Self-similarity—particular solution or asymptote?

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An attempt is made to determine the place of self-similar solutions of the first and second kinds as asymptotic solutions for partial differential equations, i.e., solutions independent of the initial conditions. © *1996 American Institute of Physics.* [S1063-7761(96)01811-2]

1. INTRODUCTION

Self-similar solutions are used in the search for approximate solutions of partial differential equations. Much work has been devoted to self-similar solutions.[1,2] However, up to now it has been uncertain to what extent self-similar solutions are asymptotically correct or whether they are nothing more than particular solutions to a problem. It is unclear how to give an answer in the most general form. A small number of analytic solutions can be used only for guidance. They show that self-similar solutions of the first kind (according to Zel'dovich's classification) with self-similarity coefficients derived from dimensional considerations can be asymptotic, i.e., independent of the initial conditions in the limit, while the picture is most likely the opposite for self-similar solutions of the second kind (although, of course, there may be exceptions).

2. SELF SIMILARITY OF THE FIRST KIND

2.1. The propagation of heat

The problem is completely solved for the linear heat conduction equation both in the case of a point, instantaneous (δ-function) source and for sources in an arbitrary region with an arbitrary time variation.

It can be shown that for $t \gg t_0$ (where t_0 is time the source is active) at distances $R \gg r_0$ (where r_0 is the characteristic size of the source) the solution actually does not depend on the initial conditions and satisfies the requirements for asymptotic behavior.

For example, for the spherical problem with an instantaneous source the correction to the temperature $T(r,t,r_0)$ owing to finite dimensions has the form

$$\frac{T_{r_0}}{T_{r_0\to 0}}=1-\frac{r_0^2}{4Dt}=1-\frac{r_0^2}{r^2}$$

for all $r<Dt/r_0$, where the overwhelming bulk of the energy is concentrated. Here D is the coefficient of thermal diffusivity.

2.2. Point explosion (Sedov problem)

In part of the discussion and numerical tables given below we follow Gubkin.[3]

Let us assume that the release of energy E_0 takes place, not at point in the air, but in some volume V_1. Because of this, a pressure P develops discontinuously, which differs from the pressure P_0 in the surrounding medium (for air $P_0=1$ atm; see Fig. 1).

In other words, the substance changes from state A to state B. Later this part of the substance equilibrates with the surroundings, i.e., returns to a pressure $P_0(C)$ through adiabatic pressure relief. At point C there will be a reduced density and elevated temperature. The internal energy in this state differs from the initial value by the difference between E and the work done by the pressure forces:

$$E(P_0,V_2)=E(P_0,V_1)+E_0-\int_{V_1}^{V_2}P\,dV.$$

The integral is taken along the adiabatic. Points C can be reached by other means, by slowly heating the material along the $P=P_0$ isobars. The heat expenditure Q then is

$$Q=E(P_0,V_2)-E(P_0,V_1)+P_0(V_2-V_1).$$

The difference E_0-Q is the energy of the hydrodynamic motion beyond the confines of volume V_2

$$E_0-Q=E'=\left[\int_{V_1}^{V_2}(P-P_0)dV\right].$$

We can introduce the concept of a hydrodynamic efficiency ε' and the fraction of "frozen" energy, q:

$$\varepsilon'=1-q=E'/E_0.$$

Then, for an ideal gas ($P=P/P_0$ is dimensionless and γ is the adiabatic index) we have

$$q=\gamma\frac{P^{1/\gamma}-1}{P-1}.$$

The dynamic effect is more marked for higher P: $q\to 0$ for $P\to\infty$ and $q\to 1$ for $P\to 1$. We now return to a point explosion. Here, also, it is possible to introduce the concept of "frozen" energy, as Taylor did.[4] Using the fact that all the entropy originates in the shock wave and the subsequent expansion is adiabatic, we obtain

$$q_p(R)=\frac{\gamma}{\gamma-1}\int_0^V\left(\frac{P_f^{1/\gamma}}{\rho_f}-1\right)dV.$$

Here the index p denotes the degree to which the explosion is localized at a point, P_f and ρ_f are the dimensionless pressure and density at the shock front, and the volume is expressed in dimensionless units, regardless of the geometry:

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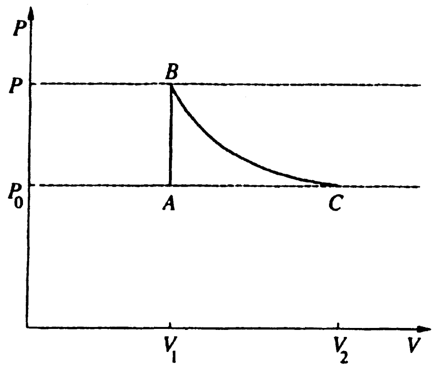

FIG. 1. The pressure change in a particle of a substance of volume V_1 with instantaneous uniform energy release over the volume. AB is the pressure jump and BC is the subsequent isotropic release.

TABLE II.

R,m	$\frac{P_f-P_0}{P_0}$	$\frac{P_f-P_0}{P_0}$	$\bar{R}=\frac{R}{R_0}$
5	54000	55000	0.023
10	6700	6800	0.046
20	840	860	0.093
30	247	250	0.14
50	55	55	0.23
100	7.7	8.7	0.46
200	1.5	1.6	0.93
300	0.66	0.69	1.4

Note. The second column is a numerical calculation using the hydrodynamic equations; the third, a calculation using the approximate formula; $\bar{R}$ is the dimensionless radius of the front.

$$V_0=\frac{E_0}{P_0}=\begin{cases}(4/3)\pi R^3 & \text{—spherical case}\\ \pi R & \text{—cylindrical case}\\ 2x & \text{—planar case}\end{cases}$$

where E_0 is taken to mean the amount of energy released by, respectively, a point, unit length of a cylinder, or unit area (from the energy entering one side) of a plane. Equating $q(P=P_f)=q_T(V)$, we obtain an integral relationship which gives the $P_f(V)$ dependence. The most surprising thing is that here the results are close to reality.

For $\gamma=2$ the solution $V(P)$ is analytic

$$V=\frac{1}{2}\frac{3P^{1/2}-1}{(P-1)(P^{1/2}-1)}.$$

Using the limiting expressions for V as $P\to\infty$ and $P\to 1$, we can write a generalized solution for arbitrary γ which is an approximation for all values, but is exact for $\gamma=2$

$$V=\frac{(\gamma^2-1)P^{(\gamma+1)/3\gamma}-(\gamma-1)}{\gamma(P-1)(P^{(\gamma+1)/3\gamma}-1)}.$$

For example, for $\gamma=1.4$

$$\frac{R^3}{\bar{R}}=V=\frac{0.96P^{0.57}-0.4}{1.4(P-1)(P^{0.57}-1)},$$

where $\bar{R}$ is the so-called reduced radius $\bar{R}=(E_0/(4/3)\pi P_0)^{1/3}$. Calculations with the approximate formula differ from a numerical calculation by less than 1–2%.

(a) The limiting (self-similar) solution for $P_f\to\infty$:

$$P_f=A/V,\quad A=(\gamma^2-1)/\gamma.$$

This solution is compared with the numerical solution to the self-similar spherical problem in Table I. The quantity $\alpha=E/E_0$ (where E is some auxiliary energy) is introduced in Ref. 2. It is calculated by introducing the quantity A for the three symmetries: spherical, cylindrical, and planar. This yields

$$\frac{\alpha_{\rm cyl}}{\alpha_{\rm sph}}=1.17,\quad \frac{\alpha_{\rm pl}}{\alpha_{\rm sph}}=1.33,$$

which is in very good agreement with Fig. 75 on p. 252 of Ref. 2.

(b) The pressure at the front of a spherical shock wave obtained numerically is compared with our approximate method in Table II ($E_0=4.2\cdot10^{19}$ erg, $\gamma=1.4$, $P_0=10^6$ erg/cm^3). The middle two columns are an expression of the universal coupling $P_f(R)/P_0$, which is independent of geometry. Based on this fact plus the circumstance that this result was attained by comparing two solutions which apply to energy release in a finite volume and from a point, we may conclude that the self-similar solution is universal (for $R\gg r_0$, $t\gg t_0$, where r_0 is a characteristic dimension which is independent of the shape and energy source and t_0 is the energy release time).

Note that the solution is roughly true in the region where the pressure is not too high, but also for moderate pressures where it is impossible to neglect the back pressure. And this is despite the difference in the behavior of the pressure at large distances for $\Delta P\to 0$, e.g.,

$$P_{\rm sph}\propto 1/R\ \ln^{1/2}R,$$

where in the approximate solution

$$P=\frac{B}{V^{1/2}},\quad B=\frac{3\gamma(\gamma-1)}{\gamma+1}.$$

For example, we have $P=0.28$ atm at $R=500$ m, while the approximate expression given above implies that $P=0.24$. Thus, a new procedure emerges for determining the pressure at a shock front over a wide range of variation in the parameters in place of the standard empirical formula of Sadovskii.[5] That the pressure curve is independent of the shape of the energy release volume expressed in dimensionless form reveals, in particular, the following fact: a pancake shaped (i.e., flat and bounded by a circle) charge creates the

TABLE I. Comparison of approximate and numerical calculations.

γ	1.1	1.2	1.3	1.4	5/3	2	3
$\tilde{A}=P_fV$	0.19	0.35	0.51	0.66	1.02	1.43	2.53
A	0.19	0.37	0.53	0.69	1.07	1.5	2.67

Note. The second row is the equivalent of the combination $\tilde{A}=P_fV$ from the numerical calculation and the third, $A=(\gamma^2-1)/\gamma$.

same pressure drop in its epicenter (at a level of $P \approx 0.3-0.5$ atm) and the same impulse as a point charge at 300 times the power.

Both of the problems examined here confirm the physical significance of self-similar solutions, their asymptotic nature, even if not proven entirely rigorously. On the other hand, not one counterexample has been found. At first it seemed that the problem of a substance expanding into vacuum might contain a counterexample.

(c) Expansion into vacuum. Let us suppose that some material loaded to a pressure P_0 is released into a vacuum. After some time the pressure in the material falls to zero and the material will move by inertia. On the one hand it might seem that all the conditions for self-similarity are satisfied when the size of the cloud greatly exceeds the initial size. On the other hand, since the expansion is adiabatic the expression for the entropy, initially specified as a function of the Lagrange point, retains its form until the end. The solution cannot "forget" the initial conditions and, in this sense, is not universal.

It turns out, however, that the solution can still be given a self-similar form. For spherical expansion it is

$$\rho = \delta(\xi)/r^3, \quad P = A(\xi)\rho^\gamma, \quad \xi = r/u_0 t$$

(if $\xi=1$ defines the boundary of the cloud, then u_0 is the velocity of the boundary). The contradiction is removed because the equations of hydrodynamics are satisfied for arbitrary functions $\delta(\xi)$ and $A(\xi)$ and it is they which carry the "stamp" of the initial conditions.

3. SELF-SIMILARITY OF THE SECOND KIND

Self-similarity of the first kind occurs during outward propagation of a wave; the perturbation encompasses only the interior of the material. In view of the finiteness of the volume encompassed by the motion, all the physical quantities (energy, impulse, mass) are also finite and have their ordinary physical significance. Since they are dimensional, they are used to determine the self-similar coefficients and powers. The problem acquires a closed, unique character.

With self similarity of the second kind, the motion is inward. The region which supports this motion extends to infinity. Because of this, the quantities which are significant for the problem diverge or go, as does the impulse, to zero. Thus, it cannot be said that they are conserved and no use can be made of their dimensionality. The self-similarity index is not derived from physical considerations, but from requiring a solution exist and the integral curve pass through a singular point, i.e., from the finiteness of the self-similar functions. In the problems of a shock wave converging to a center, cavity collapse (the Rayleigh problem), and sudden impact, the self-similarity index is transcendental and is sometimes not unique, and there is no basis for preferring one power to another. The singular point lies on the limiting characteristic separating the perturbation regions: anything outside it has no effect on the leading edge. The space enclosed between the limiting characteristic and the leading perturbation line can be considered "pure," as if shut off from external perturbations. However, this does not mean that a solution is formed every time in the same way, regardless of the initial conditions. But even if the self-similar solution does not contain in itself an element of universality, it can be reliably applied to a small region of a size which also goes to zero as the shock converges to the center (see Fig. 2).

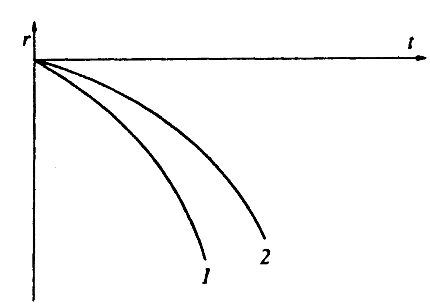

FIG. 2. An (r,t)-diagram for a self-similar solution. (1) Shock front, (2) line of singular points.

Two examples are investigated in the following: the collapse of a cavity under the two extreme assumptions of an incompressible and an infinitely compressible substance.

3.1. Incompressible substance

(a) Spherical cavity. There is no need to turn to a direct solution of the problem. It is sufficient to keep in mind the expression for the velocity,

$$u = u_1 r_1^2 / r^2,$$

(the subscript 1 refers to the inner boundary with the vacuum) and use an expression for the total kinetic energy,

$$E \propto \int_r^\infty \frac{u^2}{2} r^2 dr \propto u_1^2 r_1^3.$$

We find that

$$u_1 \propto 1/r_1^{3/2}.$$

Here the solution itself has a fully self-similar form in the limit $r_1 \to \infty$. This happens for the simple reason that the energy is finite and this determines the self-similarity index. It may be said that we have not acted entirely correctly in calling this case self-similarity of the second kind. It should be kept in mind that an incompressible substance is an abstraction and that for any, arbitrarily (but not infinitely) rigid material the expression for the energy applied to an infinite region will converge owing to the work done by the pressure forces, which vanish only in the limit $\gamma \to \infty$.

(b) Cylindrical cavity: incompressible substance. In terms of all its external features, the problem is completely analogous to the spherical problem. Because of the slow reduction in the velocity with increasing radius ($u = u_1 r_1 / r$), however, the integral of the kinetic energy,

$$E \propto \int_{r_1}^{r_2} u^2 r\, dr \propto u_1^2 r_1^2 \ln \frac{r_1}{r_2}$$

now diverges as $r_2 \to \infty$. Here r_2, the outer boundary of the shell (it is determined by the mass of the material), can no longer be displaced to infinity. But if we go over to the self-similar equations in accordance with the generally accepted procedure, then the solution is

$$u=u_1\frac{r_1}{r}=u_1\xi,\quad P=u_1^2\Pi(\xi),$$

$$\Pi(\xi)=\frac{1}{2}\int_{\xi_2}^{1}\frac{ds(s+\alpha-1)}{s},$$

where

$$\alpha=-\frac{r_1}{u_1}\frac{du_1}{dr_1}$$

is the desired self-similarity index.

In order to proceed correctly to the limit $r\to 0$, $\xi\to 0$, (without taking Π to infinity), it is necessary to set $\alpha=1$. Then $\Pi=(1-\xi^2)/2$. There is no logarithm in this solution. Why? In fact, two different problems have been solved: one is the exact problem which described the free flight of a shell of finite mass and finite dimensions and the other is a problem with a rapidly rising pressure at infinity in synchrony with the collapse, i.e., rather meaningless (this last statement follows from the fact that $\Pi|_{\xi\to 0}=1/2$ and $P\propto 1/r_1^2$). The problem with a cylinder is of interest from two standpoints. It suggests that, first, the behavior of the integral curves is not a guarantee that the problem itself is physically correct. In obtaining a set of many self-similar solutions of the second kind, which contain diverging integrals as an obligatory condition, we cannot be certain that the initial statement of the problem and the final result agree with one another. The presence of external signs of self-similarity ($r\to 0$) still does not mean that a self-similar solution actually exists (or always exists). Second, the question arises of whether the presence of a logarithm in a given solution is a specific manifestation of the particular statement of the problem or has a more general significance. It is not clear why the self-similar solution of the second kind has the form of a power law dependence

$$r\propto t^{\alpha},$$

and not of a more complicated expression

$$r\propto t^{\alpha}\ \ln^{\beta}(t).$$

The logarithm is a distinctive function. With differentiation in the limit $t\to 0$ or $t\to\infty$, when only the principal term is retained, the logarithm can be regarded as a constant. In this case, the two formulas can not be distinguished. The class of possible solutions expands incredibly: besides the fact that β is arbitrary, other functional dependences of the form $\ln\ln\ldots\ln t$ are permissible. Since a dimensionless expression must appear in the argument of the transcendental function, the presence of the logarithm means that the solution depends on the initial conditions, i.e., universality is lacking, although this dependence may be weak. Does this not explain why attempts to obtain a power law self-similar solution by numerical integration are sometimes unsuccessful? A logarithm is unacceptable in self-similarity of the first kind, since this violates the conservation laws. (In the point explosion problem, energy is not conserved.)

3.2. Infinitely compressible substance

Imagine a shell of ideal gas at temperature $T=0$ flying freely toward the center of a sphere. It is clear that until it reaches the center, the pressure inside it will be zero. The absence of feedback means that the individual gas particles do not feel one another; they are independent. The problem is essentially kinematic. Evidently, the initial distributions of density and velocity are carried right up to the last moment, when the leading edge arrives at the center. There is no need to speak of any kind of universality, although the external conditions for self-similarity would seem to exist ($r\to 0$). For example, assuming that the velocity of every point in the shell at the initial time is proportional to the distance to the center, a situation arises in which the entire shell arrives simultaneously at the center of the sphere; it might be said that superaccumulation takes place. Usually during spherical cumulation an infinitely small fraction of the material has infinite parameters. Here, on the other hand, the entire shell, i.e., a finite mass, acquires an infinite density.

Despite their artificiality, these examples of incompressible and absolutely compressible shells converging toward a center can be regarded as opposite to one another and as limiting cases relative to reality. Self-similarity is not attained in either limit and it appears that this can be regarded as proof that it is absent in the intermediate physical cases, as well.

4. CONCLUSION

This article lacks mathematical rigor and absolute proofs. At the same time, the examples imply that solutions with self-similarity of the first kind are most likely to be limiting solutions to problems with dimensional initial conditions which vanish asymptotically. On the other hand, in problems involving self-similarity of the second kind, the initial conditions are often (if not always) "present" until the end, the self-similar solution has a special significance, which solution cannot be generalized to real physical problems.

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