Vortex dynamics of a solution of two superfluid liquids

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We show that in solutions of two superfluid liquids, the number of hydrodynamic theorems triples and there can be three types of vortices. Allowing for dissipative processes, we derive the equations of vortex motion of the system using a phenomenological approach based on conservation laws. © *1996 American Institute of Physics.* [S1063-7761(96)02111-7]

1. INTRODUCTION

In this paper we derive the equations of the vortex dynamics of a solution of two superfluid liquids. We base our reasoning on the phenomenological equations for such a system obtained in Refs. 1–4.

The aim of the present paper is, first, to corroborate the existence of three types of vortex excitations in such systems on the basis of the statement (proved below) that the number of hydrodynamic theorems in this "three-velocity" system triples in comparison to that in ordinary "one-velocity" hydrodynamics, and, second, to derive, using a phenomenological approach based on conservation laws, the equations of vortex motion of the system, allowing for dissipative processes (a similar derivation for He II was done by Bekarevich and Khalatnikov[5]).

Analysis of conservation laws shows that the complete system of equations of "three-velocity" hydrodynamics has the following form[1]:

$$\frac{\partial \rho_1}{\partial t}+\operatorname{div}(\rho_1\mathbf{v}_n+\mathbf{p}_1)=0,$$
$$\frac{\partial \rho_2}{\partial t}+\operatorname{div}(\rho_2\mathbf{v}_n+\mathbf{p}_2)=0, \tag{1}$$

$$\frac{\partial j_i}{\partial t}+\frac{\partial \Pi_{ik}}{\partial x_k}=0, \tag{2}$$

$$\frac{\partial S}{\partial t}+\operatorname{div}(S\mathbf{v}_n)=0, \tag{3}$$

$$\frac{\partial \mathbf{v}_1}{\partial t}+(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1+\nabla\mu_1=0,$$
$$\frac{\partial \mathbf{v}_2}{\partial t}+(\mathbf{v}_2\cdot\nabla)\mathbf{v}_2+\nabla\mu_2=0, \tag{4}$$
$$\operatorname{curl}\mathbf{v}_1=\operatorname{curl}\mathbf{v}_2=0.$$

Here $\rho=\rho_1+\rho_2$ is the density of the solution, ρ_1 and ρ_2 are the component densities ($\rho_1=\rho c$ and $\rho_2=\rho(1-c)$, with c the solution's concentration); $\mathbf{j}=\mathbf{p}_1+\mathbf{p}_2+\rho\mathbf{v}_n$ is the liquid's momentum per unit volume; $\mathbf{p}_1$ and $\mathbf{p}_2$ are the relative momenta of the two superfluid motions in the reference frame in which the normal (nonsuperfluid) part of the liquid is at rest,

$$\mathbf{p}_1=\rho_1(\mathbf{v}_1-\mathbf{v}_n),\quad \mathbf{p}_2=\rho_2(\mathbf{v}_1-\mathbf{v}_n);$$

μ_1 and μ_2 are the chemical potentials of the solution's components; $\mathbf{v}_n$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are the velocities of the normal and two superfluid motions; and Π_{ik} is the momentum flux tensor, which has the form

$$\Pi_{ik}=\rho v_{ni}v_{nk}+(p_{1k}v_1+v_{nk}p_{1i})+(p_{2k}v_2+v_{nk}p_{2i})+P\delta_{ik},$$

where P is the pressure,

$$P=-\varepsilon+TS+\mu_1\rho_1+\mu_2\rho_2,$$

T and S are the temperature and the entropy density, and ε is the internal energy density.

Note that to simplify the formulas we assume the effect of mutual drag of the two superfluid motions predicted by Andreev and Bashkin[4] to be small, so that in what follows all quantities describing this effect are set to zero, i.e., $\rho_s^{12}=\rho_s^{21}=0$.

2. TRIPLING OF THE NUMBER OF HYDRODYNAMIC THEOREMS AND THE THREE TYPES OF VORTICES IN SOLUTIONS OF TWO SUPERFLUID LIQUIDS

One of the corollaries of Eqs. (1)–(4) is the tripling of the number of fundamental hydrodynamic theorems. Indeed, these equations for superfluid motion imply

(1) the validity of Kelvin's theorem, which states that the circulations of the superfluid velocities along the contours formed by the respective particles of the superfluid components ($l=1,2$) remain constant:

$$\frac{D_1}{Dt}\oint_{L_1}\mathbf{v}_1\cdot d\mathbf{l}_1=0,\quad \frac{D_2}{Dt}\oint_{L_2}\mathbf{v}_2\cdot d\mathbf{l}_2=0,$$

where $D_l/Dt=\partial/\partial t+\mathbf{v}_l\cdot\nabla$;

(2) the validity of Lagrange's theorem, which states that if initially curl $\mathbf{v}_l=0$ everywhere, curl $\mathbf{v}_l=0$ for all time; and

(3) the validity of Bernoulli's theorem, which states that the quantities $\partial\Phi_l/\partial t+\mu_l+\frac{1}{2}\mathbf{v}_l^2+\Omega$ remain constant along the stream lines of the corresponding superfluid component. (Here Φ_l is the phase of the corresponding wave function describing the related superfluid motion, whose gradient determines the superfluid velocity, $\mathbf{v}_l\sim\nabla\Phi_l$, and Ω is the external force potential.)

Next, representing the momentum flux tensor Π_{ik} in the form

$$\Pi_{ik}=A_iSv_{nk}+v_{1i}j_{1k}+v_{2i}j_{2k}+\delta_{ik}P,$$

where $\mathbf{A}=\rho_nS^{-1}\mathbf{v}_n-\rho_{n1}S^{-1}\mathbf{v}_1-\rho_{n2}S^{-1}\mathbf{v}_2$, $\rho_n=\rho_{n_1}+\rho_{n2}$ is the total normal density, with ρ_{n1} and ρ_{n2} the corresponding components of ρ_n, $\mathbf{j}_1=\rho_{n1}\mathbf{v}_n+\rho_1\mathbf{v}_1$, and $\mathbf{j}_2=\rho_{n2}\mathbf{v}_n+\rho_2\mathbf{v}_2$, we arrive at the following equation for $\mathbf{A}$:

$$\frac{\partial A_i}{\partial t}+(\mathbf{v}_n\cdot\nabla)A_i=-\frac{\partial T}{\partial x_i}-A_j\frac{\partial v_{nj}}{\partial x_i}, \tag{5}$$

which has the same form as the equation for $\mathbf{A}$ in two-velocity hydrodynamics.[6] One can easily verify that Eq. (5) leads to all three theorems for $\mathbf{A}$. In particular, Bernoulli's theorem has the form $T+\mathbf{v}_n\cdot\mathbf{A}=\text{const}$ along the stream lines of the normal component.

The fact that the number of hydrodynamic theorems triples when the system contains multiply-connected regions implies that there can be three types of vortex excitations coexisting in the system: two superfluid excitations with quantized circulation described in Ref. 7, and what became known as temperature $\mathbf{A}$-vortices described in Ref. 8.

3. PHENOMENOLOGICAL EQUATIONS OF VORTEX MOTION

Bearing in mind that dissipative effects violate the conservation of the circulation of the vector $\mathbf{A}$ (Ref. 8), we assume that only the vortices of the superfluid velocities are nonzero in the system. As is known,[5] vortex motion differs from vortex-free motion in terms of the dependence of the internal energy ε of the liquid on the absolute value of the curl of velocity, which can be expressed in differential form:

$$\begin{gathered}\delta\varepsilon=\lambda_1\delta\omega_1+\lambda_2\delta\omega_2,\\ \omega_1=|\text{curl }\mathbf{v}_1|,\quad \omega_2=|\text{curl }\mathbf{v}_2|,\end{gathered} \tag{6}$$

where the coefficients λ, as the microscopic treatment of vortex filaments implies,[9] are

$$\lambda_1=\rho_1\frac{\hbar}{m_1}\ln\frac{R_1}{a_1},\quad \lambda_2=\rho_2\frac{\hbar}{m_2}\ln\frac{R_2}{a_2}.$$

Here $\hbar$ is Planck's constant, m_1 is the mass of the corresponding atom, and R_i/a_i is the ratio of the vortex separation to the effective vortex radius.

Next, starting with the equations of hydrodynamics of solutions of two superfluid liquids obtained in Ref. 1 and conservation laws, we proceed as follows. Into the momentum flux tensor Π_{ik} and the energy flux $\mathbf{Q}$ we introduce additional terms, denoted by π_{ik} and $\mathbf{q}$, respectively, in such a way that all dissipation processes are included in these terms. We define the mass flux vector $\mathbf{j}$, the momentum of the liquid per unit volume, in such a way that the continuity equation retains its form. Then the equations of mass, energy, and momentum conservation become

$$\frac{\partial\rho}{\partial t}+\text{div }\mathbf{j}=0, \tag{7}$$

$$\frac{\partial E}{\partial t}+\text{div}(\mathbf{Q}+\mathbf{q})=0, \tag{8}$$

$$\frac{\partial j_i}{\partial t}+\frac{\partial}{\partial x_k}\{\Pi_{ik}+\pi_{ik}\}=0, \tag{9}$$

where

$$\begin{aligned}\mathbf{Q}=&(\mathbf{p}_1+\rho_1\mathbf{v}_n)(\mu_1-\mathbf{v}_n^2/2)+(\mathbf{p}_2+\rho_2\mathbf{v}_n)(\mu_2-\mathbf{v}_n^2/2)\\ &+\mathbf{v}_n(\mathbf{j}\cdot\mathbf{v}_n)+\mathbf{p}_1(\mathbf{v}_n\cdot\mathbf{v}_1)+\mathbf{p}_2(\mathbf{v}_n\cdot\mathbf{v}_2)\end{aligned} \tag{10}$$

is the unperturbed energy flux, and E is the liquid's energy, which by Galilean transformations can be expressed in terms of the internal energy ε of the liquid in the reference frame moving with the velocity $\mathbf{v}_n$ of the normal fraction of the liquid as follows:

$$E=\rho\mathbf{v}_n^2/2+(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{v}_n+\varepsilon. \tag{11}$$

The internal energy ε satisfies the following thermodynamic identity, which allows for the above-noted variation in the liquid's energy caused by vortex motion:

$$\begin{aligned}d\varepsilon=&\mu_1d\rho_1+\mu_2d\rho_2+TdS+\mathbf{p}_1d(\mathbf{v}_1-\mathbf{v}_n)\\ &+\mathbf{p}_2d(\mathbf{v}_2-\mathbf{v}_n)+\lambda_1d\omega_1+\lambda_2d\omega_2.\end{aligned} \tag{12}$$

In addition to the conservation equations (7)–(9), the hydrodynamic equations incorporate the equations of superfluid components and the equation for the increase of entropy:

$$\begin{gathered}\frac{\partial\mathbf{v}_1}{\partial t}+\nabla\left(\mu_1+\frac{\mathbf{v}_n^2}{2}+\mathbf{v}_n\cdot\mathbf{v}_1\right)=\mathbf{f}_1,\\ \frac{\partial\mathbf{v}_2}{\partial t}+\nabla\left(\mu_2+\frac{\mathbf{v}_n^2}{2}+\mathbf{v}_n\cdot\mathbf{v}_2\right)=\mathbf{f}_2,\end{gathered} \tag{13}$$

$$\frac{\partial S}{\partial t}+\text{div}(S\mathbf{v}_n)=\frac{R}{T}, \tag{14}$$

where the quantities $\mathbf{f}_1$, $\mathbf{f}_2$, and R (and π_{ik} and $\mathbf{q}$, for that matter) are yet to be defined.

To determine the form of the unknown functions, we find the time derivatives of the left- and right-hand sides of Eq. (11) and use the hydrodynamic equations (7), (9), and (12)–(14) to isolate all terms that are total divergences. The result is

$$\begin{aligned}\frac{\partial E}{\partial t}=&\frac{\partial\rho}{\partial t}\left(\frac{\mathbf{v}_n^2}{2}\right)+\rho\mathbf{v}_n\cdot\frac{\partial\mathbf{v}_n}{\partial t}+\mathbf{v}_n\cdot\frac{\partial(\mathbf{p}_1+\mathbf{p}_2)}{\partial t}\\ &+(\mathbf{p}_1+\mathbf{p}_2)\cdot\frac{\partial\mathbf{v}_n}{\partial t}+\mu_1\frac{\partial\rho_1}{\partial t}+\mu_2\frac{\partial\rho_2}{\partial t}+T\frac{\partial S}{\partial t}\\ &+\mathbf{p}_1\cdot\frac{\partial(\mathbf{v}_1-\mathbf{v}_n)}{\partial t}+\mathbf{p}_2\cdot\frac{\partial(\mathbf{v}_2-\mathbf{v}_n)}{\partial t}+\lambda_1\frac{\partial\omega_1}{\partial t}\\ &+\lambda_2\frac{\partial\omega_2}{\partial t}=-\frac{\partial\rho}{\partial t}\frac{\mathbf{v}_n^2}{2}+\mathbf{v}_n\cdot\frac{\partial\mathbf{j}}{\partial t}+\lambda_1\frac{\partial\omega_1}{\partial t}+\lambda_2\frac{\partial\omega_2}{\partial t}\\ &+\mu_1\frac{\partial\rho_1}{\partial t}+\mu_2\frac{\partial\rho_2}{\partial t}+T\frac{\partial S}{\partial t}+\mathbf{p}_1\cdot\frac{\partial\mathbf{v}_1}{\partial t}+\mathbf{p}_2\cdot\frac{\partial\mathbf{v}_2}{\partial t}.\end{aligned}$$

Expressing all time derivatives via Eqs. (7), (9), and (11)–(14) and using the expressions that follow from (6), namely

$$\lambda_1 \frac{\partial \omega_1}{\partial t} = \lambda_1 \mathbf{n}_1 \cdot \mathrm{curl}\, \frac{\partial \mathbf{v}_1}{\partial t} = \lambda_1 \mathbf{n}_1 \cdot \mathrm{curl}\{\mathbf{f}_1 + [\boldsymbol{\omega}_1 \times (\mathbf{v}_n - \mathbf{n}_1)]\} - \lambda_1 \mathbf{n}_1 \cdot \mathrm{curl}[\boldsymbol{\omega}_1 \times \mathbf{v}_n],$$

$$\lambda_2 \frac{\partial \omega_2}{\partial t} = \lambda_2 \mathbf{n}_2 \cdot \mathrm{curl}\, \frac{\partial \mathbf{v}_2}{\partial t} = \lambda_2 \mathbf{n}_2 \cdot \mathrm{curl}\{\mathbf{f}_2 + [\boldsymbol{\omega}_2 \times (\mathbf{v}_n - \mathbf{n}_2)]\} - \lambda_2 \mathbf{n}_2 \cdot \mathrm{curl}[\boldsymbol{\omega}_2 \times \mathbf{v}_n],$$

where $\mathbf{n}_i = \boldsymbol{\omega}_i / \omega_i$, we obtain

$$\begin{aligned}\frac{\partial E}{\partial t} = &-\left(\mu_1 - \frac{\mathbf{v}_n^2}{2}\right) \mathrm{div}\{\mathbf{p}_1 + \rho_1 \mathbf{v}_n\} - \left(\mu_2 - \frac{\mathbf{v}_n^2}{2}\right) \mathrm{div}\{\mathbf{p}_2 + \rho_2 \mathbf{v}_n\} + \mathbf{f}_1 \cdot \mathbf{p}_1 + \mathbf{f}_2 \cdot \mathbf{p}_2 - \mathbf{p}_1 \cdot \nabla\left(\mu - \frac{\mathbf{v}_n^2}{2} + \mathbf{v}_n \cdot \mathbf{v}_1\right) \\ &- \mathbf{p}_2 \cdot \nabla\left(\mu_2 - \frac{\mathbf{v}_n^2}{2} + \mathbf{v}_n \cdot \mathbf{v}_2\right) - v_{ni} \frac{\partial}{\partial x_k}\{\Pi_{ik} + \pi_{ik}\} \\ &- T\, \mathrm{div}(S \mathbf{v}_n) + R + \sum_{l=1}^{2} \{\lambda_l \mathbf{n}_l \cdot \mathrm{curl}\{\mathbf{f}_l + [\boldsymbol{\omega}_l \times (\mathbf{v}_n - \mathbf{v}_l)]\} - \lambda_l \mathbf{n}_l \cdot \mathrm{curl}[\boldsymbol{\omega}_l \times \mathbf{v}_n]\}.\end{aligned}$$

Isolating the total divergences and performing certain transformations, we find that

$$\begin{aligned}\frac{\partial E}{\partial t} + \mathrm{div}\left\{ \mathbf{Q} + (\pi \mathbf{v}_n) + \sum_{l=1}^{2} \lambda_l [\mathbf{n}_l \times \{\mathbf{f}_l + [\boldsymbol{\omega}_l \times (\mathbf{v}_n - \mathbf{v}_l)]\}] \right\} = T\left(\frac{\partial S}{\partial t} + \mathrm{div}(S \mathbf{v}_n)\right) + \frac{\partial v_{ni}}{\partial x_k}\left\{ \pi_{ik} - \sum_{l=1}^{2} \lambda_l \left\{ \omega_l \delta_{ik} - \frac{\omega_{li}\omega_{lk}}{\omega} \right\} \right\} + (\mathbf{j} - \rho \mathbf{v}_n + \mathrm{curl}\, \lambda_1 \mathbf{n}_1 \\ + \mathrm{curl}\, \lambda_2 \mathbf{n}_2) \times \sum_{l=1}^{2} \{\mathbf{f}_l + [\boldsymbol{\omega}_l \times (\mathbf{v}_n - \mathbf{v}_l)]\}.\end{aligned}$$

Here $(\pi \mathbf{v}_n)_i = \pi_{ik} v_{nk}$.

Comparing these expressions with the equations of energy conservation and increase of entropy, we find that

$$\mathbf{q} = (\pi \mathbf{v}_n) + \sum_{l=1}^{2} \{\lambda_l [\mathbf{n}_l \times \{\mathbf{f}_l + [\boldsymbol{\omega}_l \times (\mathbf{v}_n - \mathbf{v}_l)]\}]\}, \tag{15}$$

$$R = -\frac{\partial v_{ni}}{\partial x_k}\left\{ \pi_{ik} - \sum_{l=1}^{2} \lambda_l \left\{ \omega_l \delta_{ik} - \frac{\omega_{li}\omega_{lk}}{\omega} \right\} \right\} + (\mathbf{j} - \rho \mathbf{v}_n + \mathrm{curl}\, \lambda_1 \mathbf{n}_1 + \mathrm{curl}\, \lambda_2 \mathbf{n}_2) \times \sum_{l=1}^{2} \{\mathbf{f}_l + [\boldsymbol{\omega}_l \times (\mathbf{v}_n - \mathbf{v}_l)]\}. \tag{16}$$

Now we require that the dissipation functions be positive-definite quadratic forms. From Eq. (16) we then arrive at the following expressions for the vector terms:

$$\mathbf{f}_l = -[\boldsymbol{\omega}_l \times (\mathbf{v}_n - \mathbf{v}_l)] + \alpha_l [\boldsymbol{\omega}_l \times (\mathbf{j} - \rho \mathbf{v}_n + \mathrm{curl}\, \lambda_l \mathbf{n}_l)] + \beta_l [\mathbf{n}_l \times [\boldsymbol{\omega}_l \times (\mathbf{j} - \rho \mathbf{v}_n + \mathrm{curl}\, \lambda_l \mathbf{n}_l)]] - \gamma_l \mathbf{n}_l (\boldsymbol{\omega}_l \cdot (\mathbf{j} - \rho \mathbf{v}_n + \mathrm{curl}\, \lambda_l \mathbf{n}_l)). \tag{17}$$

Here $l = 1,2$, $\beta_l \geq 0$, and $\gamma_l \geq 0$ in view of the condition that $R \geq 0$.

Reasoning along similar lines, we arrive at an expression for π_{ik}:

$$\pi_{ik} = \sum_{l=1}^{2} \lambda_l \left\{ \omega_l \delta_{ik} - \frac{\omega_{li}\omega_{lk}}{\omega} \right\} + \tau_{ik}, \tag{18}$$

where τ_{ik} is, as usual, the viscous stress tensor, which is generally expressed in terms of the viscosity tensor ν_{iklm}:

$$\tau_{ik} = \nu_{iklm} \frac{\partial v_{nl}}{\partial x_m}.$$

Allowing for the fact that

$$\mathbf{j} - \rho \mathbf{v}_n = -\rho_1 (\mathbf{v}_n - \mathbf{v}_1) - \rho_2 (\mathbf{v}_n - \mathbf{v}_2),$$

and transforming the expressions for the $\mathbf{f}_l$ ($l = 1,2$), we can reduce Eq. (13) for the superfluid velocities to the form

$$\begin{aligned}&\frac{\partial \mathbf{v}_l}{\partial t} + (\mathbf{v}_l \cdot \nabla) \mathbf{v}_l + \nabla \mu_l \\ &= -\rho_l^{-1} [\boldsymbol{\omega}_l \times \mathrm{curl}\, \lambda_l \mathbf{n}_l] - (1 - \alpha_l \rho_l) [\boldsymbol{\omega}_l \times (\mathbf{v}_n - \mathbf{v}_l - \rho_l^{-1} \mathrm{curl}\, \lambda_l \mathbf{n}_l)] - \beta_l \rho_l [\mathbf{n}_l \times [\boldsymbol{\omega}_l \times (\mathbf{v}_n - \mathbf{v}_l \\ &\quad - \rho_l^{-1} \mathrm{curl}\, \lambda_l \mathbf{n}_l)]] + \gamma_l \rho_l \mathbf{n}_l (\boldsymbol{\omega}_l (\mathbf{v}_n - \mathbf{v}_l - \rho_l^{-1} \mathrm{curl}\, \lambda_l \mathbf{n}_l)).\end{aligned} \tag{19}$$

Here the first term on the right-hand side, which coincides with the one introduced in Ref. 10, is the force of mutual friction emerging as a result of the interaction between the set of straight vortex filaments and the normal liquid moving with the mean-drift velocity $\mathbf{v}_n$, the term with γ_l is the longitudinal (along the direction of the vortices) force of mutual friction between the respective superfluid vortex and the normal component, and the terms of type $\lambda_l \omega_{li} \omega_{lk} / \omega_l$ can be interpreted as vortex tension.

If we allow for the expression (18) for π_{ik} and the fact that all phenomena related to entropy and momenta transfer by vortices are effects quadratic in velocity and are therefore small and can be ignored, the variation in energy flux can be expressed as follows:

$$\mathbf{q} = \sum_{l=1}^{2} \lambda_l [\mathbf{n}_l \times \{\mathbf{f}_l - [\boldsymbol{\omega}_l \times \mathbf{v}_l]\}]. \tag{20}$$

In the absence of a longitudinal component ($\gamma_l = 0$) this can be interpreted as energy transfer in the direction perpendicular to $\boldsymbol{\omega}$.

Thus, the complete set of equations of motion of the given system consists of the equations for the superfluid velocities (13), Eqs. (7), (8), (11), and (14) (where $\mathbf{f}_l$, R, and π_{ik} are determined from Eqs. (16)–(18)), and the thermodynamic identity (12).

In conclusion we note that in the limit in which one of the superfluid components is zero, the resulting equations are simply the equations of Ref. 5 for vortex motion in He II with allowance for dissipative processes.

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