - \frac{1}{2} : (\partial\psi^a \psi_a - \psi^a \partial\psi_a) :,
\end{gathered}
\tag{36}
$$

$$
c = 3\left(d - \frac{2A_a^a}{q}\right). \tag{37}
$$

The set of currents (36) can be combined into the superfields

$$
\Gamma^{\pm} = \frac{1}{\sqrt{2}}\Sigma^+ + \Theta\left(T \mp \frac{1}{2}\partial K\right), \tag{38}
$$

so that the energy-momentum super-tensor is given by the sum

$$\Gamma = \frac{1}{2}(\Gamma^{+} + \Gamma^{-}) = -\frac{1}{q} : \langle DI, I\rangle : + \frac{2}{3q^2} : \langle I, : \{I, I\} : \rangle : . \tag{39}$$

Here I denotes Lie algebra valued super-Kac-Moody currents of the affine superalgebra $\hat{\mathbf{g}}$:

$$\begin{aligned} I &\equiv I^a E_a + I_a E^a, \\ I^a &= -\sqrt{\frac{q}{2}}\psi^a + \Theta\left(j^a + \left(\frac{1}{2} f^a_{bc} : \psi^b\psi^c : + f^{ab}_c : \psi_b\psi^c :\right)\right), \\ I_a &= -\sqrt{\frac{q}{2}}\psi_a + \Theta\left(j_a + \left(\frac{1}{2} f^{bc}_a : \psi_b\psi_c : + f^c_{ab} : \psi^b\psi_c :\right)\right). \end{aligned} \tag{40}$$

We now propose a quantum version of the PL T-duality transformation. Perhaps the most comprehensive way to find PL T-duality transformation rules for the quantum fields of the model is to quantize canonically the Sfetsos canonical transformations for PL T-duality related σ-models [21] and then define and solve the quantum version of the equations (22), (24), (26). Though developing this approach for the $N = 2$ superconformal field theory is an important problem and worth solving, it is beyond our reach at the present moment.

Instead we determine the quantum counterpart of the mapping (25) as an automorphism of the operator algebra of the quantum fields, defined by right multiplication by the rightmoving matrix-valued function $H(Z)$, which implies that $N = 2$ SWZNW model is PL self-dual. We propose a very simple way to find the matrix elements of H using super-Kac-Moody Ward identities and the Knizhnik—Zamolodchikov equation.

In the $N = 1$ superfield formalism an arbitrary conformal superfield is defined by the following OPEs [39]:

$$\begin{aligned} I^a(Z_1)F^{\Lambda}(Z_2) &= Z_{12}^{-1/2} E^a F^{\Lambda}(Z_2) + \text{reg}, \\ I_a(Z_1)F^{\Lambda}(Z_2) &= Z_{12}^{-1/2} E_a F^{\Lambda}(Z_2) + \text{reg}. \end{aligned} \tag{41}$$

Here E^a, E_a denote the generators of the $\mathbf{g}^{\mathbb{C}}$ in the representation with the highest weight Λ,

$$\Gamma(Z_1)F^{\Lambda}(Z_2) = Z_{12}^{-3/2}\Delta F^{\Lambda}(Z_2) + Z_{12}^{-1}\frac{1}{2}DF^{\Lambda}(Z_2) + Z_{12}^{-1/2}\partial F^{\Lambda}(Z_2) + \text{reg}, \tag{42}$$

where the conformal dimension Δ is given by

$$\Delta = C_{\Lambda}/q, \ C_{\Lambda} \equiv -(E^a E_a + E_a E^a), \tag{43}$$

and we have used the standard notations for even and odd world-sheet super-intervals between a pair of points $Z_i = (z_i, \Theta_i), i = 1, 2$:

$$Z_{12} \equiv z_1 - z_2 - \Theta_1\Theta_2, \ \Theta_{12} = Z_{12}^{1/2} \equiv \Theta_1 - \Theta_2, \tag{44}$$

so that

$$Z_{12}^{n+1/2} = Z_{12}^{n}\Theta_{12}, \ n \in \mathbb{Z}. \tag{45}$$

We postulate the quantum version of the formula (25):

$$t : F^\wedge(Z) \to \check{F}^\wedge(Z) = F^\wedge(Z)H(Z), \tag{46}$$

which is the quantum counterpart of (25) (here and in what follows the leftmoving coordinate dependence of the fields will be omitted for simplicity). It follows from the Sugawara formula (39) and the OPEs (41), (42) that the conformal superfield $F^\wedge(Z)$ of the model satisfies the Knizhnik—Zamolodchikov equation [39]

$$\frac{q}{2}DF^\wedge(Z) + : F^\wedge I : (Z) = 0, \tag{47}$$

which is a quantization of the classical relation $I = G^{-1}DG$. In view of (46) the dual field $\check{F}^\wedge$ satisfies the similar equation

$$\frac{q}{2}D\check{F}^\wedge(Z) = - : \check{F}^\wedge \check{I} : (Z) = - : \check{F}^\wedge H^{-1}IH : (Z) + \frac{q}{2}\check{F}^\wedge H^{-1}DH(Z). \tag{48}$$

Let us go back for a moment to the classical case and consider Eqs. (22), (24), and (26). Using them we can write

$$H^{-1}DH = 2(\check{I}^+ - H^{-1}I^-H). \tag{49}$$

As its quantum version we propose

$$\frac{q}{2}\check{F}^\wedge H^{-1}DH(Z) = -2 : \check{F}^\wedge(\check{I}^+ - H^{-1}I^-H) : (Z). \tag{50}$$

The substitution (50) converts (48) into

$$: \check{F}^\wedge(\check{I}^- - \check{I}^+) : (Z) =: \check{F}^\wedge(H^{-1}(I^+ - I^-)H) : (Z). \tag{51}$$

Using the left-invariant complex structure J on the group $\mathbf{G}$ one can rewrite it in the form

$$: \check{F}^\wedge(J\mathrm{End}(H)J\check{I}) : (Z) =: \check{F}^\wedge I : (Z), \tag{52}$$

where we have introduced the notation $\mathrm{End}(H)x = HxH^{-1}$, $x \in \mathbf{g}^{\mathbb{C}}$ and we imply that $\mathrm{End}(H)$ belongs to the group of super-Kac-Moody algebra automorphisms. The equation (52) means that $\mathrm{End}(H)$ interchanges the isotropic subalgebras of the Manin triple because it anticommutes with the complex structure J.

By virtue of (52) eq. (48) takes the form

$$\frac{q}{2}\check{F}^\wedge H^{-1}DH(Z) =: \check{F}^\wedge((\mathrm{End}(H^{-1})J\mathrm{End}(H)J - 1)\check{I}) : (Z). \tag{53}$$

Using super-Kac-Moody Ward identities [39] it is easy to see that (53) decays into the system of equations

$$\begin{aligned} &H^{-1}DH = 0, \\ &\mathrm{End}(H^{-1})J\mathrm{End}(H)J - 1 = 0. \end{aligned} \tag{54}$$

Its solution is given by the constant matrix anticommuting with J:

$$\begin{aligned} &DH = 0, \\ &J\mathrm{End}(H) + \mathrm{End}(H)J = 0. \end{aligned} \tag{55}$$

In the orthonormal basis we have chosen, any matrix which anti-commutes with J should have the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & \bar{h} \end{pmatrix}, \tag{56}$$

where h is an arbitrary complex matrix (the bar denotes complex conjugation). Let us denote by $\mathrm{Aut}(\mathbf{g}, J)$ the group of automorphisms of $\mathbf{g}$ which commute with J. It is clear that

$$\mathrm{End}(H) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{57}$$

is a solution of (55). Hence each solution of (55) should have the form:

$$\mathrm{End}(H) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & \bar{m} \end{pmatrix}, \quad \begin{pmatrix} m & 0 \\ 0 & \bar{m} \end{pmatrix} \in \mathrm{Aut}(\mathbf{g}, J). \tag{58}$$

In view of (52) $\mathrm{End}(H)$ should be also an automorphism of the algebra $\hat{\mathbf{g}}$. It imposes on the matrix m the relation

$$m^{cb}\bar{m}_{ab} = \delta^c_a. \tag{59}$$

The next condition we should demand is $t^2 = 1$ (that is, PL T-duality is an involution). It gives the second relation for m:

$$m^{cb}\bar{m}_{ba} = \delta^c_a. \tag{60}$$

Therefore the set of PL T-duality transformations in the $N = 2$ superconformal WZNW model on the group manifold $\mathbf{G}$ is given by the set of matrices (58) satisfying (59), (60). Hence, under the quantum PL T-duality the currents (40) transform as

$$t : I^a \to m^{ab} I_b, \quad I_a \to \bar{m}_{ab} I^b, \tag{61}$$

or in components,

$$t : \psi^a \to m^{ab}\psi_b, \quad j^a \to m^{ab} j_b, \quad \psi_a \to \bar{m}_{ab}\psi^b, \quad j_a \to \bar{m}_{ab} j^b. \tag{62}$$

Taking into account (36), (59), and (62) we find the PL T-duality transformation of the $N = 2$ Virasoro superalgebra currents:

$$\begin{aligned} &t : \Sigma^{\pm} \to \Sigma^{\mp}, \\ &t : K \to -K, \quad T \to T. \end{aligned} \tag{63}$$

Notice that, as in the classical case, PL T-duality acts in the leftmoving sector as an identity transformation. Therefore we may conclude that quantum PL T-duality in the $N = 2$ superconformal WZNW models is a mirror duality and has a geometric realization which is given by PL $\mathbf{G}_{\pm}$-holomorphic action on the target space of the model.

Here a remark is in order. In many examples of the $N = 2$ SWZNW models on the compact groups ($SU(3)$, $SU(2) \times U(1)$, ...) the transformations (61) coincide with Weyl reflections. In these cases mirror symmetry was interpreted by the authors of [28] as a gauge symmetry. They presented also a contradictory example, $SU(2) \times SU(2)$-SWZNW model, where the Weyl reflections failed to give mirror symmetry. It follows from our formula (61) that in this example mirror symmetry is given by an external automorphism of the Lie algebra $su(2) \times su(2)$. This example illustrates the general picture: PL T-duality is given by an automorphism (internal or external) which interchanges the isotropic subalgebras of the underlying Manin triple.

4. POISSON—LIE T-DUALITY AND MIRROR SYMMETRY IN QUANTUM KAZAMA—SUZUKI MODELS

In this section we consider PL T-duality in Kazama—Suzuki models. Kazama and Suzuki have studied [40] the conditions under which an $N = 1$ superconformal coset model can have an extra supersymmetry, giving rise to an $N = 2$ superconformal model. Then the $N = 2$ superconformal coset theories were classified more accurately in [41]. Their conclusion can be reformulated as follows. Suppose the Manin triple $(\mathbf{g}^{\mathbb{C}}, \mathbf{g}_+, \mathbf{g}_-)$ associated with the pair $(\mathbf{g}, J)$ has a Manin subtriple $(\mathbf{h}, \mathbf{h}_+, \mathbf{h}_-)$, that is, $\mathbf{h}_\pm \subset \mathbf{g}_\pm$ are subalgebras of $\mathbf{g}_\pm$ such that $\mathbf{h} \equiv \mathbf{h}_+ \oplus \mathbf{h}_-$ is a subalgebra of $\mathbf{g}^{\mathbb{C}}$ and $\tau : \mathbf{h}_+ \to \mathbf{h}_-$. Notice that the Manin subtriple specified above defines (with the help of the involution τ) a pair $(\mathbf{k}, J)$ such that $\mathbf{k}^{\mathbb{C}} = \mathbf{h}$ and $\mathbf{k} \subset \mathbf{g}$.

Assume that the basis (30) is chosen so that the subbases

$$\begin{aligned} &\{E^i, i = 1, ..., d_h\}, \\ &\{E_i, i = 1, ..., d_h\} \end{aligned} \tag{64}$$

are bases in the subalgebras $\mathbf{h}_+$ and $\mathbf{h}_-$, respectively. Let us consider a vector subspace

$$\mathbf{a} = \mathbf{g}^{\mathbb{C}}/\mathbf{h} \tag{65}$$

generated (over $\mathbb{C}$) by the vectors

$$\{E^\alpha, \alpha = d_h + 1, ..., d\}, \quad \{E_\alpha, \alpha = d_h + 1, ..., d\}. \tag{66}$$

The Manin triple construction of the Kazama—Suzuki models is given by the following.

Proposition. Suppose the isotropic subspaces

$$\mathbf{a}_\pm \equiv \mathbf{a} \cap \mathbf{g}_\pm \tag{67}$$

are Lie subalgebras. Then the currents

$$\begin{aligned} \Sigma^+_{cs} &= \frac{2}{\sqrt{q}} \left(\psi^\alpha j_\alpha + \frac{1}{2} f^\gamma_{\alpha\beta} : \psi^\alpha \psi^\beta \psi_\gamma : \right), \\ \Sigma^-_{cs} &= \frac{2}{\sqrt{q}} \left(\psi_\alpha j^\alpha + \frac{1}{2} f^{\alpha\beta}_\gamma : \psi_\alpha \psi_\beta \psi^\gamma : \right), \\ K_{cs} &= \left(\frac{2C^\beta_\alpha}{q} - \delta^\beta_\alpha \right) : \psi^\alpha \psi_\beta : - \frac{2}{q} \left(\hat{f}_c j^c - \hat{f}^c j_c \right), \\ T_{cs} &= -\frac{1}{q} : (j^a j_a + j_a j^a) : - \frac{1}{2} : (\partial \psi^a \psi_a - \psi^a \partial \psi_a) : + \frac{1}{q} : (u^k u_k + u_k u^k) :, \end{aligned} \tag{68}$$

where

$$\hat{f}^a = f^{a\gamma}_\gamma, \quad \hat{f}_a = f^\gamma_{a\gamma}, \quad C^\alpha_\beta = \hat{f}^a f^\alpha_{a\beta} + \hat{f}_a f^{a\alpha}_\beta, \tag{69}$$

$$u^k = j^k - f^{k\alpha}_\beta : \psi^\beta \psi_\alpha :, \quad u_k = j_k + f^\beta_{k\alpha} : \psi^\alpha \psi_\beta :, \tag{70}$$

satisfy the OPEs of the $N = 2$ super-Virasoro algebra with the central charge

$$c_{cs} = c_g - c_h. \tag{71}$$

This is just the $N = 2$ extension [42] of the Goggard—Kern—Olive construction formulated in terms of Manin triples and can be checked by direct calculations.

The Kazama—Suzuki model based on the coset $\mathbf{G}/\mathbf{K}$ can be obtained from the SWZNW model on the group $\mathbf{G}$ by gauging an anomaly-free subgroup $\mathbf{K}$ [43]. In view of the Manin triple construction (68), (71) this implies classically that the currents corresponding to the Manin subtriple $(\mathbf{h}, \mathbf{h}_+, \mathbf{h}_-)$ should vanish:

$$I^i(Z) = I_i(Z) = 0. \tag{72}$$

In quantizing the theory canonically one should impose in some way such constraints on physical states. We impose

$$\begin{aligned} I^i(Z_1)\Phi(Z_2) &= \text{reg}, \\ I_i(Z_1)\Phi(Z_2) &= \text{reg}, \end{aligned} \tag{73}$$

that is, the physical states of the coset are the highest vectors of the trivial $\hat{\mathbf{h}}$-representation.

Under PL T-duality (61) the set of constraints (73) will transform, in general, into an other set of constraints giving another coset model. Therefore we should define PL T-duality transformations in the Kazama—Suzuki model as the subset of (58)–(60) which stabilizes the set (73), or equivalently, as the subset which stabilizes the Manin subtriple $(\mathbf{h}, \mathbf{h}_+, \mathbf{h}_-)$. Taking into account this condition and using (61) we obtain PL T-duality transformation rules for the currents (68) of the $N = 2$ super-Virasoro algebra,

$$\begin{aligned} &t : \Sigma^{\pm}_{cs} \to \Sigma^{\mp}_{cs}, \\ &t : K_{cs} \to -K_{cs}, \ T_{cs} \to T_{cs}, \end{aligned} \tag{74}$$

which are similar to (63). It is clear that PL T-duality in the leftmoving sector is given by the identity transformation.

Let us consider an example of the Kazama—Suzuki model based on the coset $U(2)/(U(1)\times \times U(1))$ (the $N = 2$ minimal model). The complexification of $u(2)$ is the Lie algebra $gl(2, \mathbb{C})$. In this case the commutation relations (31) in the orthonormal basis (30) are given by

$$\begin{aligned} &[E^0, E^1] = E^1, \\ &[E_0, E_1] = E_1, \\ &[E^1, E_1] = -E^0 + E_0. \end{aligned} \tag{75}$$

The isotropic subalgebras $\mathbf{g}_+$ and $\mathbf{g}_-$ of the complex Manin triple are generated by the vectors E^0, E^1 and E_0, E_1 respectively. The currents of the super-Kac-Moody algebra $\hat{gl}(2, \mathbb{C})$ are characterized by the following OPEs

$$\begin{aligned} &I^a(Z_1)I^b(Z_2) = Z_{12}^{-1/2} f^{ab}_c I^c(Z) + \text{reg}, \\ &I_a(Z_1)I_b(Z_2) = Z_{12}^{-1/2} f^c_{ab} I_c(Z) + \text{reg}, \\ &I^a(Z_1)I_b(Z_2) = -Z_{12}^{-1}\frac{q}{2}\delta^a_b + Z_{12}^{-1/2}(f^a_{bc}I^c - f^{ac}_b I_c) + \text{reg}, \end{aligned} \tag{76}$$

where $a, b, c = 0, 1$ and the structure constants are given by (75). The Manin subtriple defining our coset model is given by

$$h = h_+ \oplus h_-, \quad h_+ = \mathbb{C}E^0, \quad h_- = \mathbb{C}E_0. \tag{77}$$

Thus, the Manin subtriple corresponds to the $N = 2$ $U(1)^2$-SWZNW model which is described by the pair of scalar complex free superfields $X^0(Z), X_0(Z)$ with obvious OPEs

$$X^0(Z_1)X_0(Z_2) = -2\log Z_{12}. \tag{78}$$

The currents of the super-Kac-Moody algebra $\hat{gl}(2, \mathbb{C})$ can be realized in terms of the fields $X^0(Z), X_0(Z)$ and super-parafermions $S^1(Z), S_1(Z)$ [44]:

$$\begin{aligned} &I^0 = \frac{\sqrt{q}}{2}DX^0, \quad I_0 = \frac{\sqrt{q}}{2}DX_0, \\ &I^1 = iS^1 \exp\left(-\frac{1}{\sqrt{q}}(X_0 - X^0)\right), \quad I_1 = iS_1 \exp\left(\frac{1}{\sqrt{q}}(X_0 - X^0)\right). \end{aligned} \tag{79}$$

The super-parafermion OPEs are deduced from the OPEs (76), (78) and the null-vector relation in the trivial $\hat{su}(2)$-representation.

The most general PL T-duality transformation in $U(2)$-SWZNW model is given by

$$\begin{aligned} &I^0 \to I_0, \quad I_0 \to I^0, \\ &I^1 \to \exp(i\phi)I_1, \quad I_1 \to \exp(-i\phi)I^1, \end{aligned} \tag{80}$$

where ϕ is an arbitrary real number. We see that the constraints transform into itself. From these formulas we easily find the PL T-duality transformations of the parafermions of the coset

$$S^1 \to \exp(i\phi)S_1, \quad S_1 \to \exp(-i\phi)S^1. \tag{81}$$

Thus, the PL T-duality transformation acts in the $U(1)^2$-subspace of the $U(2)$-SWZNW model as the usual $R \to 1/R$ T-duality (at the self-dual point), while the PL T-duality transformation (81) corresponds to the axial-vector duality of the coset $SU(2)/U(1)$ [29] (to see this it is enough to recover the leftmoving constraints).

It is clear that there is a direct generalization of this example to the coset models $\mathbf{G}/U(1)^r$, where r is the dimension of the maximal torus of the group $\mathbf{G}$. The PL T-duality transformation will act on the maximal torus as an Abelian $R \to 1/R$ T-duality (at the self-dual point), while in the $N = 2$ Kazama—Suzuki model it will act as an axial- vector duality [45]. In the non-Abelian coset models the PL T-duality transformation rules of the fields are given by the non-Abelian generalization of the axial-vector duality via [46]. In principle they can be found using the non-Abelian generalization of the super-parafermions (79). Some aspects of this construction in the non supersymmetric case can be found in [47].

Thus, in summary, we conclude that quantum PL T-duality in the Kazama—Suzuki models is a mirror duality also.

5. CONCLUSION

In this work we have considered the PL T-duality transformation in quantum $N = 2$ superconformal WZNW and Kazama—Suzuki models. The PL T-duality transformation rules in the quantum $N = 2$ SWZNW models are found using the Manin triple construction of the $N = 2$ SWZNW models, the Knizhnik—Zamolodchikov equation, Ward identities, and the conjecture that, as in the classical case, PL T-duality is given by constant automorphisms of the rightmoving super-Kac-Moody algebras of the models which interchange the isotropic subalgebras of the underlying Manin triples. We have shown that in these models PL T-duality is a mirror duality. We have thus given a geometric realization of the mirror symmetry in these models. Notice also that our results are in agreement with the conjecture proposed in [28] that mirror symmetry can be considered as a gauge symmetry (which is extended in some cases by the external automorphisms) of the self-dual points of the moduli space of the $N = 2$ superconformal field theories.

We have given Manin triple construction of the Kazama—Suzuki models, representing them as $(Manin\ triple)/(Manin\ subtriple)$-cosets. By means of this representation we defined PL T-duality transformations in the Kazama—Suzuki models as the subset of PL T-duality transformations of the numerator triple which stabilize the denominator triple. It was shown that, thus defined, PL T-duality is a mirror duality also. An interesting open problem is to find the corresponding geometric picture of PL T-duality and mirror symmetry in the classical Kazama—Suzuki models.

Our results are useful in discussing Calabi—Yau superstring compactifications and allow us to conjecture that PL T-duality is an adequate geometric structure underlying mirror symmetry. The extension of our results to the Gepner construction of superstring vacua [48] (see also [49]) would be a test of the conjecture.

Another interesting problem is to quantize the equations (22), (24) and determine the quantum version of (21) and (23). Moreover, their solution is important in the context of quantum PL T-duality and mirror symmetry; it may be useful also in discussing T-duality for open strings and D-branes on curved backgrounds and will be helpful in «quantization» of the existing treatments [17, 18].

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