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# MAGNETIZATION OF A DIFFUSIVE RING: BEYOND THE PERTURBATION THEORY

*V. V. Afonin*[1)], *Yu. M. Galperin*[1,2)]

[1)] *A. F. Ioffe Physico-Technical Institute, Russian Academy of Sciences*
[2)] *Department of Physics, University of Oslo, P. O. Box 1048 Blindern, 0316 Oslo, Norway*

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Average persistent current over a set of diffusive metallic rings with fixed number of electrons is considered. We study the case in which the phase breaking time is much greater than an inverse average interlevel distance. In such a case, many return events for an electron must be taken into account. As a result, one arrives at a nonperturbative problem for a cooperon mode fixed by an external magnetic field. This multi-cooperon problem has been considered previously by Altland et al. [17], and in several following papers within the framework of supersymmetric approach. Such an approach involves very tedious calculations which were performed using a computer algebraic package. Here we solve the problem in question with the help of a replica trick. It is demonstrated that the replica trick in combination with a proper analytical continuation in the replica space allows one to obtain the result in much more explicit way.

## 1. INTRODUCTION

Magnetic properties of small conductors were studied extensively during the last several years (see Refs. [1, 2] and the bibliography cited there). It has been understood that the magnetic moment (and the associated persistent current) induced by an external magnetic flux is a very specific manifestation of mesoscopic behavior. While originally predicted to appear in clean, one-dimensional, metallic rings [3], most of the recent discussions about persistent currents have been focused on metallic rings that contain impurities [4]. Static magnetic properties of small rings and dots were studied by several authors [5–11]. An important step in the understanding of magnetization of mesoscopic quantum rings took into account the difference between canonical and grand canonical ensembles [7, 10, 12–14]. It was shown that the magnetization of isolated rings with a fixed number of particles is much larger than that of the ensemble of rings kept under fixed chemical potential. As a result, the main contribution to the magnetic moment was expressed in terms of the fluctuation of the number of particles at fixed chemical potential, $\langle(\delta N)^2\rangle$. The latter quantity was analyzed in Refs. [10, 14] under the condition $\hbar/\Delta\tau_\phi \gg 1$. Here $\Delta$ is an average interlevel distance at the Fermi level, $\Delta^{-1} = \nu V$ ($\nu$ is the density of states at the Fermi level, and $V$ is the volume) and $\tau_\phi$ is the phase-breaking time.

Let us discuss the physical meaning of the parameter $\hbar/\Delta\tau_\phi$. As is well known [15], in the absence of an external magnetic field the quantum correction to the conductivity is proportional to the classical probability $W$ for an electron with a velocity $v$ and momentum $p$ to return to the vicinity of the starting point (more exactly, into the volume of the order of $v\,dt\,(\hbar/p)^2$, which is important for quantum interference). The probability $W$ is given by the expression [15]

$$W \propto \frac{v\hbar^2}{p^2}\int^{\tau_\phi} dt\, P(\mathbf{r},t)\big|_{\mathbf{r}=0}, \tag{1}$$

where $P(\mathbf{r},t)$ is the probability density. Here we employ the fact that in a diffusive regime it is a smooth function of coordinates at the scale of the mean free path $\ell$. To estimate $P(0,t)$ we take into account that the electron diffusion is restricted by a finite volume of the sample. In such a case, we have

$$P(0,t) \propto \frac{1}{V} \sum_{n,\mathbf{n}_\perp} \exp\left[-D\left(\frac{n^2}{R^2} + \frac{\mathbf{n}_\perp^2}{d_\perp^2}\right) t\right] . \tag{2}$$

Here $D$ is the diffusion constant, $R$ is the radius of the ring, and $d_\perp$ is its transverse dimension. The numbers $n$, $\mathbf{n}_\perp$ have the meaning of quantum numbers for longitudinal and transverse diffusive modes, respectively. For a thin ring, $d_\perp \ll R$, only $\mathbf{n}_\perp = 0$ is important. We can see that at $D\tau_\phi/R^2 \gg 1$ the sum over discrete $n$, $\mathbf{n}_\perp$ in (2) cannot be replaced by an integral. Otherwise, only $n = 0$ is important, and $W \sim \tau_\phi \Delta/\hbar$. If this quantity is small, we can restrict the analysis to a single return event.

Let us now concentrate on the case of external magnetic field. In a magnetic field, the number $n$ in the expression (2) must be replaced by $n - \Phi/\Phi_0$, where $\Phi$ is the magnetic flux embedded in the ring, and $\Phi_0 = \pi\hbar c/e$ [16]. It is clear that the quantum contribution is maximal if $\Phi/\Phi_0$ is close to an integer number $n_0$. If the difference $\tilde{n} \equiv n_0 - \Phi/\Phi_0 = 0$, we have the same situation as that in the absence of a magnetic field — only the mode with $n = n_0$ is important. One can expect that this property is also the case at finite $|\tilde{n}| \ll 1$. Indeed, for $n \neq n_0$

$$\delta W \propto \sum_{n \neq n_0} \frac{\Delta}{D\tilde{n}^2/R^2 + 1/\tau_\phi} \sim \frac{\Delta R^2}{D}$$

(for the last estimate we have assumed $D\tau_\phi/R^2 \gg 1$). Consequently, if $\delta W \ll 1$ one can ignore the contributions of all the modes with $n \neq n_0$ to the probability for the return. However, the corresponding contribution of the mode with $n = n_0$ is not small at $\Delta\tau_\phi/\hbar \gtrsim 1$. Hence, we arrive at the problem of calculating of the localization contribution in the case

$$D/R^2 \gg \Delta/\hbar \gtrsim 1/\tau_\phi .$$

In this region we can still use a single-mode approximation, but the perturbation theory involving a single return event fails.

The problem in question was addressed by Altland et al. [17, 18] (see also Refs. [19–21]). The authors used the so-called $Q$-Hamiltonian approach within the framework of the supersymmetric method. An intrinsic feature of this method is that one has to cancel out specific nonphysical contributions. Therefore, the supersymmetric approach involves tedious algebraic calculations. Consequently, the authors of Refs. [17, 18] extensively used a computer algebraic package. As a result, the intermediate equations have not been published, because, as it was stated, the computer printout had many pages.

On the other hand, another approach — the so-called replica method — exists [22]. According to this method, one has to replace the system under consideration by $N$ systems which are identical to the original one and at the end tend $N \to 0$. Usually, after such a procedure one obtains relatively simple expressions. The limiting transition $N \to 0$ (if done properly) automatically cancels out the nonphysical contributions, which has to be done explicitly within the supersymmetric approach.

To take the full advantage of this property, one needs a regular procedure to calculate the limit $N \to 0$. The aim of the present paper is to suggest a procedure of analytical continuation of a nonperturbative two-particle Green's function from integer $N$ to the whole complex plane which includes the point $N = 0$. Such a procedure allows one to calculate the limit rather automatically,without the need of direct cancellation of nonphysical contributions. We obtain an analytical nonperturbative expression for the persistent current in a mesoscopic diffusive ring and compare it with the results of Refs. [17, 18].

The paper is organized as follows. In Sec. 2 the basic equations for the fluctuation of the number of particles, as well as for the persistent current are analyzed. The effective action in the single-mode approximation is considered in Section 3. In Sec. 4 the particle number autocorrelation function and persistent current are calculated in the nonperturbative region, and results are summarized. In the following calculations we set $\hbar = 1$. Then $\hbar$ will be restored in the estimates and final results.

## 2. BASIC EQUATIONS

According to Ref. [10], the main contribution to the persistent current $I$ can be expressed in terms of the magnetic flux $\Phi$ embedded in the ring as follows:

$$I = \frac{c\Delta}{2} \frac{\partial}{\partial \Phi} \langle (\delta N)^2 \rangle_{\mu=\langle\mu\rangle} , \tag{3}$$

where $\langle (\delta N)^2 \rangle_{\mu=\langle\mu\rangle}$ is the particle number autocorrelation function, calculated at a given value of chemical potential. The latter can be expressed in terms of single-electron Green's functions as [23]

$$\langle (\delta N)^2 \rangle = \int_{-\mu}^{0} d\epsilon_1 d\epsilon_2 K(\epsilon_1, \epsilon_2) , \tag{4}$$

where

$$K(\epsilon_1, \epsilon_2) = \frac{1}{\pi^2} \int d\mathbf{r}_1 d\mathbf{r}_2 \times$$
$$\times \left\{ \langle \operatorname{Im} G^R_{\epsilon_1}(\mathbf{r}_1, \mathbf{r}_1) \operatorname{Im} G^R_{\epsilon_2}(\mathbf{r}_2, \mathbf{r}_2) \rangle - \langle \operatorname{Im} G^R_{\epsilon_1}(\mathbf{r}_1, \mathbf{r}_1) \rangle \langle \operatorname{Im} G^R_{\epsilon_2}(\mathbf{r}_2, \mathbf{r}_2) \rangle \right\} . \tag{5}$$

Here $\langle \ldots \rangle$ means the usual impurity average. The quantity (5) has been calculated in Ref. [10] in the limiting case $\Delta\tau_\phi \ll \hbar$. Our aim is to go beyond this limiting case, i.e. to calculate the correlation function for arbitrary $\Delta\tau_\phi/\hbar$, keeping $p_F \ell \gg \hbar$. For this purpose we employ the method used by Efetov, Larkin, and Khmelnitskii [22] with minor modifications. Namely, we will use the so-called $Q$-Hamiltonian approach within the framework of the replica trick. The confined expression for the correlation function $K(\omega)$ ($\omega = \epsilon_1 - \epsilon_2$) can be written in the form (cf. Ref. [22])

$$K(\omega) = \left[ \frac{\nu^2}{N^2 \int DQ \exp(-F)} \int d\mathbf{r}_1 \, d\mathbf{r}_2 \int DQ e^{-F} \operatorname{Tr} [\Lambda Q(\mathbf{r}_1)] \operatorname{Tr} [\Lambda Q(\mathbf{r}_2)] \right]_{N \to 0} , \tag{6}$$

where

$$F = \frac{\pi\nu}{4}\int dz\,\mathrm{Tr}\left[D\left(\nabla Q + \frac{ie}{c}\mathbf{A}[Q,\Lambda]_-\right)^2 + 2\left(i\omega - \frac{1}{\tau_\phi}\right)\Lambda Q\right]. \quad (7)$$

Here $\mathbf{A}$ is the vector-potential, and $[A,B]_- \equiv AB - BA$. Taking into account only the elastic scattering by short-range, nonmagnetic impurities, we can specify $Q$ as $2N \times 2N$ Hermitian matrices, $Q^2 = 1$, $\mathrm{Tr}\,Q = 0$, $N$ is the number of replicas, while

$$\Lambda = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix},$$

where $\hat{1}$ is the $N \times N$ unit matrix. The parameter $\tau_\phi^{-1}$ is introduced phenomenologically. We assume that the phase breaking is due to the inelastic processes. Following [22], we use the parameterization

$$Q = \Lambda\exp(W), \quad W = \begin{pmatrix} 0 & B \\ -B^+ & 0 \end{pmatrix},$$

where $B$ is an arbitrary $N \times N$ matrix.

### 3. EFFECTIVE ACTION IN SINGLE-MODE APPROXIMATION

Let us consider a ring with the radius $R$ and the width $d_\perp \ll R$. We can therefore, take into account only the dependence of the matrices $B$ on the angular coordinate $\varphi$. Expanding this dependence into the discrete Fourier series, $B = \sum_n B_n \exp(in\varphi)$, we introduce the mode number $n$. As was explained in Sec. 1, only one mode with $n = n_0$ corresponding to $\min(n - \Phi/\Phi_0)$ is important (this assumption will be justified at the end of Sec. 4A). Retaining only this mode and assuming $\nabla_\varphi = (1/R)\partial/\partial\varphi$, we obtain $W\nabla_\varphi W + \nabla_\varphi W W = 0$. Hence,

$$\nabla_\varphi Q = \frac{1}{R}\Lambda\frac{\partial}{\partial\varphi}e^W = (\nabla_\varphi W)W^{-1}\,\mathrm{sh}\,W = \frac{in_0}{R}\,\mathrm{sh}\,W.$$

We can then expand $\mathrm{sh}^2 W = (1/2)(\mathrm{ch}\,2W - 1)$ as a series in

$$W^{2k} = (-1)^k \begin{pmatrix} \sqrt{B_{n_0}B_{n_0}^+} & 0 \\ 0 & \sqrt{B_{n_0}^+B_{n_0}} \end{pmatrix}^{2k}.$$

The item $\mathrm{Tr}\,(\Lambda Q)$ can be treated in a similar way. As a result, we obtain the following expression for $F$:

$$F = \frac{\pi}{2\Delta}\left[\frac{D}{R^2}\left(n_0 - \frac{\Phi}{\Phi_0}\right)^2 \mathrm{Tr}\,\sin^2\left(\sqrt{B_{n_0}B_{n_0}^+}\right) + 2\left(i\omega - \frac{1}{\tau_\phi}\right)\mathrm{Tr}\cos\left(\sqrt{B_{n_0}B_{n_0}^+}\right)\right]. \quad (8)$$

We must now consider an important point. An arbitrary complex $N \times N$ matrix $B$ can be described by two Hermitian $N \times N$ matrices. These matrices are defined as

$$B = \rho\,\exp(i\varphi), \quad B^+ = \exp(-i\varphi)\,\rho. \quad (9)$$

The quantity $F$ depends only on the matrix $\rho$. On the other hand, an arbitrary Hermitian matrix could be diagonalized, the eigenvalues being real. One can immediately see that the

integral over $\rho$ in the expression (6), with $F$ taken from Eq. (8) for the correlation function diverges. This divergence, in fact, does not occur, because the eigenvalues of $\rho$ must be defined in a finite interval. Indeed, one has to define the variables $\rho$ in a way to obtain a one-to-one correspondence between $\rho$ and $Q$. On the other hand, one can explicitly show that

$$Q = \Lambda e^W = \begin{pmatrix} \cos\sqrt{BB^+} & \dfrac{\sin\sqrt{BB^+}}{\sqrt{BB^+}}B \\ B^+\dfrac{\sin\sqrt{BB^+}}{\sqrt{BB^+}} & -\cos\sqrt{B^+B} \end{pmatrix} = \begin{pmatrix} \cos\rho & \sin\rho\, e^{i\varphi} \\ e^{-i\varphi}\sin\rho & -\cos\rho^T \end{pmatrix}, \tag{10}$$

where $\rho^T$ denotes the transposed matrix $\rho$. Here we have employed the relation

$$\rho^T = \sqrt{e^{-i\varphi}\rho^2 e^{i\varphi}}\,, \tag{11}$$

which is a consequence of the symmetry properties of the initial replica Hamiltonian (see Appendix $A$). It is clear that the matrix $Q$ is a periodic function of $\rho$, and that one has to specify a region at least not larger than one period in order to obtain a one-to-one correspondence. Moreover, to obtain proper analytical properties (damping is the lower semi-plane of the $\omega$-variable) of the action $F$ (8), we must define the integration limits as $(-\pi/2, \pi/2)$. Finally, the action $F$ reads as

$$F = \frac{\pi}{2\Delta}\left[\frac{D}{R^2}\left(n_0 - \frac{\Phi}{\Phi_0}\right)^2 \operatorname{Tr}\sin^2\rho + 2\left(i\omega - \frac{1}{\tau_\phi}\right)\operatorname{Tr}\cos\rho\right]. \tag{12}$$

Now let us transform the variables from $B, B^+$ to $\rho, u \equiv \exp(i\varphi)$, where the Jacobian is (see Appendix B)

$$\frac{D(B, B^+)}{D(\rho, u)} = 2(\det\, u^{-1}\rho)^N\,. \tag{13}$$

We see that the variables $u$ can be integrated out and cancelled with the denominator in Eq. (6).

## 4. PARTICLE NUMBER AUTO-CORRELATION FUNCTION

### A. Eigenvalue representation

Let us now return to Eq. (6). Since $\operatorname{Tr}\Lambda Q = 2\operatorname{Tr}\cos\rho$, we see that the integrand depends only on the eigenvalues of $\rho$. Hence, we should transform the variables to the eigenvalues and some other ones which could be integrated out in the numerator and the denominator. This transform is outlined in Appendix 5. As a result, we obtain

$$K_N = \frac{(V\nu)^2 \displaystyle\int_0^1 \{d\lambda\}\,\theta(\lambda^{(i)})\left[\sum_{j=1}^{N}\cos\frac{\pi\lambda^{(j)}}{2}\right]^2}{N^2\displaystyle\int_0^1 \{d\lambda\}\,\theta(\lambda^{(i)})}, \tag{14}$$

where $\{d\lambda\} \equiv \prod_{i=0}^{N-1} d\lambda^{(i)}|\lambda^{(i)}|^{2i+N}$, and

$$\theta(x) = \exp\left[-\frac{\pi D}{2\Delta R^2}\left(n_0 - \frac{\Phi}{\Phi_0}\right)^2 \sin^2\left(\frac{\pi x}{2}\right) - \frac{\pi}{\Delta}\left(i(\epsilon_1 - \epsilon_2) - \frac{1}{\tau_\phi}\right)\cos\left(\frac{\pi x}{2}\right)\right]. \quad (15)$$

The expression (15) contains three dimensionless parameters:

$$\gamma \equiv \frac{\pi\hbar}{\Delta\tau_\phi}, \quad \Omega \equiv \frac{\pi(\epsilon_1 - \epsilon_2)}{\Delta}, \quad \mathscr{E} \equiv \frac{\hbar\pi D\tilde{n}^2}{2R^2\Delta}, \quad (16)$$

where $\tilde{n} \equiv (n_0 - \Phi/\Phi_0)$. It is important to keep in mind the following. If $\max(\gamma, \Omega) \gg 1$, only the small $\lambda$'s are important. Hence, one obtains at the result which is found in the framework of the perturbation theory [10, 14]. However, if both $\gamma$ and $\Omega$ are small one must sum the multicooperon contributions, which cannot be done in the framework of the perturbation theory. There is a substantial simplification in the case

$$\tilde{n} \ll 1, \quad \text{but} \quad \mathscr{E}_c \equiv \frac{\hbar\pi D}{2R^2\Delta} \gg 1. \quad (17)$$

In this case, only one mode with $|n_0 - \Phi/\Phi_0| \ll 1$ is important; this is the case in which Eq. (15) is valid. Consequently, we consider the case in which the inequalities (17) hold, but the quantities $\gamma$ and $\Omega$ can be arbitrary. In fact, the mode $n_0$ must be considered in a nonperturbative manner, while the other modes can be treated in the framework of the perturbation theory.

### B. Analytical continuation

We are not able to analytically calculate the expression (14) for an arbitrary $N$. Instead, we will perform analytical continuation of this expression to arbitrary $N$, and then calculate its limit as $N \to 0$.

We introduce the quantity

$$Z_N = \prod_{k=0}^{N-1} \int_0^1 dx_k x_k^{2k+N} \theta(x_k), \quad (18)$$

where

$$\theta(x) = \exp\left[-\mathscr{E}\sin^2(\pi x/2) - (i\Omega - \gamma)\cos(\pi x/2)\right]. \quad (19)$$

It is convenient to define

$$\zeta_N \equiv \ln Z_N \equiv \zeta^R + \zeta^A,$$

where

$$\zeta^R \equiv \frac{1}{2}\sum_{k=\frac{N+1}{2}}^{\frac{3N-1}{2}} \ln \int_0^1 dx\, x^{2k-1+\delta}\theta(x),$$

$$\zeta^A \equiv \frac{1}{2}\sum_{k=-\frac{N+1}{2}}^{-\frac{3N-1}{2}} \ln \int_0^1 dx\, x^{-2k-1+\delta}\theta(x), \quad (20)$$

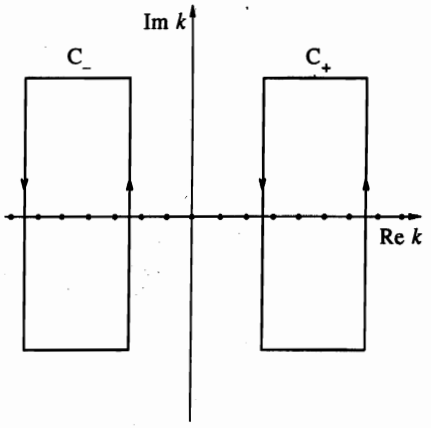


**Fig. 1.** The integration contours $C_{\pm}$ for $N = 4$

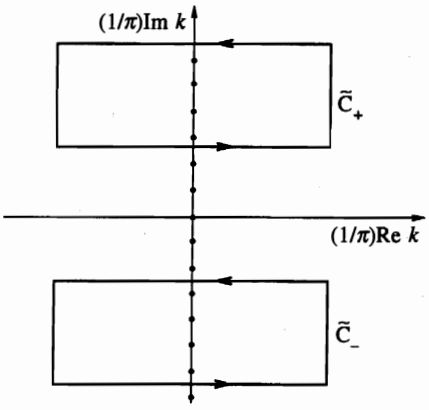


**Fig. 2.** The integration contours $\tilde{C}_{\pm}$ for $N = 4$

$\delta$ is a small positive number which later will vanish. We introduce this parameter to keep the important integrals convergent in the limit $N \to 0$. The first step is to express the sum over $k$ in terms of the contour integral over complex $k$. For this purpose we should keep in mind that the derivative $\partial f[2\pi i(k + 1/2)]/\partial k$ [where $f(z) = (e^z + 1)^{-1}$] has second-order poles at integer numbers. Consequently, one can express $\zeta_{R(A)}$ as

$$\zeta_{R(A)} = \int\limits_{C^{\pm}} dk \left(\frac{\partial f(2\pi ik)}{\partial k}\right) F^{\pm}(k), \tag{21}$$

$$F^{\pm}(k) = \frac{1}{2}\int^{k} dk' \ln\left[\int\limits_{0}^{1} dx\, x^{\pm 2k' - 1 + \delta}\theta(x)\right]. \tag{22}$$

The contours $C^{\pm}$ are shown in Fig. 1. These expressions are correct only if other singularities, except for the poles of $f$, are not important. We can show that the function $F^{+}(k)$ has singularities only in the left-hand semi-plane of the complex variable $k$, while the function $F^{-}$ has singularities only in the right-hand semi-plane of $k$. To prove this statement, one must expand the function $\theta(x)$ in a Taylor series. For the following, it is convenient to rotate the $k$-plane through $\pi/2$ by introducing a new variable, $k_1 \equiv 2\pi ik$. The transformed contours $\tilde{C}^{\pm}$ are shown in Fig. 2.

Making use of exponential convergence of the integral due to the properties of $\partial f/\partial k$, we transform the contour integrals to the integrals along the real axis. For simplicity let us assume that $N$ is even. As a result, we have

$$\zeta_{R(A)} = \mp \int\limits_{-\infty}^{\infty} dk \left(\frac{\partial f(k)}{\partial k}\right)\left[F^{\pm}\left(\frac{k \pm i\pi N}{2\pi i}\right) - F^{\pm}\left(\frac{k \pm 3i\pi N}{2\pi i}\right)\right]. \tag{23}$$

We can now perform an analytical continuation over $N$. For the functions $F^{\pm}$ the continuation must be done in a different way for the reason to be discussed later. For this purpose we replace

$iN$ by $\pm N_0$ in the functions $F^{\pm}$, respectively. Here $N_0$ is a real, positive quantity which will eventually tend to zero. Finally, we have

$$\zeta_{N_0} = \int\limits_{-\infty}^{\infty} dk \left[\left(\frac{\partial f(k-\pi N_0)}{\partial k}\right) - \left(\frac{\partial f(k-3\pi N_0)}{\partial k}\right)\right]\left[F^{-}\left(\frac{k}{2\pi i}\right) - F^{+}\left(\frac{k}{2\pi i}\right)\right] . \quad (24)$$

As a result, the lowest-order term in the $N_0$-expansion of the function $\zeta_{N_0}$ is $\propto N_0^2$. Finally, we obtain

$$\zeta = 2\pi i N_0^2 \int\limits_{-\infty}^{\infty} dk \left(\frac{\partial^2 f(k)}{\partial k^2}\right) \ln\left[\int\limits_0^1 dx\, x^{-ik/\pi - 1 + \delta}\theta(x)\right] . \quad (25)$$

The reason for splitting the function $\zeta_N$ into $\zeta^R$ and $\zeta^A$ with the replacements $N \to \pm iN_0$ is as follows. As $N$ tends to zero, the integration contour comes infinitely close to the cut of the logarithm functions in the expressions for $F^{\pm}$. Such a situation is not the case for any finite $N$, and it leads to a nonphysical pinch which should be subtracted. Within the above mention procedure such a contribution is purely imaginary, while the one of interest to us is real. The imaginary contributions to $F^{+}$ and $F^{-}$ have opposite signs. Thus, the nonphysical contribution is automatically cancelled in the sum $\zeta^R + \zeta^A$. We note that these terms are of the first order in $N_0$, and that they must vanish; otherwise, the two-particles Green's function would be divergent.

In fact, a similar trick has been used by Matsubara to formulate the thermal Green's function technique (see, e.g. Ref. [24]). Let us compare our analytical continuation of the function $\zeta$ to the analytical continuation of the two-particle Matsubara Green's function $K(\Omega_m)$, where $\Omega_m$ is the external Matsubara frequency. In each case one must use two functions, which are regular in the upper (retarded) and the lower (advanced) semi-plane, respectively. The two above-mentioned functions can then be combined into one with a cut in the complex plane. The physical reason for such a splitting into $R$ and $A$ parts is to cancel the nonphysical contributions. In the Matsubara case the nonphysical contributions to $K(\Omega_m)$ arise at the point $\Omega_m \to 0$ and cancel out after a similar continuation $\Omega_m \to i\Omega$ of the sum over $\Omega$.

### C. Persistent current

Following Ref. [10], we express the current according to Eqs. (3) and (4). On the other hand,

$$K(\epsilon_1, \epsilon_2) \propto e^{-\zeta} \frac{\partial^2}{\partial\epsilon_1\, \partial\epsilon_2} e^{\zeta} .$$

Finally, we obtain

$$I = -\frac{c\Delta}{2} \frac{\partial}{\partial\Phi} \lim_{N_0 \to 0} \frac{\zeta_{N_0}(\epsilon_1 = \epsilon_2 = 0)}{N_0^2} = -J_0 \tilde{n} \mathscr{F} , \quad (26)$$

where $J_0 = \hbar c D / R^2 \Phi_0 = eD/\pi R^2$, and

$$\mathscr{G} = i \int_{-\infty}^{\infty} dk \, \frac{\partial^2 f(k)}{\partial k^2} \, \frac{\int_0^1 dx \, \sin^2(\pi x/2) x^{-ik/\pi - 1 + \delta} \theta(x, \Omega = 0)}{\int_0^1 dx \, x^{-ik/\pi - 1 + \delta} \theta(x, \Omega = 0)} . \tag{27}$$

One can check directly that at

$$\Delta \ll \frac{\hbar}{\tau_\phi} \ll \frac{\hbar D}{R^2}, \quad |\tilde{n}| \ll 1$$

the above expressions lead to the expressions obtained in Refs. [10, 14]. One can calculate the integrals with the help of the steepest-descent method to find [10, 14]

$$I = -J_1 \tilde{n}, \quad J_1 = \frac{e\Delta}{\pi^3} \frac{D\tau_\phi}{R^2} . \tag{28}$$

In the region

$$\gamma \ll 1, \quad \mathscr{E}_c \gg \mathscr{E} \gg 1$$

one can also develop a perturbation theory. Indeed, only small values of $x$ in the integrals in Eq. (27) are important. The physical reason for this is the magnetic-field-induced phase breaking. In this region we obtain the result

$$\mathscr{G} = \frac{1}{\pi^2 \tilde{n}^2} \frac{R^2 \Delta}{\hbar D}, \quad I = -\frac{1}{\pi^3 \tilde{n}} \frac{e\Delta}{\hbar} . \tag{29}$$

This result agrees with the asymptotic result of Ref. [14] for $\gamma \gg 1$, $\sqrt{R^2/D\tau_\phi} \ll \tilde{n} \ll 1$. Note that the result (29) obtained for $\gamma \ll 1$ is valid in the region

$$\sqrt{R^2\Delta/\hbar D} \ll \tilde{n} \ll 1 .$$

For the case $\gamma \ll 1$, $\mathscr{E} \ll 1$, where the perturbation theory is not applicable, we set $\theta(x, \Omega = 0) = 1$. As a result, we obtain $\mathscr{G} = 0.21$, and the current being

$$I = -0.21 \frac{eD}{\pi^3 R^2} \tilde{n} . \tag{30}$$

We observe a maximum at $\tilde{n} \sim \sqrt{R^2\Delta/\hbar D}$, where the maximal current is

$$I_{\max} \sim e\sqrt{\Delta D/\hbar R^2} . \tag{31}$$

Expressions (29) and (30) are fully consistent with the curve calculated in Refs. [17, 18] with the help of a computer algebraic package. Let us consider the dependence of the maximal current on $\gamma \approx \hbar/\Delta\tau_\phi$. At $\gamma \gg 1$ the perturbation theory [10, 14] predicts the maximum of the current at $\tilde{n} \sim \sqrt{R^2/D\tau_\phi}$; the maximum value is

$$I_{\max} \sim e \frac{\Delta}{\hbar} \sqrt{\frac{\tau_\phi D}{R^2}} . \tag{32}$$

Consequently, $I_{\max} \propto \gamma^{-1/2}$ at $\gamma \gg 1$, and it is $\gamma$-independent at $\gamma \ll 1$. In this region we estimate the persistent current to

$$J_{\max} \sim \frac{ev_F}{R}\sqrt{\frac{\Delta\tau_{el}}{\hbar}},$$

where $\tau_{el}$ is the elastic relaxation time. The quantity $\Delta\tau_{el}/\hbar$ for a typical metal can be estimated as $(\ell/R)(a^2/A)$ where $\ell$ is the mean free path, $a$ is a typical interatomic distance, and $A$ is the cross section of the ring. Equation (30) shows that at $\Delta\tau_\phi/\hbar \gg 1$ the phase-breaking time $\tau_\phi$ is not contained in the expression for the persistent current. Returning to Eq. (1) we have to conclude that at $t \gtrsim \hbar/\Delta$ the electronic wave packet is not smeared in space. This means that the $n_0$-mode of the cooperon is localized in a sense; the localization length is of the order of $\sqrt{\hbar D/\Delta}$. Of course, this does not mean localized, because other modes are still under weak localization conditions.

The range of parameters in which the theory is applicable and where it leads to non-trivial results can be expressed as follows

$$1 \ll R/\ell \ll K,\ (1/K)(\tau_\phi/\tau_{el}),$$

where $K \sim (pd_\perp/\hbar)^2$ is the number of transverse channels. The left inequality is the criterion for a diffusive motion, the first right inequality is the Thouless criterion $\mathscr{G}_c \gg 1$, and the last right inequality is the condition $\Delta\tau_\phi/\hbar \gg 1$. We see that one needs low temperatures to satisfy the inequality $\tau_\phi/\tau_{el} \gg K \gg 1$, and also samples of very small size. No previous experiments, to the best of our knowledge, satisfy this set of conditions.

## 5. DISCUSSION

As one can see from the preceding sections, the results of the replica procedure which are complicated for arbitrary integer $N$, are rather simple in the limit $N \to 0$. In this limit the nonphysical contributions are cancelled automatically, while in the supersymmetric method this has been done explicitly. An important feature which leads to such a simplification is the procedure of the analytical continuation which is done before direct calculations. Specifically, one has two functions which are analytical in the upper (lower) semi-plane of the complex plane of $N$, respectively. The proper analytical continuation is a combination of these two functions. Consequently, it has a cut at $\mathrm{Im}\, N = 0$. The procedure used above allows one to cancel automatically the nonphysical pinch in the two-particle Green's function, which otherwise would exist at $N_0 = 0$. We believe that such a construction is important, in general, for the calculations involving the replica trick. In such a way we reproduce analytically and rather simply the results obtained in Refs. [17, 18] by a computer algebraic package.

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## APPENDIX A

### Effective action–derivation

Here we rederive the expression (6) following Ref. [22], in order to clarify important symmetry properties. Following Ref. [22] we use the replica trick and introduce field operators:

$$\Psi = \{\psi_1, \ldots, \psi_N, \psi_1^+, \ldots, \psi_N^+\}, \quad \Psi^+ = \begin{pmatrix} \psi_1^+ \\ \cdots \\ \psi_N^+ \\ -\psi_1 \\ \cdots \\ -\psi_N \end{pmatrix}.$$

Here $\psi_i\psi_i^+ + \psi_i^+\psi_i = 0$. The action can be written as follows:

$$\begin{aligned} F &= i\int (d\mathbf{r})\,\Psi^+(\mathbf{r})(\hat{E} - \hat{\mathscr{H}})\Psi^+(\mathbf{r}), \\ \hat{E} &= E\hat{I}, \\ \hat{\mathscr{H}} &= [H_0 + U_{el}(\mathbf{r})]\,\hat{I} - \left(\frac{\omega}{2} + i\delta\right)\Lambda. \end{aligned} \tag{A.1}$$

Here $H_0$ is the free-electron Hamiltonian, $\hat{I}$ is the $2N \times 2N$ unit matrix, and $U_{el}(\mathbf{r}) = U_0\sum_i^M \delta(\mathbf{r} - \mathbf{r}_i)$, where $M$ is the total number of impurities. The first $N$ rows of $\hat{\mathscr{H}}$ describe the evolution of the retarded Green's functions, while the last $N$ rows describe an evolution of the advanced Green's functions. The following step is the averaging over the positions of the impurities. We have

$$\Sigma \equiv \prod_{i=1}^{M}\int \frac{d\mathbf{r}_i}{V}\exp\left[iU_0\sum_{i=1}^{M}\Psi^+(\mathbf{r}_i)\Psi(\mathbf{r}_i)\right] = \left[\int\frac{d\mathbf{r}}{V}\prod_{f=1}^{2N}\left(1 + iU_0\psi_f^+(\mathbf{r})\psi_f(\mathbf{r})\right)\right]^M. \tag{A.2}$$

Here we have taken into account that only the linear terms in $\psi_f^+$ and $\psi_f$ can enter the continual integral for the correlation function (Grassman algebra). For the same reason, one must allow for only the terms with different $f$ when calculating the product. For a weak scattering and in the thermodynamic limit $M, V \to \infty, M/V = \text{const}$,

$$\begin{aligned} \Sigma &\approx \exp(\delta\mu + i\Gamma), \\ \delta\mu &= \frac{MU_0}{V}\int d\mathbf{r}\,\Psi^+(\mathbf{r})\Psi(\mathbf{r}), \\ \Gamma &= \frac{g_0^2}{2}\int d\mathbf{r}\sum_{f\neq g}\psi_f^+(\mathbf{r})\psi_f(\mathbf{r})\psi_g^+(\mathbf{r})\psi_g(\mathbf{r}). \end{aligned} \tag{A.3}$$

Here $\delta\mu$ is a shift in the chemical potential, and $g_0^2 = 2MU_0^2/V$ ($g_0$ is the coupling constant). By analogy with Ref. [22], we introduce an auxiliary scalar field which is represented by Hermitian matrices $Q$. As a result, the effective $\psi^4$ interaction can be decoupled as follows:

$$\exp\left[-\frac{g_0^2}{2}\int d\mathbf{r}\sum_{f\neq g}\psi_f^+(\mathbf{r})\psi_f(\mathbf{r})\psi_g^+(\mathbf{r})\psi_g(\mathbf{r})\right] =$$
$$= \frac{\int \mathscr{D}Q\,\exp\left[-\mathrm{Tr}\int d\mathbf{r}(\pi\nu/4\tau_{el})Q^2(\mathbf{r}) - (1/2\tau_{el})\Psi^+(\mathbf{r})Q\Psi(\mathbf{r})\right]}{\int \mathscr{D}Q\,\exp\left[-\mathrm{Tr}\int d\mathbf{r}(\pi\nu/4\tau_{el})Q^2(\mathbf{r})\right]}. \qquad \text{(A.4)}$$

Here we used the definition $2\pi\nu g_0^2\tau_{el} = 1$. This expression is the same as Eq. (17) from Ref. [22]. To analyze the symmetry properties of the impurity-averaged Hamiltonian, we take into account that the initial Hamiltonian possesses the property $\mathscr{H}_{ij}|_{i,j\leq N} = \mathscr{H}^*_{i+N,j+N}$. This property must be kept after impurity averaging and after introducing the field $Q$. In terms of $Q$ it reads as $iQ_{ij} = -iQ^*_{i+N,j+N}$. Taking into account Eq. (10) we obtain the relation (11). The following steps are exactly the same as those in Ref. [22].

## *APPENDIX B*

### Calculation of the Jacobian

Let us arrange the columns of the $2N^2 \times 2N^2$ matrix $\partial(B, B^+)/\partial(\rho, u)$ as

$$\{B_{11}, \ldots, B_{N1}; B_{12}, \ldots, B_{NN}; B^+_{11}, \ldots, B^+_{NN}\}$$

and the rows as

$$\{\rho_{11}, \ldots, \rho_{N1}; \rho_{12}, \ldots, \rho_{N2}, \ldots, \rho_{NN};$$
$$u_{11}, \ldots, u_{N1}; u_{12}, \ldots, u_{N2}, \ldots, u_{NN}\}.$$

Taking into account the matrix identities

$$dB = d\rho\, u + \rho\, du,$$
$$dB^+ = u^{-1}\, d\rho - u^{-1}\, du\, u^{-1}\rho \qquad \text{(B.1)}$$

we express $\partial(B, B^+)/\partial(\rho, u)$ as follows:

$$\frac{\partial(B, B^+)}{\partial(\rho, u)} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22,} \end{pmatrix}, \qquad \text{(B.2)}$$

where $\hat{A}_{ik}$ are the $N^2 \times N^2$ matrices. One can show that

$$\hat{A}_{11} = u \times \hat{1}, \quad \hat{A}_{22} = -(u^{-1}\rho) \times u^{-1},$$
$$\hat{A}_{21} = \underbrace{\rho \otimes \rho \otimes \ldots \rho}_{N},$$
$$\hat{A}_{12} = \underbrace{u^{-1} \otimes u^{-1} \otimes \ldots u^{-1}}_{N}. \qquad \text{(B.3)}$$

Here $\times$ means the Kronecker product, and $\otimes$ means the direct product [25]. Making use of the identity [25]

$$\det(A \times B) \equiv (\det A)^p\,(\det B)^q$$

(where $q$ and $p$ are the ranges of the matrices $A$ and $B$, respectively), and of the Laplace expansion of the determinant [25], we obtain Eq. (13).

## APPENDIX C

### Variable transformation

Let us consider the set of variables which includes the eigenstates $\lambda_{(b)}$ and $N$ eigenvectors $\mathbf{X}^{(b)}$. One can see from the definition $\rho_{ik}X_k^{(b)} = \lambda^{(b)}\delta_{ik}X_i^{(b)}$ that any vector of the type $e^{i\chi^{(b)}}\mathbf{X}^{(b)}$ (where $\chi^{(b)}$ is an arbitrary phase) satisfies the equation with the same $\lambda^{(b)}$ and $\rho_{ik}$. Consequently, one must exclude $N$ extra variables $\chi^{(b)}$. We therefore require the diagonal elements $X_i^{(i)}$ to be real. Consequently, the matrix $X_k^{(i)}$ can be constructed according to the following procedure. The first column, $X_i^{(1)}$, contains $N-1$ variables $X_i^{(1)}$, $i \neq 1$, while the last (real) column, $X_1^{(1)}$, is calculated from the requirement of normalization. In the next column, $X_i^{(2)}$, the last $N-2$ variables are chosen independently. The element $X_1^{(2)}$ is determined by the orthogonality of the vectors $\mathbf{X}^{(2)}$ and $\mathbf{X}^{(1)}$, while the last element, $X_2^{(2)}$ is determined by the normalization of $|\mathbf{X}^{(2)}|$. The following elements are determined by continuation of this procedure. Note that since all the off-diagonal elements are complex, we can consider the real ($U_k^{(i)}$) and the imaginary ($V_k^{(i)}$) parts. In this way we can present $N^2$ independent elements of the matrix $\rho$ in terms of $N$ eigenvalues $\lambda^{(i)}$, and $N^2 - N$ independent variables $U_k^{(i)}$ and $V_k^{(i)}$. From the definition, $\rho_{ij} = \sum_k X_i^{(k)}\lambda^{(k)}X_j^{(k)*}$, we can express $\rho$ in terms of $\{U, V, \lambda\}$ as

$$\begin{aligned}
\frac{\partial\rho_{ij}}{\partial U_i^{(k)}} &= \lambda^{(k)}\left[U_j^{(k)}(1+\delta_{ij}) - iV_j^{(k)}(1-\delta_{ij})\right], \\
\frac{\partial\rho_{ij}}{\partial V_i^{(k)}} &= \lambda^{(k)}\left[iU_j^{(k)}(1-\delta_{ij}) + V_j^{(k)}(1+\delta_{ij})\right], \\
\frac{\partial\rho_{ij}}{\partial\lambda^{(k)}} &= U_i^{(k)}U_j^{(k)} + V_i^{(k)}V_j^{(k)} + i\left(V_i^{(k)}U_j^{(k)} - U_i^{(k)}V_j^{(k)}\right).
\end{aligned} \tag{C.1}$$

Note that the above formulas do not contain summation over repeated superscripts. To calculate the Jacobian, we arrange the corresponding $N^2 \times N^2$ transformation matrix in the following way. The columns are labeled by $N^2$ «old» variables

$$\{\rho_{i1}, \rho_{i2}, \ldots, \rho_{iN}\}.$$

The rows are labeled by $N^2$ «new» variables

$$\underbrace{\{U_{k>1}^{(1)}\}}_{N-1}, \underbrace{\{U_{k>2}^{(2)}\}}_{N-2}, \ldots, \underbrace{\{U_N^{(N-1)}\}}_{1}; \underbrace{\{V_{k>1}^{(1)}\}}_{N-1}, \underbrace{\{V_{k>2}^{(2)}\}}_{N-2}, \ldots, \underbrace{\{V_N^{(N-1)}\}}_{1}; \underbrace{\{\lambda^{(k)}\}}_{N}.$$

Consequently, as follows from Eq. (33), the first $(N-1)$ rows contain the common factor $\lambda^{(1)}$ times the quantities which depend only on $\{U, V\}$. The next $(N-2)$ lines contain the factor $\lambda^{(2)}$, etc. The last $N$ lines are $\{\lambda\}$-independent. As a result, the Jacobian can be expressed as $\prod_{i=1}^{N}\left[\lambda^{(i)}\right]^{2(N-i)} \times$(some function of $\{U, V\}$). This expression has to be multiplied by $(\det\rho)^N = \left(\prod_{i=1}^{N}\lambda^{(i)}\right)^N$, and we obtain at Eq. (14).

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