Nonlinear generalization of Fock's approach to analyzing quantum systems with a point spectrum

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We generalize Fock's description of the harmonic oscillator to the case where the photon-number operator N is a nonlinear function of the Hamiltonian operator H. Taking the N vs H dependence in the form of a cubic polynomial as an example, we classify the types of energy spectra and study the corresponding regular one-dimensional potentials. We also describe situations in which the N vs H dependence is not a polynomial. Finally, we examine the possibility of generalizing the method to the case of spectra for which the second finite difference of the eigenvalues of H is constant. © *1996 American Institute of Physics.* [S1063-7761(96)00412-X]

1. INTRODUCTION

Over the years researchers have shown unflagging interest in the search and study of quantum mechanical models that either allow constructing exact solutions explicitly or provide a fairly complete qualitative and numerical analysis of the solutions.[1–5] The explanation lies in the development of the general spectral theory of the Hamiltonian operators and in the progress achieved in new branches of physics, such as nonlinear quantum optics[6] and nanotechnology involving quantum-size elements. Of obvious interest to these avenues of research are the analysis of the problem of reconstructing potential patterns from a given point energy spectrum (in particular, from an equidistant spectrum spoiled by a set of energy gaps of fixed sizes and positions) and the study of the bifurcations of production and collapse of such gaps and their relationship to the bifurcations of the potential pattern.

Various approaches to analyzing the direct and inverse Schrödinger problems have been adopted in studying these problems. Among these are the approaches based on using supersymmetry properties[7,8], on studying the simplest nonlinear algebras[9,10] and q-deformed oscillators[11,12] and on investigating the higher dynamical symmetries of the Schrödinger operator.[13] We also note new approaches that go back to the Darboux transformation method,[14] such as the factorization method[15,16] and the dressing chain method,[17] or are its direct development.[18]

On the other hand, in the theory of the harmonic oscillator there is the well-known Fock approach,[19] which used not only creation and annihilation operators but also the concept of the photon-number operator. We believe that the fruitfulness of this approach has not been fully appreciated yet.

It is well known[20,21] that in the Fock representation for the case of the harmonic oscillator the creation and annihilation operators are linear functions of the momentum and position operators and the photonnumber operator is a linear function of the Hamiltonian operator.

In this paper we develop an approach suggested in Refs. 22–25 and based on introducing analogs of the photon-number operator that are nonlinear functions of the Hamiltonian operator. We believe that this is a natural generalization of Fock's approach and follows the basic logic of describing quantum states in the Fock space for the harmonic oscillator, which preserves the graphic nature of the physical interpretation of the results.

Note that although the main results of this work refer to the spatially one-dimensional case, the approach can be generalized to incorporate spatially non-one-dimensional states (or systems with several degrees of freedom). In particular, by basing our reasoning on the relationship that exists between the Schrödinger problem in the spatially one-dimensional case and the Schrödinger problem for s states in the three-dimensional case we can specify classes of spherically symmetric potentials that for s states lead to an energy spectrum with a fixed pattern of gaps in the equidistant spectrum.

The plan of the paper is as follows. Section 2 is devoted to the general scheme of a nonlinear generalization of Fock's approach. In Sec. 3 we give the result of applying this method in the case of a simple nonlinear (more precisely, polynomial) dependence of the photon-number operator on the Hamiltonian operator. Finally, in Sec. 4 we discuss the possible realizations of more complicated functions $N(H)$ and a generalization of the approach to the case of quadratic spectra.

2. NONLINEAR ANALOGS OF THE PHOTON–NUMBER OPERATOR: A FORMAL SCHEME

Let us examine a quantum dynamical system with a Hamiltonian operator H. We define the analogs of the photon-number operators, N and $\tilde{N}$, and the analogs of the creation and annihilation operators, L and $L^\dagger$, by the following relationships:

$$N \equiv LL^\dagger, \quad \tilde{N} \equiv L^\dagger L, \tag{2.1}$$

$$[N,H]=0, \quad [\tilde{N},H]=0, \tag{2.2}$$

which coincide in form with the corresponding relationships used to describe the harmonic oscillator in the Fock space.[20]

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In view of (2.1), the self-conjugate operators N and $\tilde{N}$ are nonnegative. The conditions (2.2) are met if N and $\tilde{N}$ are functions of the Hamiltonian operator H:

$$N=N(H), \quad \tilde{N}=\tilde{N}(H), \tag{2.3}$$

(the reader will recall that for the harmonic operator both N and $\tilde{N}$ are linear functions of H). Fixing such a functional dependence determines a class of quantum dynamical systems.

The first condition in (2.2) can be met if we put

$$[H,L]=L\Omega, \quad \Omega=\Omega^{\dagger}, \tag{2.4}$$

where Ω is an arbitrary self-adjoint operator. The second condition in (2.2) then leads to the following restriction:

$$[\Omega,\tilde{N}]=0. \tag{2.5}$$

In what follows we assume that the operator Ω is a function of the Hamiltonian operator, or $\Omega=\Omega(H)$; in this case both conditions in (2.2) are met. For the case of the harmonic oscillator we have $\Omega\equiv\omega=$ const.

Let us now establish how N and $\tilde{N}$ are linked to the operator Ω. To this end we write the obvious relationship $NL=L\tilde{N}$ is the following form:

$$[N,L]=L(\tilde{N}-N)\equiv L\Delta(H). \tag{2.6}$$

It can be shown that for an arbitrary operator function $N(H)$ the following is true:

$$[N(H),L]=N(K+H)-N(H). \tag{2.7}$$

Here the function $N(K+H)$ is understood to be a series in powers of the argument, and the powers are interpreted in the sense of binomial series,

$$(K+H)^n\equiv\sum_{m=0}^{n}C_n^m K^{n-m}H^m,$$

with the powers of the symbol K defined as follows:

$$K^1=[H,L], \quad K^2=[H,[H,L]], \ldots, \quad K^{n+1}=[H,K^n]. \tag{2.8}$$

If the operators H and L are related by Eq. (2.4), then $K^m=L\Omega^m$, and Eq. (2.7) assumes the form

$$[N(H),L]=L\{N(\Omega+H)-N(H)\}, \tag{2.9}$$

from which we find that the characteristic operator $\Omega(H)$ is related to the analogs of the photon-number operators, N and $\tilde{N}$, as

$$N(\Omega(H)+H)=\tilde{N}(H)\equiv N(H)+\Delta(H). \tag{2.10}$$

In view of (2.2), the operators N, $\tilde{N}$, and H can have a common system of eigenfunctions. Let $(\psi,E,\nu(E))$ be a solution of the eigenvalue problem

$$H\psi=E\psi, \quad N\psi=\nu\psi, \quad \nu=\nu(E)\geqslant 0. \tag{2.11}$$

Then, using Eqs. 2.4, (2.7), and (2.10), we find that the function $\psi'=L\psi$ satisfies the following equations:

$$\begin{aligned} &H\psi'=(E+\Omega(E))\psi', \\ &N\psi'=N(\Omega(E)+E)\psi'=(\nu+\Delta(E))\psi', \\ &\nu(E+\Omega(E))-\nu(E)=\Delta(E). \end{aligned} \tag{2.12}$$

Thus, the operator L maps a solution of the Schrödinger problem into another solution, with the two not necessarily belonging to the eigenfunction space. In the latter case we can pose the problem of building an operator L that maps an eigenelement of the Schrödinger problem into another eigenelement after a finite number of iterations. Here, however, we restrict our discussion to operators acting in the eigenvalue space of the Schrödinger problem. In this case, if we have $\nu'\equiv\nu+\Delta(E)\geqslant 0$, the function ψ' is an eigenfunction, and L is the spectrum–shift operator defining the following mapping of the eigenelements of the Schrödinger problem:

$$\begin{aligned} &L\colon \{\psi,E,\nu\}\to\{\psi'\equiv L\psi,\ E'\equiv E+\Omega(E), \\ &\nu'\equiv\nu+\Delta(E)\}\to\ldots, \quad \nu=n(E'). \end{aligned} \tag{2.13}$$

Following Fock's approach, we define the initial element of the mapping as the solution of the problem

$$H\psi_0=E_0\psi_0, \quad N\psi_0=0\leftrightarrow L^{\dagger}\psi_0=0, \quad \nu(E_0)=0. \tag{2.14}$$

In contrast to the harmonic-oscillator case, for fixed functions $N(H)$ and $\tilde{N}(H)$ this problem can be solved for a certain set of real roots of the equation $\nu(E_0)=0$ by defining the multiplet of the system's "ground" states (in the operator N), i.e., we have "degeneracy" in the eigenstates of N. The L-map of each of these states can generate a related subsequence of eigenvalues of H. Combining these subsequences makes it possible to determine the energy spectrum allowed by the given functions $N(H)$ and $\tilde{N}(H)$.

The next important difference from the harmonic-oscillator case is the fact that the requirement that the eigenvalues of N and $\tilde{N}$ be nonnegative when the N vs H dependence is nonmonotonic may lead to the appearance of subsequences of the eigenelements of the problem (2.11) consisting of a finite number of elements. If at the $(n+1)$st step the L-map (2.13) leads to a negative value ν_{n+1}, then $\psi_{n+1}=0$ and the subsequence is terminated at the nth step. Here, for the other subsequences, the "ground" state (in N) may have an energy higher than the last, or upper, level of the truncated sequence. This results in an energy gap in the spectrum.

The mapping (2.13) defines sequences of different types. The most interesting from the standpoint of physics are the equidistant and quadratic sequences, realized at $\Omega(H)=\omega=$const and $\Omega(H)=\alpha\sqrt{H+\beta}$, respectively (the latter case is studied in Sec. 4). With allowance for possible multivalued solutions of Eq. (2.14), for the ground state (in N) classes of spectra can appear that are formed by combining several finite or infinite sequences. For instance, these sequences may not intersect, which leads to spectra with energy gaps, as noted earlier. In Sec. 3 we thoroughly study quantum systems with such types of spectra. There we show that both the size and position of a gap can be controlled by varying the structural parameters of the system.

If we formally go to the classical limit ($\hbar \to 0$), the following equations for the complex-valued functions $\mathscr{S}$ and $\mathscr{S}^*$ are the analogs of Eq. (2.4):

$$\{\mathscr{H},\mathscr{S}\}=i\mathscr{S}\Omega(\mathscr{H}), \quad \{\mathscr{H},\mathscr{S}^*\}=-i\mathscr{S}^*\Omega(\mathscr{H}), \tag{2.15}$$

where $\mathscr{H}$ is the classical Hamiltonian function, and $\{\cdots,\cdots\}$ stands for the classical Poisson bracket. Here the selected class of dynamical systems is determined by the way in which the functions $\mathscr{N}=\mathscr{N}(\mathscr{H})$ and $\delta(\mathscr{H})=\lim_{\hbar\to 0}\Delta(H)/\hbar$ are specified, and the characteristic function $\omega(\mathscr{H})=\lim_{\hbar\to 0}\Omega(H)/\hbar$ is given by the following relationship:

$$\frac{\partial\mathscr{N}}{\partial\mathscr{H}}\omega(\mathscr{H})=\delta(\mathscr{H}) \tag{2.16}$$

(the limit of (2.10) as $\hbar\to 0$).

In the quantum case the functional equation (2.10) for fixed $N(H)$ and $\widetilde{N}(H)$ can lead to several solutions for the characteristic operator $\Omega(H)$, of which only one remains when we go to the classical limit. Hence the solutions of Eqs. (2.4) and (2.10) can define essentially quantum dynamical systems.

Note that in the case of a single degree of freedom and a Hamiltonian function of the form $\mathscr{H}=p^2/2+U(x)$ there is a direct relationship between the solution of Eq. (2.16) in the action–angle variables and the classical inverse problem of determining the potential $U(x)$ from the energy dependence of the oscillation period $2\pi/\Omega(\mathscr{H})$ (Ref. 26). In particular, for $\Omega(H)=\omega=\text{const}$ an equation for the class of isochronous potentials. arises

Thus, by following the logic of describing the quantum harmonic oscillator in the Fock space we arrived at a peculiar formulation of the inverse spectral problem for the Hamiltonian operator, defined by Eqs. (2.4) and (2.10), for the analogs of the creation and annihilation operators, and for the characteristic operator $\Omega(H)$. This makes it possible, knowing the functional dependence of the analogs of the photon-number operators, N and $\widetilde{N}$, on the Hamiltonian operator H, to determine the admissible structure of the energy spectrum and in some cases to obtain the complete solution of the inverse spectral problem. Obviously, it is possible to establish how this approach is linked to other approaches (say, to the approach related to the assessment of the higher symmetries of the Schrödinger equation[13] or, in the particular case where the operators L and $L^\dagger$ are polynomials in the momentum operator, to the factorization method[16]).

Up to this point our discussion concerned a nonlinear generalization of the Fock method that did not resort explicitly to the analogs of the creation and annihilation operators, the spectrum-shift operators L and $L^\dagger$. Let us examine the conclusions that follow from the assumption that these operators are polynomials (recall that in the harmonic-oscillator case they are first-degree polynomials in p). Let

$$L(p,x)=\sum_{m=0}^{M}L_m(x)p^m, \quad L^\dagger(p,x)=\sum_{m=0}^{M}p^mL_m(x), \tag{2.17}$$

and let the Hamiltonian operator have the natural form

$$H=\frac{1}{2}p^2+U(x). \tag{2.18}$$

Then the operators $N\equiv LL^\dagger$ and $\widetilde{N}\equiv L^\dagger L$ can be reduced to

$$N=LL^\dagger=\sum_{m=0}^{M}C_m(x)H^m+\sum_{m=0}^{M-1}D_m(x)H^mp, \tag{2.19}$$

$$\widetilde{N}=L^\dagger L=\sum_{m=0}^{M}\widetilde{C}_m(x)H^m+\sum_{m=0}^{M-1}\widetilde{D}_m(x)H^mp. \tag{2.20}$$

Here $C_m(x)$, $\widetilde{C}_m(x)$, $D_m(x)$, and $\widetilde{D}_m(x)$ are functions of the coefficients $L_m(x)$ in the expansions (2.17), the potential $U(x)$, and their derivatives. The condition $[N,H]=[\widetilde{N},H]=0$ leads to the following system of equations:

$$C_m(x)=\text{const}, \quad \widetilde{C}_m(x)=\text{const}, \quad D_m(x)=\widetilde{D}_m(x)=0. \tag{2.21}$$

The solution of this system yields the explicit form of the $L_m(x)$, the nonlinear differential equation for $U(x)$, and the expression for additional integrals of this equation.

Note that we arrive at the same expressions for the coefficients and the same equation for the potential by solving Eq. (2.4), where we must put $\Omega(H)=\omega=\text{const}$ and seek the spectrum-shift operators L and $L^\dagger$ in the form (2.17). The spectral recursion formula in this case has the form $E_{n+1}=E_n+\omega$. Thus, by choosing the structure of the operators L and $L^\dagger$ in the of finite-degree polynomials in the momentum operator we found that Eq. (2.4) is solvable for the class of quantum dynamical systems with a equidistant spectrum (with, possibly, a finite number of energy gaps).

In conclusion of this section we note that realizing systems with a nonequidistant type of spectra requires, on the one hand, introducing characteristic operators $\Omega(H)$ that are not constants and, on the other, discarding the polynomial form of the spectrum-shift operators (examples of systems that allow for spectrum-shift operators that are not polynomials are examined in Sec. 4).

3. SYSTEMS WITH PHOTON-NUMBER OPERATORS OF THIRD DEGREE IN H

Let us examine a situation in which the analogs of the photon-number operators are polynomials in the Hamiltonian operator:

$$N(H)=\sum_{m=0}^{M}h_mH^m, \quad \widetilde{N}(H)=\sum_{m=0}^{M}\widetilde{h}_mH^m, \quad M\geqslant 1. \tag{3.1}$$

The general relationship (2.7) assumes the form

$$[N(H),L]=\sum_{m=0}^{M}h_m(K+H)^m-\sum_{m=0}^{M}h_mH^m\equiv L(\widetilde{N}-N), \tag{3.2}$$

and Eq. (2.10) becomes an algebraic equation in Ω:

$$\sum_{m=0}^{M} h_m(\Omega+H)^m=\sum_{m=0}^{M}\tilde{h}_m H^m. \tag{3.3}$$

Here the quantum dynamical system is determined solely by the set of structural constants $\{h_m,\tilde{h}_m\}$. The L-mapping is determined by the following expressions:

$$E'=E+\Omega(E),$$

$$\nu'=\nu+\left\{\sum_{m=0}^{M} h_m(\Omega(E)+E)^m-\sum_{m=0}^{M} h_m E^m\right\}. \tag{3.4}$$

Let $M=3$. Equation (2.4) in this case determines no more than three characteristic operators $\Omega(H)$. The requirement that one of these operators be independent of H (say, $\Omega_1(H)\equiv\omega$) leads to the following first recursion formula for the eigenvalues of H for this branch:

$$E'=E+\omega,\quad \nu(E)>0,\quad \nu(E')\geqslant 0. \tag{3.5}$$

The problem (2.14), which determines the initial elements for building the L-mapping, has a solution for the eigenvalue triplet $\{E_0^{(1)},E_0^{(2)},E_0^{(3)}\}$ corresponding to the zero eigenvalue of N. The structure of the spectrum of H can be one of the following:

A) The spectrum is a combination of infinite equidistant (with a step ω) sequences obtained via the L-mapping of one, two, or three "ground" states (in N) $\{E_0^{(1)},E_0^{(2)},E_0^{(3)}\}$.

B) The spectrum is a combination of a lower finite equidistant group of levels and an upper infinite equidistant sequence of levels, separated by a gap. The size of the gap is generally incommensurate with the level separation ω in both equidistant parts.

The realization of either of these situations depends on the values of the structural parameters h_i. It can be shown[27] that their values are linear functions of the constants I_1 and I_2 of the first integrals of the equation for the potential $U(x)$:

$$I_1=x\left\{-\frac{1}{2}V_{xxx}+3(V^2)_x+2(\omega x)^2V_x-2kV_x\right\}+\frac{V_{xx}}{2}-3V^2+2kV, \tag{3.6}$$

$$I_2=-\frac{1}{4(\omega x)^2}\left\{-\frac{1}{2}V_{xx}+3V^2-2kV+I_1\right\}^2+\frac{(V_x)^2}{4}-V^3+kV^2-I_1V, \tag{3.7}$$

and the equation for the potential $V(x)\equiv U(x)-(\omega x)^2/2$ is

$$\tfrac{1}{4}V_{xxxx}-\tfrac{3}{2}(V^2)_{xx}-(\omega x)^2V_{xx}-3\omega^2xV_x+kV_{xx}=0. \tag{3.8}$$

In what follows we set k equal to zero—in this case the expression for the shift operators acquires the simplest form. Note that the expression for I_2 can also be interpreted as an equation for the potential, containing the constants of the integrals, I_1 and I_2, as structural parameters.

In view of (3.1) and (3.5), the recursion relation $E\Rightarrow E'$ is defined by the following relationships:

$$E'=E+\omega,\quad \nu(E)=(2E-\omega)^3+I_1(2E-\omega)-I_2\geqslant 0. \tag{3.9}$$

For $I_1>0$ the ground state in N with the energy $E_0(I_1,I_2)$ is a singlet and the spectrum is strictly equidistant. This case, realized on asymmetric potentials, was actually thoroughly studied by McKean and Trubowitz[28] and Levitan[29] and in the classical limit amounts to a situation in which all isochronous potentials (differing from the harmonic–oscillator potential) are asymmetric.[26]

For $I_1<0$ the ground state in N is a triplet with energies $\{E_0^{(1)},E_0^{(2)},E_0^{(3)}\}$. Here the wave function for each of these states is determined by the solution of the equation $L^\dagger\psi_0=0$ and is given by the expression

$$\psi_0=\exp F_0(x,I_1,I_2)\exp\left(-\frac{\omega x^2}{2}\right), \tag{3.10}$$

where

$$F_0\equiv\int^x\frac{[(V_{x'}/2)^2-V^3(x')-I_1V(x')-I_2]^{1/2}+V_{x'}/2}{V(x')+2E_0^{(i)}-\omega}dx',$$

$$i=1,2,3. \tag{3.11}$$

Note that generally the recursion relation is multivalent: at $E_c^{\pm}=-\omega/2\pm\sqrt{-I_1/3}$, in addition to the equidistant branch (3.9), other branches of the first return function appear:

$$E_{n+1}=-\frac{1}{2}E_n+\frac{\omega}{4}\pm\frac{1}{2}\sqrt{-I_1-3\left(E_n+\frac{\omega}{2}\right)^2}. \tag{3.12}$$

The origin of such multivalence is related to the existence in these conditions of an additional pair of characteristic operators $\Omega^{\pm}(H)$ (see Sec. 4). However, if the operators L, and $L^\dagger$ obey Eq. (2.4) with $\Omega=\omega=\text{const}$, the multivalence disappears.

We found that in the $\{I_1<0,\ I_2\}$ plane there exists a denumerable set of bounded domains for which the spectrum of H is an equidistant continuation of the triplet of ground states in N. An equidistant spectrum with a gap is realized on the curves that are the boundaries of these domains. The curves are determined by the condition that the eigenvalue of N vanish at the nth step of the L-mapping of the lowest of the "ground" states. More precisely, the nth step in the L-mapping for the initial point $E_0^{(1)}=E_0^{(1)}(I_1,I_2)$ leads to $\nu(E_n)\equiv 0$ on the family of curves

$$27I_2^2=-4[I_1+4(\omega n)^2]^2[I_1+(\omega n)^2],$$

$$n=0,1,2,\ldots. \tag{3.13}$$

The corresponding ground–state energy is

$$E_0^{(1)}=\frac{1}{2}\sqrt{-\frac{I_1+\omega n}{3}}-\frac{\omega}{2}(n-1),$$

$$n=0,1,2,\ldots. \tag{3.14}$$

The action of a mapping leading to an equidistant spectrum with a gap because of a nonmonotone ν vs E dependence is depicted in Fig. 1.

Note that here, in contrast to the harmonic-oscillator case, the eigenvalues of N are not integers (more precisely,

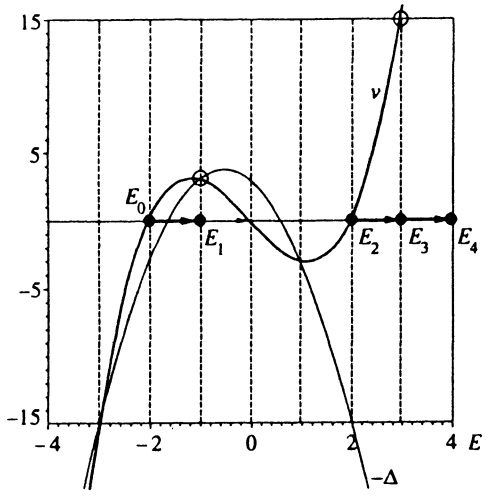

FIG. 1. An example of an L-mapping for a cubic dependence of ν on E that leads to a gap between the second and third energy levels.

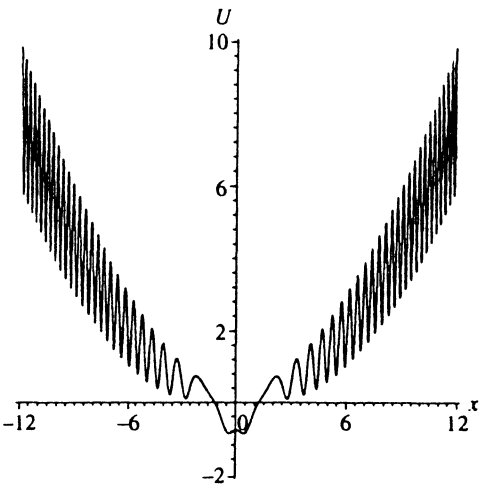

FIG. 2. A potential with a oscillatory asymptotic behavior corresponding to a triplet equidistant spectrum.

within the chosen notation are not integral multiples of 2ω). Nevertheless, the expression for ν_n retains its simple structure: on the mth curve of the (3.13) type the expression for the "photon number" corresponding to the nth energy level has the form

$$\nu_n=(2\omega)^3 n(n-m)\left(n+m+3\left(\frac{E_0}{\omega}-\frac{1}{2}\right)\right), \tag{3.15}$$

or, with allowance for (3.14),

$$\nu_n=(2\omega)^3 n(n-m)\left(n-\frac{m}{2}\pm\sqrt{-\frac{3}{4}\left(\frac{I_1}{\omega^2}+m^2\right)}\right) \tag{3.16}$$

(in this notation $\nu_n=2\omega n$ corresponds to the harmonic oscillator). Note that the first factor ensures that ν vanishes in the ground state, while the vanishing of the second factor leads to a termination of the lower sequence at the mth step of the map and the formation of a gap in the equidistant spectrum.

Numerical calculations fully support the above qualitative results. Let us now turn to systems with regular symmetric potentials ($U(x)=U(-x)$). Potentials realizing a triplet spectrum (and corresponding to the interior points of the above-mentioned bounded domains in the parameter plane $\{I_1,I_2\}$) are characterized, in the limits $x\to\pm\infty$, by oscillations with linearly growing frequency and amplitude (Fig. 2):

$$U(x)_{x\to\pm\infty}\sim\frac{1}{2}\left(\frac{\omega x}{3}\right)^2+\gamma|x|\sin\left(\frac{\omega}{\sqrt{3}}x^2+\phi_0\right). \tag{3.17}$$

The interior of each domain contains only one point at which the oscillation amplitude vanishes ($\gamma=0$), and the structural parameters corresponding to this point are

$$I_1=-\frac{4\omega^2}{3}\left[n^2+n+\frac{1}{3}\right],\quad I_2=-\frac{8\omega^3}{27}[n(n+1)(2n+1)]. \tag{3.18}$$

At these points the energy values for the three ground states (in the operator N) are linked by the following relationships:

$$E_0^{(2)}=E_0^{(1)}+(n+1/3)\omega,\quad E_0^{(3)}=E_0^{(1)}+(n+2/3)\omega, \tag{3.19}$$

with $n=0,1,2,\ldots$. As a result, at each such point the spectrum can be described as consisting of two equidistant parts: the lower part, with a step ω, consisting of n levels, and the adjacent upper infinite sequence, which is equidistant with a step $\omega/3$. The latter is quite natural if one takes into account the fact that such a spectrum for highly excited states yields the semiclassical approximation for the potential with the asymptotic behavior (3.17) at $\gamma=0$.

As we move closer to the domain boundary, the oscillation amplitude increases without limit, and the levels of two of the three equidistant sequences converge pairwise. At the boundary proper these levels merge pairwise and disappear

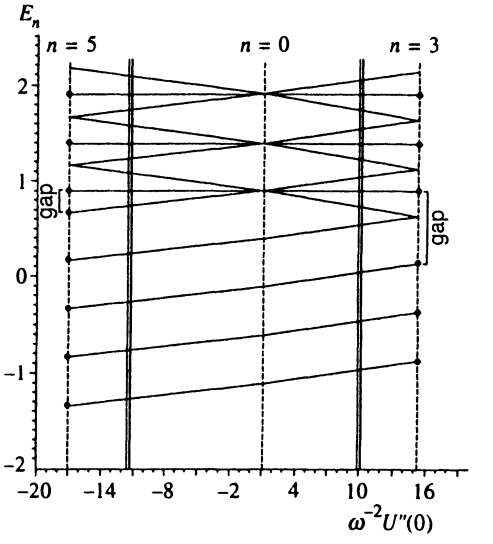

FIG. 3. Transformation of a triplet spectrum due to variation of the structural parameters that corresponds to the passage through the third "domain of oscillating potentials" between its boundaries, the two anharmonic oscillator curves with $n=3$ and $n=5$ levels in the lower equidistant group, respectively. The double vertical lines mark the parameter values at which the oscillations disappear and the spectrum in the upper part becomes strictly equidistant with a step $\omega/3$. The central dotted line corresponds to the harmonic-oscillator case.

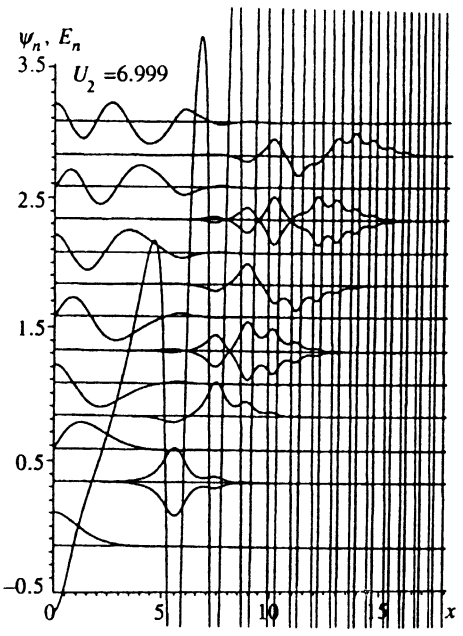

FIG. 4. Oscillating potential, energy spectrum, and wave functions near the boundary of one of the domains of existence of such potentials. Two of the three equidistant level subsequences have almost merged; the corresponding wave functions are localized at the minima, which extend to infinity, narrowing and deepening as the domain boundary is approached.

(a detailed analysis of this and other bifurcations of the potential pattern and the spectrum in the system is given in Ref. 24). As a result there appears an equidistant spectrum with a gap (Fig. 3), while the type of potential pattern drastically changes: the potential corresponding to the points of the curve (3.13) is the potential of an anharmonic oscillator with a finite number of local minima.[22,25] The alteration in the wave functions accompanying this bifurcation is shown in Figs. 4 and 5.

The half-closed sections of the curves (3.13) on which such potentials are realized successively branch out. The branch points correspond to the following values of the parameters:

$$I_1=-\frac{4\omega^2}{3}[n^2+n+1],$$

$$I_2=-\frac{8\omega^3}{27}[(n-1)(n+2)(2n+1)]. \tag{3.20}$$

The branching from the nth curve of a new $(n+1)$st curve is characterized by the following change in the spectrum and potential:

1. A pair of two new levels appear in the spectrum (Fig. 6a). For $n=0$ these levels appear at the point $E=E_0-\omega$; for $n>0$ they appear in the middle of the gap, which at the bifurcation point of the nth curve has a value 2ω, near $E=(\omega/3)(n-1/2)$. The lower of the levels augments the lower equidistant group (or, when $n=0$, is simply this group), while the upper level augments the upper equidistant group. Now on the $(n+1)$st curve the energy gap corresponds to the distance between the newly formed levels. Thus, the number of levels in the lower equidistant group increases by one as we move from curve to curve and coincides with its ordinal number n. As we move along the curve, the size of the gap increases from zero (at the point at which the given curves branches away from the previous curve) to infinity.

2. Two narrow deep wells symmetric with respect to one another are formed in the potential pattern, and asymptotically (near the branch point) these wells are described by the expression

$$V_{\rm loc}(x)\approx-\frac{(\omega X_0)^2}{\cosh^2[\omega X_0(x\pm X_0)]}, \tag{3.21}$$

with their separation X_0 from the potential's symmetry axis tending to infinity (logarithmically) as we move along the curve to its origin (Fig. 6b).

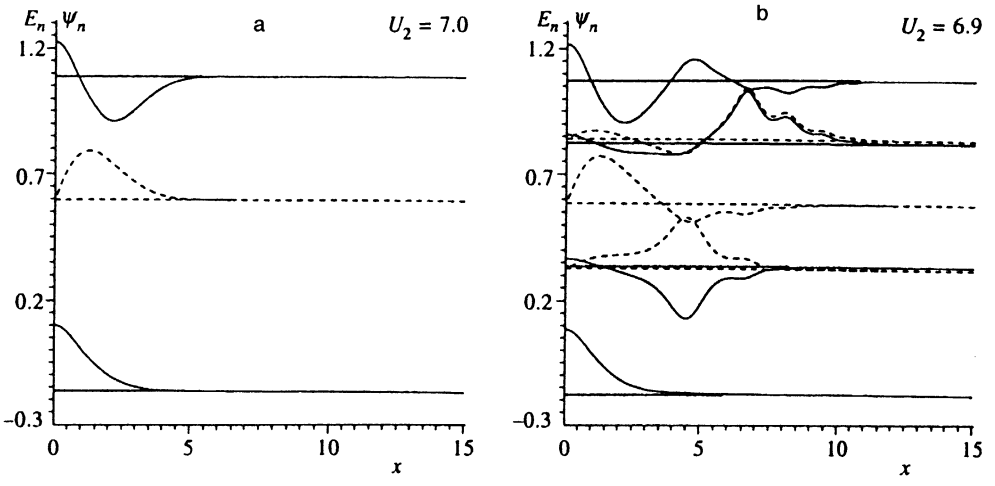

FIG. 5. The wave functions for two potentials of different types corresponding to close values of the structural parameter: (a) for an anharmonic oscillator (the spectrum contains a gap between the ground and first excited states) on the curve that is the boundary of the domain of existence of oscillating potentials; (b) for an oscillating potential near a boundary point corresponding to case a. The wave functions of the states surviving in the transition to the anharmonic potential transform, shedding the necessary number of nodes.

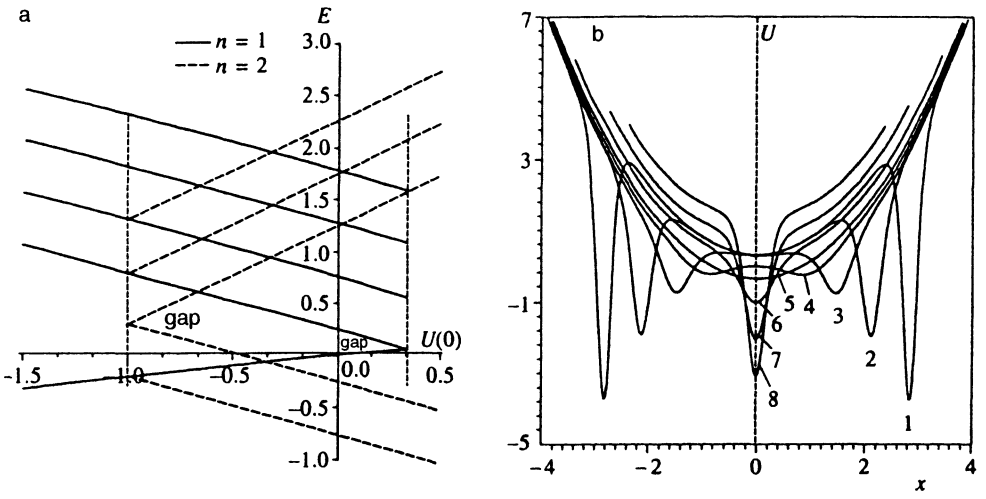

FIG. 6. Transformation of the spectrum (a) and a potential of the anharmonic type (b) along the curve $n=1$. In Fig. a, in addition, the dashed lines depict the variation of the spectrum levels along the curve $n=2$. In Fig. b the increase in the label number on the potential corresponds to a departure along the curve from the bifurcation point (its branching from the harmonic-oscillator line $n=0$).

Note that the expression (3.21) is the exact solution of the equation

$$\left\{-\frac{1}{2}V_{xx}+3V^2+I_1\right\}^2-I_1^2=4(\omega X_0)^2\left\{\frac{1}{4}(V_x)^2-V^3-I_1V\right\}, \tag{3.22}$$

which differs from (3.7) in that the nonautonomous factor $(\omega x)^2$ is replaced by $(\omega X_0)^2$ and a constant, $(-I_1^2-4(\omega X_0)^2 I_2)$, is added.

Interestingly, the approximate potential (3.21) contains the only localized quantum state and is reflectionless for any X_0 (the spectrum consists of two levels: $E_0=-1/2(\omega X_0)^2/2$ and $E_1=0$). This makes it possible to interpret the two new levels in the spectrum as related to the states localized in two new local wells (allowing for their position on the "walls" of the generating anharmonic oscillator at a height of approximate $(\omega X_0)^2/2$). The localization of the wave functions corresponding to the two new states is depicted in Fig. 7b.

Now let us study the relationship between the asymptotic behavior at zero and infinity of the symmetric anharmonic oscillators that are solutions of Eq. (3.8), as functions of the system's structural parameters $\{I_1,I_2\}$. The series expansions of the corresponding solutions have the form

$$V\approx V_0+V_2x^2+O(x^4), \quad x\sim 0, \tag{3.23}$$

$$V\approx V_\infty+\beta(2\omega x)^{-2}+O((2\omega x)^{-4}), \quad x\to\infty. \tag{3.24}$$

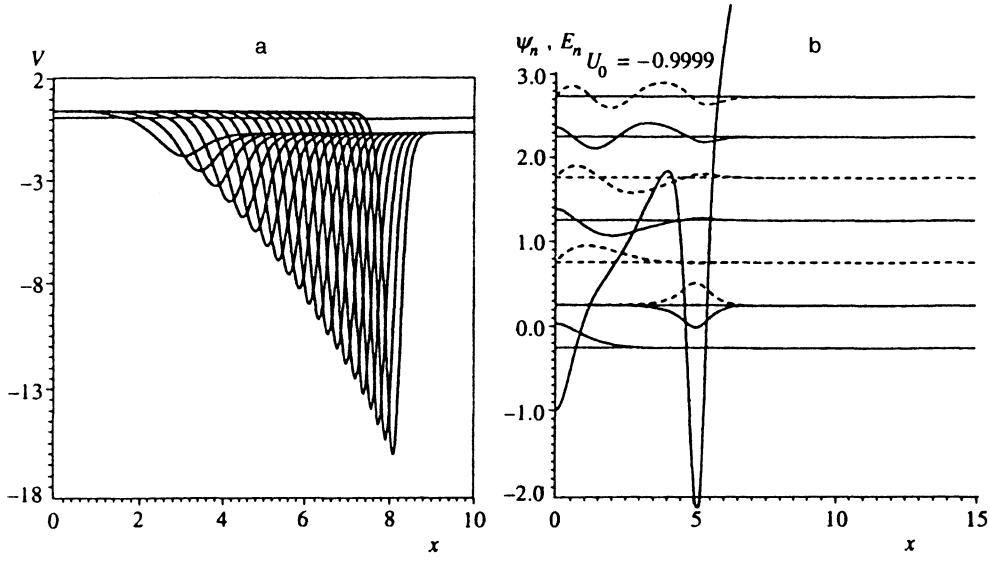

FIG. 7. (a) Variation of the pattern of the potential $V(x)$ along the curve $n=1$ in the neighborhood of its point of branching from the harmonic-oscillator line $n=0$: for all profiles a transition between the weakly varying values $V(0)$ and $V(\infty)$ is characteristic. (b) Potential, energy spectrum, and the wave functions corresponding to a point on the curve $n=2$ near its point of branching from the curve $n=1$. The two new closely lying levels that emerge in the middle of the gap correspond to wave functions localized in wells that extend to infinity at the branch point. The wave functions of all above-gap states are transformed during bifurcation in such a way that the number of nodes for each increases by two.

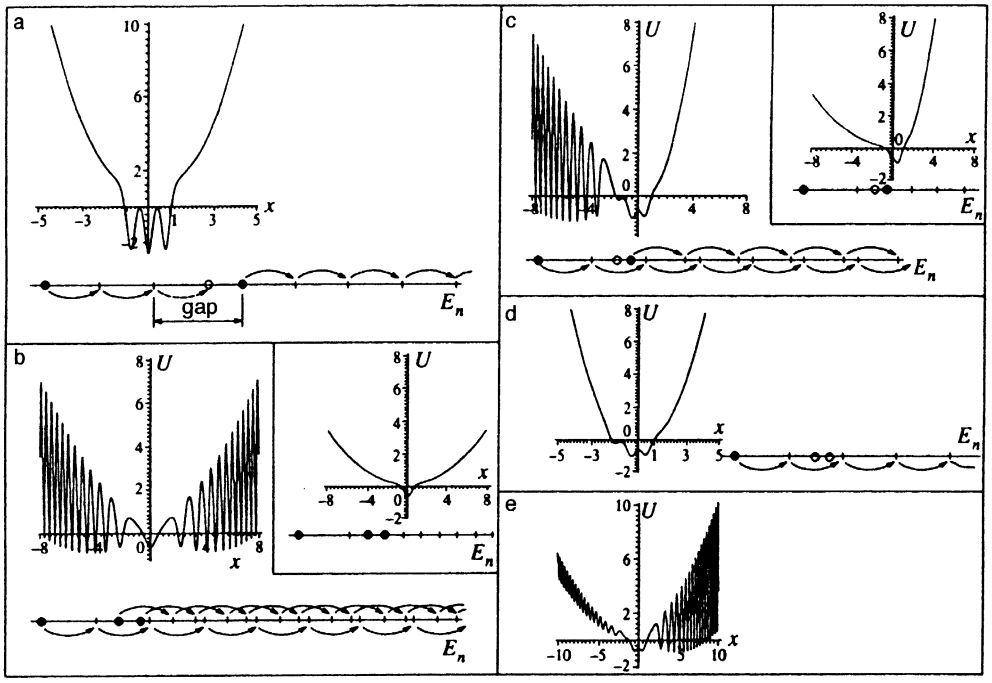

FIG. 8. Five types of potentials and the correspond spectra realized for spectrum-shift operators L that are cubic in momentum.

The pairs V_0, V_2 (or $\{V_\infty, \beta\}$) uniquely determine the4 potential and hence can be used as the parameters of the potential instead of the pair (I_1, I_2). What is interesting is that the potential parameters at zero, V_0 and V_2, and the potential parameter at infinity, V_∞ and β, are related to I_1 and I_2 in the same way:

$$I_1 = V_2 - 3V_0^2 = \beta - 3V_\infty^2,$$
$$I_2 = -V_0^3 - I_1 V_0 = -V_\infty^3 - I_1 V_\infty. \tag{3.25}$$

The transition from (I_1, I_2) to (V_0, V_2) in the expression for the curves (3.13) simplifies the latter considerably and leads to factorization:

$$\left[V_2 - \frac{9}{4}V_0^2 + (\omega n)^2\right][V_2 + 6(\omega n)V_0 + 4(\omega n)^2][V_2 - 6(\omega n)V_0 + 4(\omega n)^2] = 0. \tag{3.26}$$

The transition to the "parameters at infinity" (V_∞, β) also factorizes the expression (3.13).

Numerical analysis has shown that in the (V_0, V_2) plane an anharmonic oscillator with a single-gap spectrum corresponds to a family of alternating half-lines and half-parabolas that successively branch off from each other: for n odd the expression in the first square brackets in (3.26) vanishes, and for n even it is the expression in the second square brackets that vanishes. However, in the (V_∞, β) plane (the plane of the parameters at infinity) the solution of interest is realized only on half-lines for all values of n, with the half-lines having no common points. This means that an infinitesimal change in the parameters V_0 and V_2 brought on by the transition from one curve to the next is accompanied by a jump in the asymptotic behavior of the potential at infinity. We write the relationship between the parameters V_0 and V_∞ in the following form:

$$[V_0 - V_\infty][V_0^2 + V_0 V_\infty + V_\infty^2 + I_1] = 0. \tag{3.27}$$

The fact that V_∞ can have three different values for a single value of V_0 makes possible the jump in the value of V_∞ at the point of transition from one curve to another. More precisely, at the bifurcation point of the generating nth curve $V_\infty^b = -2\omega(n-1)/3$, while at the same point on the new $(n+1)$st curve $V_\infty^b = -2\omega(n+2)/3$, which corresponds to a jump by 2ω. From the potential–pattern viewpoint, this is related to a discontinuous drop by 2ω in the potential height at the different edges of each emerging well (3.21) (Fig. 7b).

Thus, it is possible to distinguish two types of potentials and two types of corresponding spectra in the class of symmetric potentials:

1. Potentials of the anharmonic-oscillator type that behave like $(\omega x)^2/2$ at infinity. These potentials correspond to equidistant spectra with a gap whose position and size clearly depend on the magnitude of the system's structural parameters (Fig. 8a).

2. Potentials that oscillate at infinity about the parabola $(\omega x/3)^2/2$. The corresponding spectra are combinations of three equidistant sequences with arbitrarily shifted ground states (for which $\nu = 0$). A particular case is that of potentials with a zero amplitude of oscillations about the above-mentioned parabola or, in other words, anharmonic oscillators with a frequency equal to $\omega/3$ and a corresponding equidistant spectrum (which begins, however, at n gaps of size ω) (Fig. 8b).

By numerical means we also discovered and studied three additional types of asymmetric potentials and corresponding spectra:

1. Potentials with a mixed type of asymptotic behavior (Fig. 8c): an asymptotic behavior of the type $(\omega x)^2/2$ at one infinity and an oscillating asymptotic behavior at the other.

The spectrum realizes two infinite subsequences, related to an L-map of two of the three ground states. Just as before, a particular case is that of potentials with a zero oscillation amplitude. A semiclassical approach to the spectrum of highly excited states in such potentials predicts a behavior of the type $E_n \sim \omega n/2$. This result is corroborated by numerical calculations: potentials of this type correspond to a situation in which the upper ground state is exactly in the middle between two levels belonging to the sequence constructed using the lower ground state; the spectrum can be thought of as consisting of a lower equidistant part with a step equal to ω and an upper equidistant part with a step equal to $\omega/2$.

2. Asymmetric potentials of the anharmonic-potential type (Fig. 8d). Here only one L-sequence is realized in the spectrum, which results in a strictly equidistant spectrum.

3. Asymmetric oscillating potentials (Fig. 8e). The spectrum type is the same as in the corresponding symmetric case.

Thus, we have described all possible types of one-dimensional regular potentials and spectra in physical models that allow for the existence of the simplest nonlinear analogs (cubic in H) of photon-number and spectrum-shift operators.

Concluding this section, we note that there is a certain correspondence between the Schrödinger problem in the one-dimensional case and the Schrödinger problem for s states in the spatially three-dimensional case. This correspondence makes it possible, in particular, to specify classes of spherically symmetric potentials that lead for s states to an energy spectrum with a given pattern of gaps in the equidistant spectrum. Here, because eigenfunctions even in x, $\psi_n(-x)=\psi_n(x)$, have been excluded, the spectrum-shift operator in the Schrödinger problem in $\mathscr{R}^3$ is the square of the spectrum-shift operator in the Schrödinger problem in $\mathscr{R}^1$ acting in the subspace of odd eigenelements.

4. QUANTUM SYSTEMS WITH A MORE COMPLICATED STRUCTURE

Let us now go back to analyzing the general structure of the spectrum for $N(H)$ and $\tilde{N}(H)$ that are third-degree polynomials. After isolating the root that does depend on H, Eq. (3.3) leads to the following equation:

$$h_3\Omega^2+(h_3\omega+h_2+3h_3H)\Omega+[h_3\omega^2+\omega h_2+h_1 +(2h_2+3\omega h_3)H+3H^2]=0. \tag{4.1}$$

The real roots of this equation determine the remaining pair of possible characteristic operators $\Omega_\pm(H)$ and, correspondeingly, a pair of equations for the operators $L_+\equiv L$ and $L_-\equiv L^\dagger$:

$$[H,L_\pm]=L_\pm\Omega_\pm(H). \tag{4.2}$$

A typical situation here is when the two branches of the recursion relation $E_\pm(n+1)=E_\pm(n)+\Omega_\pm\{E_\pm(n)\}$ form an ellipse in the (E_{n+1},E_n) plane. This ellipse is the support of the point spectrum of the quantum dynamic system defined by (2.4). Depending on the values of the structure parameters (ω,h_i), this ellipse may either intersect the bisectrix of the plane or not. When it does intersect the bisectrix, the L-map has fixed points. This suggests the formal possibility that the spectrum of the Hamiltonian operator H can have a condensation point. The question of what quantum dynamical systems realize this possibility has yet to be resolved.

Let us simplify the situation by assuming $h_3\equiv 0$. Then Eq. (3.3) defines the characteristic operator

$$\Omega(H)=-\omega-\frac{h_1}{h_2}-2H, \tag{4.3}$$

which is a linear function of the Hamiltonian operator, and the equation for the operators H and L assumes the form

$$[H,L]_+\equiv HL+LH=L\left(\omega+\frac{h_1}{h_2}\right). \tag{4.4}$$

The first return function corresponding to this equation describes a cycle of order 2 (i.e., two possible energy values). The operator equation (4.4) can be solved if we assume the operators L, $L^\dagger$, and H to be matrices of the following type:

$$L=\begin{pmatrix}0 & 0\\ \Lambda & 0\end{pmatrix},\quad L^\dagger=\begin{pmatrix}0 & \Lambda^\dagger\\ 0 & 0\end{pmatrix},\quad H=\begin{pmatrix}H_1 & 0\\ 0 & H_2\end{pmatrix}. \tag{4.5}$$

Plugging these expressions into Eq. (4.4) yields the following relationship:

$$H_1\Lambda+\Lambda H_2=-(\omega+\omega_0)\Lambda,\quad \omega_0\equiv\frac{h_1}{h_2}, \tag{4.6}$$

Assuming that

$$H_1=\frac{1}{2}p^2+U_1(x),\quad H_2=-\frac{1}{2}p^2+U_2(x), \tag{4.7}$$

we find that the operator $\Lambda(x,p)$ and the potentials $U_1(x)$ and $U_2(x)$ obey the following equation:

$$\frac{1}{2}[p^2,\Lambda]+U_1\Lambda+\Lambda U_2=-(\omega+\omega_0)\Lambda. \tag{4.8}$$

This equation allows solutions that are polynomials in the momentum operator. The conditions for its solvability in this class of solutions lead to a relationship between $U_1(x)$ and $U_2(x)$. For instance, for $\Lambda=\Lambda_0(x)+\Lambda_1(x)p$ the potentials are related as follows:

$$\frac{d}{dx}\left(\frac{1}{U_+}\frac{dU_-}{dx}\right)=2U_+,\quad U_\pm\equiv U_1\pm U_2+\omega+\omega_0, \tag{4.9}$$

or, in integral form,

$$U_-(x)=2\int^x dx'U_+(x')\int^{x'}dx''U_+(x''). \tag{4.10}$$

Thus, in the one-dimensional case, the solvability of Eqs. (2.4) for the essentially quantum characteristic operators defined by Eq. (3.3) can be related to a matrix representation of the operators L and H. Such a generalization, however, is capable of resolving the problem only in the case of recursion relations leading to cycles of finite order (and, accordingly, to spectra of H with a finite number of elements).

Earlier Dubov *et al.*[30] suggested that the solution of the operator equation

$$[[H,L],L]=cL^2, \quad c=\text{const}, \tag{4.11}$$

incorporates the class of quantum models for which the second finite difference of the eigenvalues of H is constant,

$$E_{n+2}-2E_{n+1}+E_n=c, \tag{4.12}$$

just as the equation $[H,L]=\omega L$ specifies the class of models for which the first finite difference of the eigenvalues is constant ($E_{n+1}-E_n=\omega$). But while in the latter case L is the shift operator in the solutions of the Schrödinger equation (not necessarily eigensolutions), in the former case the result of the action of the operator L obeying Eq. (4.11) on a function that is a solution of the Schrödinger problem may be more complicated.

To analyze the possible situations, we write Eq. (4.11) in the form of the following system:

$$[H,L]=L\Omega, \quad [\Omega,L]=cL. \tag{4.13}$$

Here Ω is an arbitrary operator (not necessarily self-adjoint), and we assume that there is an inverse of L. Let (ψ,E) be a solution of the Schrödinger problem with the operator H, or $H\psi=E\psi$. Then the function $\psi'\equiv L\psi$ is the solution of the Schrödinger equation with the operator $H'\equiv H-\Omega$ corresponding to the energy value $E'=E-c$:

$$(H-\Omega)\psi'=(E-c)\psi', \quad \psi'\equiv L\psi. \tag{4.14}$$

Thus, in the general case the operator L maps the solutions of the Schrödinger equation with the Hamiltonian operator H into solutions of the Schrödinger equation with another Hamiltonian operator (not necessarily self-adjoint). If $\Omega=\Omega^\dagger$, then $L^\dagger$ maps the solutions of the Schrödinger equation with the operator H into solutions of the Schrödinger equation with the operator $H'=H+\Omega$ and the energy value $E'=E$. Here $LL^\dagger\psi$ and $L^\dagger L\psi$ satisfy the initial Schrödinger equation with the operator H and the initial value of parameter E.

Now let us go back to the case where $\Omega=\Omega(H)$ and determine the characteristic operator $\Omega(H)$ in (2.4) for which this equation is compatible with (4.11). In this case L maps the solution ψ of the equation $H\psi=E\psi$ into the function $\psi'\equiv L\psi$ that is the solution of the equation $H\psi'=E'\psi'$. The system of equations (4.13) implies

$$E'=E+\Omega(E), \quad \Omega(E')-\Omega(E)=c, \tag{4.15}$$

and it can easily be verified that the condition (4.12) is met. Excluding E', we arrive at the function equation $\Omega(E+\Omega(E))-\Omega(E)=c$, with a corresponding similar operator equation for $\Omega(H)$:

$$\Omega(H+\Omega(H))-\Omega(H)=c \tag{4.16}$$

(this equation can be obtained directly from (4.13) without resorting to the basis of eigenfunctions of H). Writing the analog of the system (4.13) for the operator $L^\dagger$, we arrive at the second functional relationship for $\Omega(H)$:

$$\Omega(H-\Omega(H)+c)-\Omega(H)=-c. \tag{4.17}$$

Thus, Eq. (2.4) together with conditions (4.16) and (4.17) for the function $\Omega(H)$ leads to models associated with the constancy of the second finite difference.

The particular solution Eqs. (4.16) and (4.17) given in Ref. 31 has the form

$$\Omega(H)=c/2\pm\sqrt{2cH}, \quad E>c/2. \tag{4.18}$$

In this case the split system (4.13) is

$$[H,L]=L(c/2\pm\sqrt{2cH}), \tag{4.19}$$

$$[\sqrt{H},L]=\pm\sqrt{c/2}\,L. \tag{4.20}$$

The second equation is a formal analog of the equation $[H,L]=\omega L$ studied earlier: the mapping of the values of E generated by it has the form

$$\sqrt{E'}-\sqrt{E}=\pm\sqrt{c/2}=\text{const}, \quad E>c/2, \tag{4.21}$$

and realizes the requirement that the second finite difference (4.12) be constant if the map involves the eigenelements of the Schrödinger problem.

On the other hand, being more general than Eq. (4.20), Eq. (4.19) can describe systems with spectral sequences that do not meet the condition of constancy of the second finite difference. As can easily be shown, in addition to the monotonic sequences $\sqrt{E'}=\sqrt{E}\pm\sqrt{c/2}$, $E>c/2$, Eq. (4.19) admits of a sequence $\sqrt{E'}=\sqrt{c/2}-\sqrt{E}$, $E<c/2$, which describes a cycle of two possible energy values. This points to the possibility (characteristic of supersymmetry approaches) of using this approach to describe systems whose spectrum contains an infinite or finite number of elements.

In Ref. 31 it was shown that Eq. (4.19) is solvable in the one-dimensional case for operators L of the form $L(p,x)=L_{1/2}(x)\sqrt{H}+L_1(x)p$. This leads to a quantum system with a Hamiltonian

$$H=\frac{1}{2}p^2+\frac{l(l-1)}{\cos^2\xi}, \tag{4.22}$$

a quadratic spectrum of the Schrödinger problem,

$$E_n=\alpha(n+l)^2, \quad \psi_n(x)=\cos^l\xi C_n^{(l)}(\sin\xi), \tag{4.23}$$

and a shift operator

$$L_{(\pm)}=\mp\sqrt{2}\,\sin\xi\sqrt{H}+i\sqrt{\alpha}\cos\xi\,p. \tag{4.24}$$

Here $\alpha\equiv c/2$, $\xi\equiv\sqrt{2\alpha}x$, $p=-id/dx$, and $C_n^{(l)}(z)$ are ultraspherical (or Gegenbauer) polynomials.

The analogs of the photon-number operators, $N_{(\pm)}\equiv L_{(\pm)}L_{(\pm)}^\dagger$ and $\tilde{N}_{(\pm)}\equiv L_{(\pm)}^\dagger L_{(\pm)}$, are not polynomials in H and are given by the following expressions:

$$N_{(\pm)}=H\mp2\sqrt{\alpha H}-(U_0-\alpha)\pm\sqrt{\alpha}\frac{U_0}{\sqrt{H}}, \tag{4.25}$$

$$\tilde{N}_{(\pm)}=H-U_0+\sqrt{\alpha}\frac{U_0}{\sqrt{\alpha}\pm\sqrt{H}}, \tag{4.26}$$

where $U_0\equiv l(l-1)$.

One can easily verify that these operators satisfy both the general relationship $[N,L]=L(\tilde{N}-N)$ and its particular case for the reduced equations $\tilde{N}(H)=N(\Omega+H)$.

The eigenvalue of the operator $N_{(+)}$ vanishes at a unique point $E=\alpha l^2$, which corresponds to the ground state of the given system. The L-mapping constructed with the help of

the operator (4.24) on this state, generates an infinite quadratic sequence of E_n that exhausts the energy spectrum of the system. Since we have $l>1$, all points of this sequence lie in the region $E>\alpha\equiv c/2$, which corresponds to sequences that keep the second finite difference constant.

Thus, the approach based on introducing and analyzing analogs of the photon-number operators can also be used for spectral sequences that differ from equidistant sequences.

5. CONCLUSION

In this paper we examined one way to generalize the description of the quantum harmonic oscillator in the Fock space to the case of a nonlinear dependence of the photon-number operator N on the Hamiltonian operator H. We illustrated the generalization by a detailed study of systems that allow for the existence of an analog of the photon-number operator, $N(H)$, in the form of a third-degree polynomial. We did a complete classification of the types of energy spectra that arise in such models and studied the corresponding regular one-dimensional potential. We found that the adopted generalization makes it possible to describe systems, interesting from the standpoint of physics, with an energy gap superposed on an equidistant spectrum. By varying the values of the system's structural parameters one can control the position and size of the gap.

We also gave examples of realizing photon-number operators with a more complicated, nonpolynomial, dependence of N on H.

Finally, we investigated certain aspects of using this method to describe the simplest spectra that satisfy the condition that the second finite difference of the eigenvalues of H be constant.

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