Nonlocal plasma electron hydrodynamics

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A system of generalized equations is formulated for electron hydrodynamics in a plasma under conditions such that the electron mean free path is not small in comparison with the inhomogeneity space scale. The principal assumption underlying the derivation of these equations is that the deviation of the electron distribution function from local thermodynamic equilibrium is small. Electron collisions are investigated in the large ionic charge limit. The system of hydrodynamic equations is closed by the solution of the kinetic equation for electrons. Expressions for the Fourier components of the electron fluxes are given in terms of generalized forces, and for the first time nonlocal expressions are systematically derived for all the electron transport coefficients: electrical resistance, thermocurrent coefficient, thermal diffusivity, electron viscosity, friction force, and the ionic convection transport coefficients. Expressions are obtained for the longitudinal and transverse components of the electron susceptibility tensor over the entire range of perturbation scale lengths from the strongly collisional limit to the collisionless limit. These expressions are used to find the damping rate of an ion-acoustic wave and the surface impedance of a semi-infinite plasma in the intermediate range of scale lengths. © *1996 American Institute of Physics.* [S1063-7761(96)01210-3]

1. INTRODUCTION

Hydrodynamic equations provide an effective tool for the investigation of a wide range of phenomena in plasmas. Even though they are not as rigorous as the more fundamental kinetic equations, the hydrodynamic equations are far simpler, involve fewer variables, and for this reason are often used far afield of their domain of formal validity. The success achieved in using the hydrodynamic equations is largely contingent upon the accuracy of their closure technique, i.e., upon the way in which the higher, nonhydrodynamic moments of the distribution function are expressed in terms of the hydrodynamic moments: density, average velocity, temperature and energy flux. The best-known derivation of hydrodynamic equations from kinetic theory is associated with the Chapman–Enskog method, which calls for the deviation of the distribution function from thermodynamic equilibrium to be expanded in the gradients of the hydrodynamic moments.[1,2] However, this method introduces sizable errors, even when the spatial scale of the perturbations is of the order of a hundred times the electron mean free path. In practice, thermonuclear investigations involve much smaller-scale plasma inhomogeneities, making it necessary to develop alternative schemes for closing the hydrodynamic equations.

Attempts have been made to improve the accuracy of the hydrodynamic equations by smoothing the transport coefficients on scales of the order of the electron mean free path.[3,4] However, this approach lacks any kind of rigorous theoretical foundation. The choice of smoothing parameters is often empirical and depends on the particular problem, detracting measurably from the value of this method. Another approach is to compute corrections to the transport coefficient from higher orders in the gradient expansion of the distribution function in the Chapman–Enskog method.[5-7] However, these corrections contain large numerical factors and do not really extend the domain of validity of the Chapman–Enskog method. Still another approach proposed in recent papers[8] proceeds from the collisionless plasma approximation and then takes into account phenomena associated with weak collisional effects.[9,10] However, this approach is incapable of providing a systematic transition to the strongly collisional limit. Several authors[11-14] have proposed approximate expressions suitable for describing the transport coefficients over a wide range of parameters. Even though such expressions have correct asymptotic forms in both the strong and the weakly collisional limits, they are merely interpolation formulas and do not fit any systematic theory of electron transport in the intermediate range of the collisionality parameter, i.e., the ratio of the electron mean free path to the perturbation wavelength. Moreover, only phenomena related to electron thermal conductivity are discussed in all the preceding papers; the issue of other electron transport coefficients is not even raised.

Here we propose and develop a new method of derivation of nonlocal hydrodynamic equations, which are found to be exactly equivalent to the kinetic description of small-amplitude perturbations in a plasma, but include only the lower moments of the electron distribution function. The proposed method can be used to systematically obtain exact expressions for the electron transport coefficients in Fourier representation and to investigation the dispersion properties of a plasma for arbitrary values of the collisionality param-

eter. The most significant approximations of this theory are the assumptions that the perturbation amplitudes are small, the plasma is completely ionized, and the ionic charge is high: $Z \gg 1$. The method has already been described[15] briefly and in application to potential plasma flows. In the present article it is elaborated and extended to the more general case of nonpotential perturbations.

The theory set forth below is based on two fundamental notions. The first has been formulated in Refs. 3 and 4 and asserts that the spatial diffusion of electrons makes the variation of their distribution function significant in comparison with the influence of electron collisions, even if the characteristic scale length of the inhomogeneity is much greater than the mean free path of a thermal electron. This assertion is attributable to the fact that energy is transported in the plasma mainly by fast electrons with velocities several times that of the thermal electron, and the energy relaxation of the electrons is associated mainly with electron–electron (ee) collisions, which occur Z times less frequently than electron–ion (ei) collisions. Luciani et al.[3,4] have introduced the concept of the electron energy delocalization length λ_ϵ, which characterizes the scale length within which the spatial diffusion rate is equal to the frequency of ee collisions of thermal electrons. This length is $\sqrt{Z}$ times the electron mean free path for ei collisions λ_{ei}. The second fundamental notion is identified with Refs. 11 and 16 and asserts that the accurate quantitative description of the transition from strong collisionality to the collisionless limit rests on a correct description of the higher angular moments of the electron distribution function, which depend on the competition between the convective spatial transport of electrons and their Coulomb scattering by ions. For thermal electrons these two processes balance each other over the mean free path λ_{ei}. Consequently, a systematic theory must include description of two scales, λ_{ei} and λ_ϵ.

The method used to solve the kinetic equation in this article was presented earlier[17] and is valid for any ratio between the scale length of the perturbation and the above-mentioned collisional relaxation lengths. This approach differs from the usual Chapman–Enskog method in that it permits the contributions of all angular harmonics of the electron distribution function to be taken into account, and it can be used to quantitatively describe the transition from ordinary collisional hydrodynamics to the collisionless limit. A special procedure for solving the initial problem for the perturbed electron distribution function is used to find relations between the electron fluxes and the generalized hydrodynamic forces, including the gradients of the plasma density and temperature, along with the plasma flow velocity and the electromagnetic fields. In the Fourier representation these relations have a form similar to the classical expressions[1,2] and give the nonlocal electrical conductivity, the nonlocal thermal diffusivity, and the thermocurrent coefficient. New transport coefficients associated with convection flows arise in the process.

This approach has made it possible for the first time to uniquely determine the nonlocal kinetic coefficients, as is demonstrated below in the example of the thermal conductivity. It is shown that the contradictions in the thermal con-

ductivity expressions proposed in several previous papers are attributable first and foremost to the incorrect definition of the thermal conductivity due to neglecting the contribution of the plasma convective motions to the electron fluxes. To illustrate the application of the nonlocal hydrodynamic equations, in this article we obtain expressions for the longitudinal and transverse components of the electron susceptibility tensor, which are valid for any ratio between the wavelength of the electromagnetic perturbation and the electron mean free path. They are used in the problems of ion-acoustic wave damping and the scanning of an electromagnetic field in a semi-infinite plasma in the domain of parameters of interest for present-day experiments on the interaction of high-intensity picosecond laser pulses with a plasma.

2. KINETIC DESCRIPTION OF NONLOCAL ELECTRON TRANSPORT

2.1. Kinetic equation for the electron distribution function

As the ground state we consider a homogeneous plasma with a Maxwellian electron distribution function $F_0(v)$ with density n_0 and temperature T_0. We assume that the plasma is fully ionized and contains ions with a high ionization multiplicity $Z \gg 1$. Being concerned primarily with phenomena linked to the transport of electrons, we assume for simplicity that the ions are cold, and we ignore ion–ion collisions. We also disregard mutual energy transfer in electron–ion collisions.

We consider the response of the plasma to a small-amplitude, low-frequency, periodic perturbation with wave vector $\mathbf{k}$. The assumption that the perturbation is quasistationary implies that the characteristic time constant of its variation is much longer than the characteristic time of ei collisions, $\omega \ll \nu_{ei}$, or that its phase velocity is small in comparison with the thermal electron velocity, $\omega/k \ll v_{Te}$. The basic kinetic equation

$$\frac{\partial f_e}{\partial t} + \mathbf{v}\frac{\partial f_e}{\partial \mathbf{r}} - \frac{e}{m_e}\mathbf{E}\frac{\partial f_e}{\partial \mathbf{v}} = C_{ei}[f_e] + C_{ee}[f_e, f_e], \qquad (2.1)$$

where e and m_e are the mass and charge of the electron, and $\mathbf{E}$ is the electric field, takes both ei and ee collisions into account. If energy transfer in the ion reference frame is ignored, the ei collision operator has the form

$$C_{ei}[f_e] = \frac{1}{2}\nu_{ei}(v)\frac{\partial}{\partial(\cos\theta)}\sin^2\theta\frac{\partial f_e}{\partial(\cos\theta)}, \qquad (2.2)$$

where $\nu_{ei}(v) = 4\pi Z n_0 e^4 \Lambda/m_e^2 v^3$ is the velocity-dependent frequency of ei collisions, and Λ is the Coulomb logarithm. The explicit form of the ee collision operator (the Landau collision integral) is given in Appendix A.

We linearize Eq. (2.1) with respect to a small perturbation $\delta f_e(\mathbf{r},t,\mathbf{v}) = f_e - F_0(v)$ of the electron distribution function. Then, forming the spatial Fourier transform, we expand δf_e in the spherical harmonics $Y_{lm}(\theta,\phi)$, which are the eigenfunctions of the ei collision integral:

$$\delta f_e = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}(k,t,v)Y_{lm}(\theta,\phi), \qquad (2.3)$$

where θ and ϕ are the polar and azimuthal angles characterizing the direction of the electron velocity relative to the vector $\mathbf{k}$. The indicated operations reduce the electron kinetic equation to an infinite system of equations in the angular harmonics f_{lm} of the electron distribution function. By virtue of the assumption that the perturbation is quasistationary, the time derivatives in the equations for harmonics with $l \geqslant 1$ can be ignored, along with ee collisions, whose frequency is $1/Z$ times that of ee collisions. In general, however, the time derivative and the ee collision integral must be retained in the equation for the symmetric part $f_{00}(v)$ of the electron distribution function, ei collisions, which are Z times more frequent, affect only the anisotropic part of the distribution function and are not represented in the equation for f_{00}. In their final form the equations for the harmonics of the electron distribution function must also take into account the fact that the ei collision integral (2.2) is written in a reference frame where ions are at rest. The average ion velocity therefore enters into the equations for f_{lm}, which then acquire the form

$$\frac{\partial f_{00}}{\partial t} + ikv\,\frac{1}{\sqrt{3}}f_{10} - i(\mathbf{k} \cdot \mathbf{u})v\,\frac{\sqrt{4\pi}}{3}\frac{\partial F_0}{\partial v}$$

$$= C_{ee}[f_{00}] \quad (l=0), \tag{2.4}$$

$$ikv\,\frac{1}{\sqrt{3}}f_{00} + ikv\,\frac{2}{\sqrt{15}}f_{20} - \frac{e}{km_e}(\mathbf{k}\cdot\mathbf{E})\sqrt{\frac{4\pi}{3}}\frac{\partial F_0}{\partial v}$$

$$= -\nu_{ei}f_{10} \quad (l=1,m=0),$$

$$ikv\,\sqrt{\frac{3}{15}}f_{2\pm1} + \frac{e}{m_e}E\sqrt{\frac{2\pi}{3}}\sin\theta_E\,\exp(\mp i\phi_{\mathbf{E}})\frac{\partial F_0}{\partial v}$$

$$= -\nu_{ei}f_{1\pm1} \quad (l=1,m=\pm1), \tag{2.5}$$

$$ikv\,\frac{2}{\sqrt{15}}f_{10} + ikv\,\frac{3}{\sqrt{35}}f_{10} - i(\mathbf{k}\cdot\mathbf{u})v\,\sqrt{\frac{16\pi}{45}}\frac{\partial F_0}{\partial v}$$

$$= -3\nu_{ei}f_{20} \quad (l=2,m=0),$$

$$ikv\,\sqrt{\frac{3}{15}}f_{1\pm1} + ikv\,\sqrt{\frac{8}{35}}f_{3\pm1} + ikuv\,\sqrt{\frac{2\pi}{15}}\sin\theta_u$$

$$\times\exp(\mp i\phi_u)\frac{\partial F_0}{\partial v} = -3\nu_{ei}f_{2\pm1}, \quad (l=2,m=\pm1),$$

$$\tag{2.6}$$

$$ikv\,\frac{1}{\sqrt{7}}f_{3\pm2} = -3\nu_{ei}f_{2\pm2} \quad (l=2,m=\pm2),$$

$$ikv\,\sqrt{\frac{l^2-m^2}{4l^2-1}}f_{l-1,m} + ikv\,\sqrt{\frac{(l+1)^2-m^2}{4(l+1)^2-1}}f_{l+1,m}$$

$$= -\frac{l(l+1)}{2}\nu_{ei}f_{lm} \quad (l>2,|m|\leqslant l). \tag{2.7}$$

Here $\mathbf{u}$ is the Fourier harmonic of the hydrodynamic flow velocity of the plasma (average ion velocity), $\mathbf{E}$ is the Fourier harmonic of the electric field vector (the renormalization of the electric field in the transition to the comoving frame is
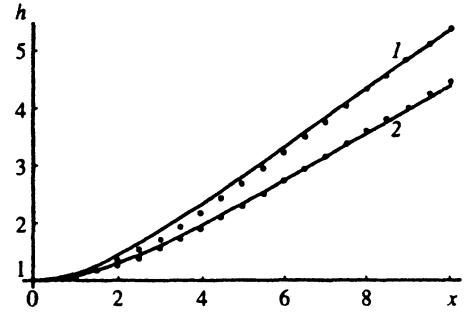

FIG. 1. Comparison of the approximate expressions (2.9) for the modified electron–ion collision frequencies (solid curves) with the exact expression (dots). $1)$ h_{10}; $2)$ h_{11}.

a second-order effect and is disregarded in the linear theory), and $\theta_E(\theta_u)$ and $\phi_E(\phi_u)$ denote the polar and azimuth angles of the vector $\mathbf{E}$ ($\mathbf{u}$) relative to the perturbation wave vector $\mathbf{k}$. The explicit form of the linearized ee collision operator C_{ee} in Eq. (4) is given in Appendix A. We note that similar equations have been derived previously[15,17–19] for potential perturbations $(m=0)$.

2.2. Summation of the angular harmonics of the distribution function

The standard approach to the solution of the infinite system of equations (2.4)–(2.7) is to assume that the higher angular harmonics are small and that quite reasonable accuracy can be attained if only two of them, f_{00} and f_{10}, are retained. This is fully justified in the strongly collisional limit, but to describe the plasma electrons in the collisionless and weakly collisional domains, we need to include a large number of angular harmonics of the distribution function δf_e. This follows qualitatively from the mere fact that the correct description of the Landau damping of low-frequency waves $(\omega \ll kv_{Te})$ requires summation of the entire infinite series of angular harmonics. The procedure used to sum the angular harmonics is similar to the one described in Refs. 16, 18–20. The principal idea underlying such summation is to solve the equations for higher harmonics by introducing a modified collision frequency ν_{lm}, which obeys the recursion relation

$$\nu_{lm}=\frac{1}{2}l(l+1)\nu_{ei}+\frac{(l+1)^2-m^2}{4(l+1)^2-1}\frac{k^2v^2}{\nu_{l+1,m}}. \tag{2.8}$$

To close the hydrodynamic equations, it is necessary to compute only two modified frequencies $\nu_{10}(k,v)=\nu_{ei}h_{10}(kv/\nu_{ei})$ and $\nu_{11}(k,v)=\nu_{ei}h_{11}(kv/\nu_{ei})$, because all the other required function are expressed in terms of these two. In Appendix B it is shown that the functions h_{10} and h_{11} have simple asymptotic limits and can be accurately approximated by the expressions

$$h_{10}(x)=\sqrt{1+(\pi x/6)^2}, \quad h_{11}(x)=\sqrt{1+(4x/3\pi)^2}, \tag{2.9}$$

as illustrated in Fig. 1. We note that an approximation for h_{10} has been proposed in Ref. 20 and later used in Ref. 19 to calculate the damping of ion-acoustic waves. A similar pro-

cedure for summing the contributions of higher angular harmonics has been discussed recently in another paper.[21]

This summation procedure can be used to find expressions for the second angular harmonics f_{2m} from Eqs. (2.6). Their substitution into Eqs. (2.5) yields expressions for an anisotropic increment to the electron distribution function $f_1 = \sum_{m=-1}^{1} f_{1m} Y_{1m}$:

$$f_1 = f_{10} \cos\theta \sqrt{\frac{3}{4\pi}} + \sqrt{\frac{3}{8\pi}} \sin\theta \sum_m f_{1m} e^{im\phi}$$

$$= -\frac{i(\mathbf{k}\cdot\mathbf{v})}{\nu_{10}} f_0 + \frac{e}{m} \frac{(\mathbf{E}_\parallel\cdot\mathbf{v})}{\nu_{10}} \frac{1}{v} \frac{\partial F_0}{\partial v} + \frac{\nu_{10} - \nu_{ei}}{\nu_{10}}$$

$$\times (\mathbf{u}_\parallel\cdot\mathbf{v}) \frac{1}{v} \frac{\partial F_0}{\partial v} - \frac{e}{m} \frac{(\mathbf{E}_\perp\cdot\mathbf{v})}{\nu_{11}} \frac{1}{v} \frac{\partial F_0}{\partial v} - \frac{\nu_{11} - \nu_{ei}}{\nu_{11}}$$

$$\times (\mathbf{u}_\perp\cdot\mathbf{v}) \frac{1}{v} \frac{\partial F_0}{\partial v}, \qquad (2.10)$$

where the following notation has been introduced for the longitudinal and transverse components of the vector $\mathbf{A}$:

$$\mathbf{A}_\parallel = \frac{\mathbf{k}(\mathbf{A}\cdot\mathbf{k})}{k^2}, \qquad \mathbf{A}_\perp = \frac{\mathbf{k}\times(\mathbf{A}\times\mathbf{K})}{k^2}.$$

There now follows from Eq. (2.4) an equation for the symmetric part of the perturbation of the electron distribution function $f_0 \equiv f_{00} Y_{00} = f_{00} / \sqrt{4\pi}$:

$$\frac{\partial f_0}{\partial t} + \frac{k^2 v^2}{3\nu_{10}} f_0 - i \frac{(\mathbf{k}\cdot\mathbf{u})}{3} v \frac{\nu_{ei}}{\nu_{10}} \frac{\partial F_0}{\partial v}$$

$$+ i \frac{(\mathbf{k}\cdot\mathbf{E})}{3} \frac{ev}{m_e \nu_{10}} \frac{\partial F_0}{\partial v} = C_{ee}[f_0]. \qquad (2.11)$$

The anisotropic increment to the distribution function contains both potential and nonpotential components of the perturbation. The nonpotential components create solenoidal components in the electron distribution function f_1, which differ from the potential part in not requiring knowledge of the isotropic component f_0, and an explicit expression for them is obtained at once from Eq. (2.10), since the contribution of the ei collisions is taken into account by the modified collision frequency ν_{11}. To find the symmetric part of the function we must also include the contribution of the ee collisions, which is described by the integral term on the right-hand side of Eq. (2.11).

2.3. Initial-value problem for the electron distribution function

The perturbation of the isotropic part of the electron distribution function in Eq. (2.11) is created by driving forces proportional to the potential components of $\mathbf{E}$ and $\mathbf{u}$, and it does not depend explicitly on the other two hydrodynamic variables: the perturbations of the density δn and the temperature δT. At first glance, therefore, this equation cannot be used to analyze the hydrodynamic moments δn and δT of the electron distribution function as variables independent of $\mathbf{E}$ and $\mathbf{u}$ as is required in the general formulation of electron transport theory.

The Chapman–Enskog method easily surmounts this problem, because the left-hand side of Eq. (2.4) is regarded as a small perturbation. Accordingly, it follows from the equation $C_{ee}[f_0] = 0$ that f_0 contains its first two hydrodynamic moments as arbitrary constants by virtue of conservation of the number of particles and their energy in ee collisions:

$$f_0^{(CE)}(v,t) = \left[\frac{\delta n(t)}{n_0} + \frac{\delta T(t)}{T_0}\left(\frac{v^2}{2v_{Te}^2} - \frac{3}{2} \right) \right] F_0(v), \qquad (2.12)$$

where $v_{Te} = \sqrt{T_0/m_e}$ is the thermal electron velocity. Accordingly, all the higher moments can be expressed in terms of δn and δT according to Eqs. (2.5), as is necessary in order to close the hydrodynamic equations. However, the generalization of the hydrodynamic equations to the case of weak collisions rests on the assumption that the term proportional to $ikv f_{10}$ on the left-hand side of Eq. (2.4), which is the term describing spatial transfer, can be comparable in value with, or even exceed, the ee collision integral. Consequently, the assumption that δn and δT are independent of the higher moments of the distribution function is no longer valid, and a different procedure must be found for closing the hydrodynamic equations.

Generally speaking, this problem evinces the need for a kinetic description of the motions of the plasma in the weakly collisional domain. However, the abbreviated, hydrodynamically similar description of the plasma dynamics can still be preserved, even for weakly colliding electrons, if we assume that the distribution function of the latter can be regarded as a small perturbation of the Maxwellian equilibrium distribution function. Although this assumption severely restricts the class of possible motions, it looks reasonable for many practical applications.

Following Ref. 15, we adopt the following procedure for introducing the hydrodynamic moments δn and δT as independent variables in the solution of Eq. (2.11). We assume that the initial perturbation of the electron distribution function at some distant time $t = 0$ is a Maxwellian distribution similar to (2.12) with certain initial density and temperature perturbations $\delta n(0)$ and $\delta T(0)$. The solution of Eq. (2.11) then determines the electron distribution function at any time t, which depends on four quantities: $\mathbf{E}$, $\mathbf{u}$, $\delta n(0)$, and $\delta T(0)$. The hydrodynamic moments of the perturbation of the distribution function at a certain arbitrary time t,

$$\delta n(t) = 4\pi \int_0^\infty dv\, v^2 f_0,$$

$$\delta T(t) = \frac{4\pi m_e}{3n_0} \int_0^\infty dv\, v^2 (v^2 - 3v_{Te}^2) f_0, \qquad (2.13)$$

can be represented by linear combinations of $\delta n(0)$ and $\delta T(0)$. Consequently, the initial perturbations can be eliminated by expressing them in terms of the instantaneous perturbations and solving the system of two linear algebraic equations. The electronic distribution function can therefore be expressed in terms of its instantaneous hydrodynamic moments $\delta n(t)$ and $\delta T(t)$ as independent variables.

Pursuing the solution of the kinetic equation along this line, we form the temporal Laplace transform in (2.11). The equation for the symmetric part of the electron distribution function assumes the form

$$\left(\frac{k^2 v^2}{3\nu_{10}} + p\right)\left(f_0 + i\frac{e(\mathbf{k}\cdot\mathbf{E})}{m_e k^2 v}\frac{\partial F_0}{\partial v}\right)$$

$$= ip\frac{e(\mathbf{k}\cdot\mathbf{E})}{m_e k^2 v}\frac{\partial F_0}{\partial v} + i(\mathbf{k}\cdot\mathbf{u})\frac{\nu_{ei}}{3\nu_{10}}v\frac{\partial F_0}{\partial v}$$

$$+ C_{ee}[f_0] + f_0(v,0). \tag{2.14}$$

In keeping with our assumption, the initial perturbation $f_0(v,0)$ has the hydrodynamic form (2.12), i.e., $f_0(v,0) = f_0^{(CE)}(v,0)$. Equation (2.14) is a linear inhomogeneous equation, whose general solution can be written as a linear combination of three basic solutions:

$$f_0 = i\frac{e(\mathbf{k}\cdot\mathbf{E})}{k^2 T_0}F_0 + \left(\frac{\delta n(0)}{n_0} - ip\frac{e(\mathbf{k}\cdot\mathbf{E})}{k^2 T_0}\right)\psi^N F_0$$

$$+ \frac{3}{2}\frac{\delta T(0)}{T_0}\psi^T F_0 - i(\mathbf{k}\cdot\mathbf{u})\psi^R F_0, \tag{2.15}$$

where the basis functions $\psi^A(v)$ are independent of the form and the amplitude of the perturbation and satisfy three $(A=N,T,R)$ equations with different sources S_A:

$$\left(\frac{k^2 v^2}{3\nu_{10}} + p\right)\psi^A = F_0^{-1}C_{ee}[F_0\psi^A] + S_A. \tag{2.16}$$

Here $S_N = 1$, $S_T = v^2/3v_{Te}^2 - 1$, and $S_R = v^2/(3v_{Te}^2 h_{10})$ are unitary sources corresponding to the perturbations of the number density (N), the temperature (T), and the ion velocity (R). The equation of the form (2.16) has been analyzed in detail.[19] It was solved numerically by expanding the solution in Sonin–Laguerre polynomials $L_n^{1/2}(v^2/2v_{Te}^2)$ and analytically in the strong and weakly collisional limits. In fact, all the results needed in order to close the hydrodynamic equations can be expressed in terms of the matrix of moments of the functions ψ^A:

$$J_B^A = \frac{4\pi}{n_0}\int_0^\infty v^2 dv\,\psi^A F_0 S_B. \tag{2.17}$$

This matrix J_B^A is symmetric, as shown in Appendix C.

Taking the first two moments of Eq. (2.15), we find the instantaneous density and temperature perturbations. According to (2.13), they depend on $\mathbf{E}$, $\mathbf{u}$ and also on the initial perturbations:

$$\frac{\delta n(t)}{n_0} = \frac{\delta n(0)}{n_0}J_N^N + \frac{3}{2}\frac{\delta T(0)}{T_0}J_N^T + i\frac{e(\mathbf{k}\cdot\mathbf{E})}{k^2 T_0}(1 - pJ_N^N)$$

$$- i(\mathbf{k}\cdot\mathbf{u})J_N^R, $$

$$\frac{\delta T(t)}{T_0} = \frac{\delta n(0)}{n_0}J_T^N + \frac{3}{2}\frac{\delta T(0)}{T_0}J_T^T - ip\frac{e(\mathbf{k}\cdot\mathbf{E})}{k^2 T_0}J_T^N$$

$$- i(\mathbf{k}\cdot\mathbf{u})J_T^R. \tag{2.18}$$

Solving this pair of linear algebraic equations for $\delta n(0)$ and $\delta T(0)$ and substituting the result into Eq. (2.15), we obtain an expression for the isotropic part of the electron distribution function:

$$f_0 = i\frac{e(\mathbf{k}\cdot\mathbf{E})}{k^2 T_0}F_0 + \left(\frac{\delta n}{n_0} - i\frac{e(\mathbf{k}\cdot\mathbf{E})}{k^2 T_0}\right)\frac{J_T^T\psi^N - J_T^N\psi^T}{D_{NT}^{NT}}F_0$$

$$+ \frac{\delta T}{T_0}\frac{J_N^N\psi^T - J_N^T\psi^N}{D_{NT}^{NT}}F_0 - i(\mathbf{k}\cdot\mathbf{u})$$

$$\times\left(\psi^R - \frac{D_{NT}^{RT}}{D_{NT}^{NT}}\psi^N - \frac{D_{NT}^{NN}}{D_{NT}^{NT}}\psi^T\right)F_0, \tag{2.19}$$

where $D_{AB}^{CD} = J_A^C J_B^D - J_T^D J_B^C$ Equation (2.19) is written in terms of the hydrodynamic moments and basic solutions of Eq. (2.16) and can be used to search for the anisotropic increment to the distribution function f_1 and to close the system of hydrodynamic equations.

2.4. Isotropic part of the distribution function

2.4.1. General solution

To determine the isotropic part of the electron distribution function, we need to solve Eqs. (2.16). We seek solutions by expanding the basis functions ψ^A in Sonin–Laguerre polynomials:

$$\psi^A = Z\frac{\lambda_{ei}}{v_{Te}}\sum_{n=0}^\infty c_n^A L_n^{1/2}\left(\frac{v^2}{2v_{Te}^2}\right). \tag{2.20}$$

Here $\lambda_{ei} = 3\sqrt{\pi/2}v_{Te}/\nu_{ei}(v_{Te})$ is the electron mean free path.[1] Substituting the expansion (2.20) into the basic equations (2.16) and introducing the dimensionless variable $x = v^2/2v_{Te}^2$ we obtain a system of linear equations for the coefficients c_n^A. This system is solved with the help of the *Mathematica* software package.[22] For $k\lambda_{ei}\lesssim 1$ a sufficiently accurate solution of the system can be obtained using $N\lesssim 10$ polynomials. In the case $k\lambda_{ei} > 1$ from 30 to 60 polynomials are needed to obtain a solution within $\sim 1\%$ error limits. Note that an increase in the collisionality parameter shifts the perturbation of the electron distribution function toward lower velocities, and this, in turn, causes the kinetic coefficients to diminish.

2.4.2. Hydrodynamic limit for the distribution function

In the strongly collisional domain $Zk^2\lambda_{ei}^2 \ll 1$, keeping the first two terms in the expansion of ψ^A, we obtain the classical hydrodynamic expressions for the isotropic and anisotropic increments to the electron distribution function and, accordingly, for the transport coefficients. Let us write ψ^A in the form

$$\psi^A(x) = Z\frac{\lambda_{ei}}{v_{Te}}[c_0^A L_0^{1/2}(x) + c_1^A L_1^{1/2}(x)], \tag{2.21}$$

and then evaluate the coefficients c_0^A and c_1^A, invoking the properties of the collision integral

$$\int_0^\infty C_{ee}v^2 dv = 0, \quad \int_0^\infty C_{ee}v^4 dv = 0,$$

which correspond to the laws of conservation of particle number and energy

$$c_0^N = \frac{123\pi}{512}\frac{1}{Zk^2\lambda_{ei}^2}, \quad c_1^N = \frac{15\pi}{256}\frac{1}{Zk^2\lambda_{ei}^2} = -c_0^T,$$

$$c_1^T = -\frac{3\pi}{128}\frac{1}{Zk^2\lambda_{ei}^2},$$

$$c_0^R = \frac{93\pi}{512}\frac{1}{Zk^2\lambda_{ei}^2}, \quad c_1^R = \frac{9\pi}{256}\frac{1}{Zk^2\lambda_{ei}^2}. \tag{2.22}$$

To describe the deviation from the hydrodynamic limit, we need to include the next two terms in the expansion of ψ^A. However, the inclusion of only three polynomials does not yield an adequate approximation, even in the long-wavelength limit; we need at least seven terms in the expansion in Sonin–Laguerre polynomials.[19] We can then obtain more accurate expressions for the first two coefficients of the expansion and use them to find all the kinetic coefficients:

$$c_0^N = \frac{123\pi}{512}\frac{1}{Z(k\lambda_{ei})^2}(1 + 104Zk^2\lambda_{ei}^2),$$

$$c_1^N = \frac{15\pi}{256}\frac{1}{Z(k\lambda_{ei})^2}(1 + 211Zk^2\lambda_{ei}^2),$$

$$c_0^T = -c_1^N, \quad c_1^T = -\frac{3\pi}{128}\frac{1}{Z(k\lambda_{ei})^2}(1 + 264Zk^2\lambda_{ei}^2),$$

$$c_0^R = \frac{93\pi}{512}\frac{1}{Z(k\lambda_{ei})^2}(1 + 69Zk^2\lambda_{ei}^2),$$

$$c_1^R = \frac{9\pi}{256}\frac{1}{Z(k\lambda_{ei})^2}(1 + 176Zk^2\lambda_{ei}^2). \tag{2.23}$$

The factor 264 in the expression for c_1^T coincides with the analytical solution of the kinetic equation for the electron distribution function in the given limit, obtained in Refs. 6 and 17 without expanding in spherical harmonics. We note that in all the expressions (2.23) the parameter of the expansion $Zk^2\lambda_{ei}^2$ is preceded by large numerical factors, hence the departure from the hydrodynamic limit already begins at very small wave numbers $k\lambda_{ei} \ll Z^{-1/2} \ll 1$.

2.4.3. The distribution function in the kinetic limit

In the opposite short-wavelength limit $Zk^2\lambda_{ei}^2 \gg 1$ we can disregard electron–electron collisions and, on the basis of Eq. (2.16), write the asymptotic expression

$$\psi_0^A \simeq \frac{S_A}{k^2v^2/3\nu_{10} + p}. \tag{2.24}$$

In the limit $p \to 0$ the solution (2.24) diverges in the low-velocity range as v^{-5} for the functions ψ_0^N and ψ_0^T and as v^{-3} for the function ψ_0^R. Allowance for time dependence eliminates the divergence, but the contribution of electron–electron collisions in the low-velocity range can also be significant.

For $v \ll v_{Te}$ the collision integral in Eq. (2.16) can be written in the form

$$\frac{1}{F_0}C_{ee}[F_0\psi_0^A] = \frac{1}{3Z}\sqrt{\frac{2}{\pi}}\nu_{ei}(v)\frac{v}{v_{Te}}\frac{d}{dv}\left(v^2\frac{d\psi_0^A}{dv}\right). \tag{2.25}$$

Substituting $\psi_0^N \propto v^{-5}$ into this expression, we find that the collision integral becomes commensurate with the diffusion term on the left-hand side of (2.16) in the velocity range where

$$\left(\frac{v}{v_{Te}}\right)^7 Zk^2\lambda_{ei}^2 \lesssim 1. \tag{2.26}$$

Comparing the contributions of nonstationarity and electron–electron collisions in this velocity range, we confirm that nonstationary effects can be significant for sufficiently fast processes having a characteristic time constant

$$\tau \sim |p|^{-1} < \frac{\lambda_{ei}}{v_{Te}}\frac{Z^{5/7}}{(k\lambda_{ei})^{4/7}}.$$

For the parameters of a laser plasma this represents extremely short times of the order of a few picoseconds at $k\lambda_{ei} \sim 1$.

In the case of slower processes, electron–electron collisions are dominant in the low-velocity range, even for $k\lambda_{ei} \gg 1$. To construct an approximate solution of Eq. (2.16) with the collision integral in the form (2.25), it is convenient to use the dimensionless variable $w = v/v_*$ and the function $Y^A = \psi_0^A/\psi_*^A$ where

$$v_* = v_{Te}\left(\frac{9\sqrt{\pi/2}}{Zk^2\lambda_{ei}^2}\right)^{1/7}, \quad \psi_*^{N(T)} = \frac{Z^{5/7}}{kv_{Te}}\left(9\sqrt{\frac{\pi}{2}}\right)^{2/7}(k\lambda_{ei})^{3/7},$$

$$\psi_*^R = \frac{Z^{3/7}}{3kv_{Te}}\left(9\sqrt{\frac{\pi}{2}}\right)^{4/7}\left(\frac{1}{k\lambda_{ei}}\right)^{1/7}.$$

Equation (2.16) then acquires the form

$$\frac{d^2Y}{dw^2} + \frac{2}{w}\frac{dY}{dw} + w^5Y = 1. \tag{2.27}$$

where $Y^R = Y^T = Y$ and $Y^N = -Y$. The solution of this equation can be expressed in terms of modified Bessel functions of order $1/7$ (Ref. 9). However, it is more practical to find a numerical solution, which can be approximated by the simple functional relation[19]

$$\psi^T(v) = -\psi^N(v) \approx Zk\lambda_{ei}\left(\frac{v_*}{v_{Te}}\right)^2\frac{c_\psi}{1 + c_\psi(v/v_*)^5}. \tag{2.28}$$

where the coefficient $c_\psi = 0.432$ was found numerically. The approximation (2.28) agrees with the exact solution of Eq. (2.16) within 20%. This rather large error is attributable to the fact that the parameter of the expansion $v_*/v_{Te} \approx (Zk^2\lambda_{ei}^2)^{-1/7}$, though assumed to be small, is in fact appreciable in value as a result of the small negative power exponent.[19]

The relations (2.23) and (2.28) obtained above in Secs. 2.4.2 and 2.4.3 to describe the electron distribution function will be used below to derive analytical expressions for the transport coefficients in the long-wavelength and short-wavelength limits.

3. NONLOCAL PLASMA HYDRODYNAMICS

3.1. Nonlocal transport equations

The first two moments of Eq. (2.4) yield evolution equations of continuity and energy balance for electrons:

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0,$$

$$\frac{\partial T_e}{\partial t} + \frac{2}{3n_e}\nabla \cdot \mathbf{q} + \mathbf{u}_e \cdot \nabla T_e + \frac{2}{3}T_e \nabla \cdot \mathbf{u}_e = 0, \tag{3.1}$$

where $\mathbf{u}_e = \mathbf{u} - \mathbf{j}/e n_e$ is the electron drift velocity, and

$$\mathbf{j} = -e \int d^3 v \, \mathbf{v} f_e, \quad \mathbf{q} = T_e \int d^3 v \, \mathbf{v} \left(\frac{5}{2} - \frac{v^2}{v_{Te}^2} \right) f_e \tag{3.2}$$

are the electric current and electronic heat flux in the plasma.

The derivation of expressions for the fluxes $\mathbf{j}$ and $\mathbf{q}$ in terms of the hydrodynamic variables, i.e., the closure of the hydrodynamic equations, poses the basic problem of transport theory. The problem is solved in our approach, because Eqs. (2.19) and (2.10) give expressions for f_0 and f_1 in terms of the hydrodynamic variables. Substituting f_1 (2.10) into relations (3.2), we express the electric current $\mathbf{j}$ and the heat flux $\mathbf{q}$ in terms of generalized thermodynamic forces—the Fourier component of the effective electric field:

$$\mathbf{E}^* = \mathbf{E} + i\mathbf{k}\frac{T_0}{e}\left(\frac{\delta n}{n_0} + \frac{\delta T}{T_0} \right)$$

and the temperature gradient $i\mathbf{k}\delta T$. These electron fluxes also contain terms proportional to the gradient of the average ion velocity $\mathbf{u}$.

Since f_1 contains terms corresponding to vectors $\mathbf{E}$ and $\mathbf{u}$ directed both along and across the vector $\mathbf{k}$, the fluxes contain both transverse and longitudinal components. The longitudinal components, directed along $\mathbf{k}$, coincide with those obtained previously[15] for potential perturbations:

$$\mathbf{j}_\| = \sigma \mathbf{E}_\|^* + \alpha i\mathbf{k}\delta T + \beta_j e n_0 \mathbf{u}_\|,$$

$$\mathbf{q}_\| = -\alpha T_0 \mathbf{E}_\|^* - \chi i\mathbf{k}\delta T - \beta_q n_0 T_0 \mathbf{u}_\|, \tag{3.3}$$

where σ is the electrical conductivity, α is the thermocurrent coefficient, χ is the thermal diffusivity, and $\beta_{j,q}$ are the ionic convection transport coefficients. We note that these coefficients are functions of k and p and are therefore nonlocal and nonstationary:

$$\sigma = \frac{e^2 n_0}{k^2 T_0}\left(\frac{J_T^T}{D_{NT}^{NT}} - p \right), \quad \alpha = -\frac{e n_0}{k^2 T_0}\left(\frac{J_T^N + J_T^T}{D_{NT}^{NT}} - p \right),$$

$$\beta_j = 1 - \frac{D_{NT}^{RT}}{D_{NT}^{NT}}, \quad \beta_q = \frac{D_{NT}^{RT} + D_{NT}^{RN}}{D_{NT}^{NT}}, \tag{3.4}$$

$$\chi = \frac{n_0}{k^2}\left(\frac{2J_T^N + J_T^T + J_N^N}{D_{NT}^{NT}} - \frac{5}{2}p \right).$$

The transport equations (3.3) reflect the Onsager symmetry properties: The coefficient α is identical in the expressions for $\mathbf{j}$ and $\mathbf{q}$. It can be shown (see Appendix C) that this symmetry corresponds to the relations $J_N^T = J_T^N$, $J_T^R = J_R^T$, and $J_R^N = J_N^R$ in the general case of an arbitrary collisionality pa-

rameter. We therefore conclude that the Onsager symmetry is a general property of linear kinetic systems and holds not only for collisional hydrodynamics,[1,2] but also in the nonlocal, nonstationary case.

We now write expressions for the electron fluxes transverse to the vector $\mathbf{k}$:

$$\mathbf{j}_\perp = \sigma_\perp \mathbf{E}_\perp + e n_0 \beta_{\perp j} \mathbf{u}_\perp, \tag{3.5}$$

$$\mathbf{q}_\perp = -\alpha_\perp T_0 \mathbf{E}_\perp - \beta_{\perp q} n_0 T_0 \mathbf{u}_\perp. \tag{3.6}$$

Here we have introduced the transverse transport coefficients for the current and heat flux vectors:

$$\sigma_\perp = \frac{4\pi e^2}{3 T_0}\int dv \, v^4 \frac{F_0}{\nu_{11}}, \tag{3.7}$$

$$\beta_{\perp j} = \frac{4\pi}{3 n_0}\frac{1}{v_{Te}^2}\int dv \, v^4 \frac{\nu_{11} - \nu_{ei}}{\nu_{11}} F_0,$$

$$\alpha_\perp = \frac{4\pi e}{3 T_0}\int dv \left(\frac{v^2}{2v_{Te}^2} - \frac{5}{2} \right) v^4 \frac{F_0}{\nu_{11}},$$

$$\beta_{\perp q} = \frac{4\pi}{3 n_0}\frac{1}{v_{Te}^2}\int dv \left(\frac{v^2}{2v_{Te}^2} - \frac{5}{2} \right) v^4 \frac{\nu_{11} - \nu_{ei}}{\nu_{11}} F_0. \tag{3.8}$$

In contrast with the expressions for potential flows, the expressions for the transverse transport coefficients do not depend on electron–electron collisions and do not require solving the equation for the isotropic part of the distribution function, and electron collisions are taken into account automatically through the introduction of the modified collision frequency ν_{11}.

Also used in hydrodynamics are the electron stress tensor Π_{lj} and the friction force $\mathbf{R}_{ie}$ between ions and electrons:

$$\Pi_{lj} = \int d^3 v \, m_e (v_l v_j - \delta_{lj} v^2/3) f_e,$$

$$\mathbf{R}_{ie} = \int d^3 v \, m_e \mathbf{v} \nu_{ei} f_e. \tag{3.9}$$

The electron stress tensor is related to the second angular harmonic of the electron distribution function. Equation (2.5) is convenient for determining it if Π_{lj} is expressed in terms of the friction force:

$$\Pi_{lj} = \frac{3i}{4k^2}\left(k_l \hat{R}_j + k_j \hat{R}_l - \frac{2}{3}\delta_{lj}\mathbf{k}\cdot\hat{\mathbf{R}} \right),$$

$$\hat{\mathbf{R}} = \mathbf{R}_{ie} = e n_0 \mathbf{E}^* - 2n_0 i\mathbf{k}\delta T. \tag{3.10}$$

The longitudinal and transverse components of the friction force can be written as functions of the generalized forces and ion velocity, i.e.,

$$\mathbf{R}_{ie}^\| = -(1 - \beta_j) n_0 e \mathbf{E}_\|^* + \beta_q n_0 i\mathbf{k}\delta T - \beta_r m_e n_0 \mathbf{u}_\| \frac{v_{Te}}{\lambda_{ei}}, \tag{3.11}$$

$$\beta_r = 1 + k^2 v_{Te}\lambda_{ei}[J_R^R - (1 - \beta_j)(J_R^N + J_R^T) + \beta_q J_R^T]$$

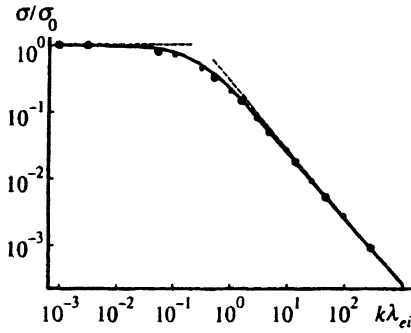$$- (2\pi)^{3/2}\frac{v_{Te}}{n_0}\int_0^\infty \frac{dv \, v F_0}{h_{10}}, \tag{3.12}$$

FIG. 2. Longitudinal electrical conductivity σ vs perturbation wavelength for a plasma with $Z=8$ (small dots) and $Z=64$ (large dots). The solid curve corresponds to the approximation (3.17), and the dashed lines correspond to the classical strongly collisional asymptotic behavior and the collisionless limit.
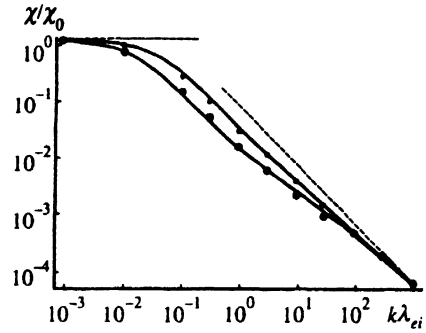

FIG. 3. Thermal diffusivity χ vs perturbation wavelength for a plasma with $Z=8$ (small dots) and $Z=64$ (large dots). The solid curves correspond to the approximation (3.18), and the dashed lines correspond to the classical strongly collisional asymptotic behavior and the collisionless limit.

$$\mathbf{R}_{ie}^{\perp}=-(1-\beta_{1j})n_0e\mathbf{E}_{\perp}-\beta_{\perp r}m_e n_0 \mathbf{u}_{\perp}\frac{v_{Te}}{\lambda_{ei}}, \qquad (3.13)$$

$$\beta_{\perp r}=\frac{4\pi}{n_0}v_{Te}\sqrt{\frac{\pi}{2}}\int dv\, v\,\frac{v_{11}-v_{ei}}{v_{11}}F_0. \qquad (3.14)$$

Here Eqs. (3.11) and (3.13) contain the two new [not found in the electron fluxes (3.3)] coefficients β_r and $\beta_{\perp r}$. This is also a consequence of the Onsager symmetry of the kinetic coefficients.

3.2. Potential components of the fluxes

We now undertake a more detailed investigation of the longitudinal transport coefficients. Here we confine the analysis to slow processes, $k^2 v_{Te}^2/v_{ei}\gg|p|$, when the value of p can be disregarded in Eqs. (2.16) and (3.4). In this approximation we substitute the solutions of Eqs. (2.16) obtained for ψ^A into Eqs. (3.4) and (3.12) and find the longitudinal transport coefficients. The results of these calculations are shown in Figs. 2–7.

The coefficients of the longitudinal electrical conductivity σ, the thermocurrent α, and thermal diffusivity χ in the strongly collisional limit $\sqrt{Z}k\lambda_{ei}\ll 1$ correspond to their classical values:

$$\sigma_0=\frac{32e^2 n_0\lambda_{ei}}{3\pi m_e v_{Te}}, \qquad \alpha_0=\frac{16en_0\lambda_{ei}}{\pi m_e v_{Te}}, \qquad \chi_0=\frac{200n_0 v_{Te}\lambda_{ei}}{3\pi}.$$

They all have similar long-wavelength asymptotic representations

$$\sigma=\sigma_0(1-19Zk^2\lambda_{ei}^2), \qquad \alpha=\alpha_0(1-107Zk^2\lambda_{ei}^2),$$

$$\chi=\chi_0(1-239\,Zk^2\lambda_{ei}^2). \qquad (3.15)$$

which have been found using the expression for the isotropic part of the distribution function in the hydrodynamic limit (2.23). We note that the thermal diffusivity deviates quite rapidly from the classical limit, because the main heat-transfer contribution is from high-energy electrons, which are not as prone to collisions. Figures 2–4 disclose significant deviations of all the transport coefficients in the intermediate range $k\lambda_{ei}\sim 1$ from their classical values.

In the short-wavelength limit ($k\lambda_{ei}\gg 1$) all the transport coefficients are inversely proportional to the wave number: $\sigma=5e^2 n_0 v_{Te}/\sqrt{8\pi kT_0}, \alpha=-en_0 v_{Te}/\sqrt{2\pi kT_0}, \chi=4n_0 v_{Te}/\sqrt{2\pi k}$. The appropriate corrections to these asymptotic expressions can be found by means of Eq. (2.28). They exhibit a fractional-power dependence on k, similar to that found in Refs. 9 and 17 for the electronic thermal diffusivity:

$$\sigma=\frac{5e^2 n_0 v_{Te}}{\sqrt{8\pi kT_0}}\frac{1+9/5\,\xi}{1+2\xi}, \qquad \alpha=-\frac{en_0 v_{Te}}{\sqrt{2\pi kT_0}}\frac{1}{1+2\xi},$$

$$\chi=\frac{4n_0 v_{Te}}{\sqrt{2\pi k}}\frac{1}{1+2\xi}. \qquad (3.16)$$

The function $\xi=1.9Z^{2/7}(k\lambda_{ei})^{-3/7}$ has been found previously[17] from the asymptotic solution of the equation for ψ^T in the range $Zk^2\lambda_{ei}^2\gg 1$.

We note that the electrical conductivity (see Fig. 2) is almost independent of the ionic charge and exhibits better agreement than the other transport coefficients with the asymptotic limits. Over the entire range of the collisionality parameter the electrical conductivity is well approximated by the equation
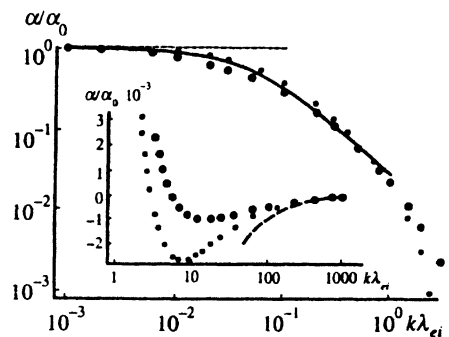

FIG. 4. Longitudinal thermocurrent coefficient α vs perturbation wavelength for a plasma with $Z=8$ (small dots) and $Z=64$ (large dots). Inset: short-wavelength range, where $\alpha(k)$ changes sign. The solid curves correspond to the approximation (3.19), and the dashed lines correspond to the classical strongly collisional asymptotic behavior and the collisionless limit.
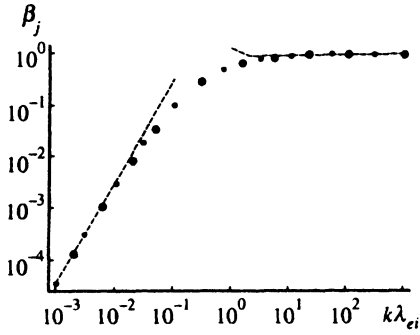
FIG. 5. Ionic convection transport coefficient β_j vs perturbation wavelength for a plasma with $Z=8$ (small dots) and $Z=64$ (large dots). The dashed traces correspond to the asymptotic representations (3.20) and (3.21).
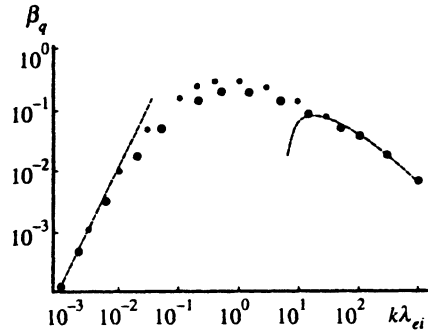

FIG. 6. Ionic convection transport coefficient β_q vs perturbation wavelength for a plasma with $Z=8$ (small dots) and $Z=64$ (large dots). The dashed traces correspond to the asymptotic representations (3.20) and (3.21).

$$\sigma = \sigma_0 \left(1 + \frac{128}{\sqrt{2}\,\pi} k\lambda_{ei}\right)^{-1}. \tag{3.17}$$

The line corresponding to this approximate expression is also shown in Fig. 2. It is evident from Fig. 2 that the conductivity begins to depart appreciably from the classical value for $k\lambda_{ei} \sim 0.4/\sqrt{Z}$.

The thermal conductivity is the most sensitive function to the ionic charge Z and the inhomogeneity scale $k\lambda_{ei}$. A departure from the classical limit sets in already at $k\lambda_{ei} \sim 0.06/\sqrt{Z}$, and the fractional asymptotic proportionality $\propto (k\lambda_{ei})^{-4/7}$ is also clearly visible in Fig. 3. We adopt the following approximation for the thermal conductivity coefficient:

$$\chi = \chi_0 \left[1 + k\lambda_{ei} \frac{100(1 + 1.6\xi)Z^{0.25}(k\lambda_{ei})^{0.5}}{3\sqrt{2\pi}(30 + Zk^2\lambda_{ei}^2)^{0.25}}\right]^{-1}, \tag{3.18}$$

which is a good approximation over the entire range of the collisionality parameter and is shown in Fig. 3.

The most unusual behavior is exhibited by the wavelength dependence of the thermocurrent coefficient. It changes sign in the intermediate range $k\lambda_{ei} \sim 1{-}10$ (see Fig. 4). In the range $k\lambda_{ei} \lesssim 1$ the coefficient α is almost independent of the ionic charge and can be characterized by the simple approximation

$$\alpha = \frac{\alpha_0}{1 + 35(k\lambda_{ei})^{1.2}}, \quad k\lambda_{ei} \lesssim 1. \tag{3.19}$$

In the range $k\lambda_{ei} > 1$ the thermocurrent coefficient α changes sign at a value of k that depends on Z. For example, α becomes equal to zero at $k\lambda_{ei} = 2.6$ and 5 for $Z=8$ and 64, respectively.

The transport coefficients in Eq. (3.4) also depend on the ion velocity $\mathbf{u}$. This dependence stems from the inclusion of higher $(l > 1)$ angular harmonics of the electron distribution function. It cannot be found by the approach proposed in Refs. 4 and 9, where only the first angular harmonic is taken into account. The behavior of the coefficients β is illustrated in Figs. 5–7. They vanish in the classical collisional limit. Their long-wavelength asymptotic $(k\lambda_{ei} \ll 1)$ representations have the form

$$\beta_j = 33k^2\lambda_{ei}^2, \quad \beta_q = 133k^2\lambda_{ei}^2, \quad \beta_r = 3.7k^2\lambda_{ei}^2 \tag{3.20}$$

and, in contrast with (3.15), do not depend explicitly on the ionic charge. In the short-wavelength limit $k\lambda_{ei} \gg 1$ the coefficients β_j and β_r tend to unity, and the coefficient β_q vanishes:

$$\beta_j = 1 - \frac{0.4\ln(k\lambda_{ei}) - 0.1}{k\lambda_{ei}}, \quad \beta_q = \frac{1.4\ln(k\lambda_{ei}) - 2.6}{k\lambda_{ei}},$$

$$\beta_r = 1 - \frac{13\ln(k\lambda_{ei}) - 41}{k\lambda_{ei}}. \tag{3.21}$$

This is consistent with the fact that the motions of electrons and ions are independent in the collisionless limit, and the ions do not contribute to the electrical current or to the thermal flux of electrons. Consequently, in the limit $k\lambda_{ei} \gg 1$ the friction force is proportional to the frequency of ei collisions and does not depend on the wavelength. We note that the coefficients β_j and β_r depend very weakly on the ionic charge. They vary by less than $(10{-}20)\%$ when Z changes from 8 to 64. The strongest dependence on Z is exhibited by the coefficient β_q, which varies by $(50{-}60)\%$ in the range $k\lambda_{ei} \lesssim 1$. The coefficients β are needed to correctly describe the electron heat flux and Landau damping of ion-acoustic waves.
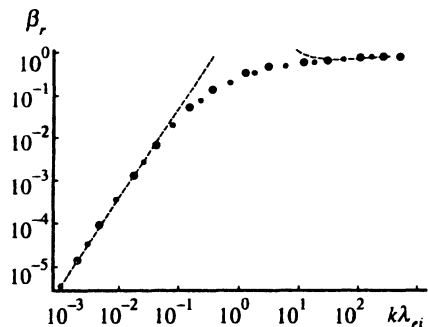

FIG. 7. Ionic convection transport coefficient β_r vs perturbation wavelength for a plasma with $Z=8$ (small dots) and $Z=64$ (large dots). The dashed lines correspond to the asymptotic representations (3.20) and (3.21).
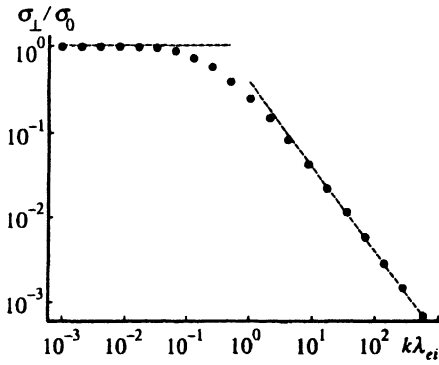
FIG. 8. Transverse electrical conductivity $\sigma_\perp$ vs wavelength. The dashed traces correspond to the classical strongly collisional asymptotic behavior and the collisionless limit.

3.3. Nonpotential components of the fluxes

In discussing the nonpotential flux components, we re-emphasize that the transverse transport coefficients do not depend on electron–electron collisions or on the isotropic correction to the distribution function. Consequently, Eqs. (3.7) and (3.8) give explicit expressions for them, which are plotted in Figs. 8–10..

When nonlocal behavior is taken into account in the general case, the transverse transport coefficients do not coincide with the longitudinal coefficients, i.e., the electron fluxes become anisotropic. The anisotropy vanishes in the classical hydrodynamic limit, when the transverse transport coefficients have the same asymptotic representations as the longitudinal.

The transverse conductivity and the transverse thermocurrent coefficient have long-wavelength asymptotic forms similar to those of the longitudinal, but their small parameter is $k^2\lambda_{ei}^2 \ll 1$, distinguishing them from the potential coefficients, for which the parameter of the expansion is $Zk^2\lambda_{ei}^2$ [cf. (3.15)]:
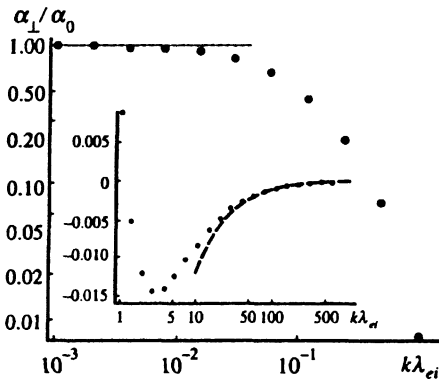

FIG. 9. Transverse thermocurrent coefficient $\alpha_\perp$ vs wavelength. Inset: short-wavelength range, where $\alpha_\perp(k)$ changes sign. The dashed traces correspond to the classical strongly collisional asymptotic behavior and the collisionless limit.
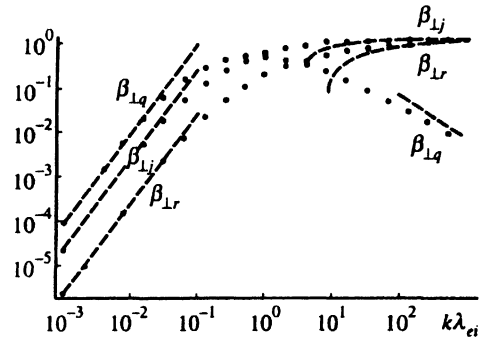

FIG. 10. Ionic convection transport coefficients $\beta_{\perp j}$, $\beta_{\perp q}$, and $\beta_{\perp r}$ vs wavelength. The dashed traces correspond to the asymptotic representations (3.24) and (3.25).

$$\sigma_\perp = \sigma_0(1 - 86k^2\lambda_{ei}^2), \quad \alpha_\perp = \alpha_0(1 - 314k^2\lambda_{ei}^2),$$
$$k^2\lambda_{ei}^2 \ll 1. \tag{3.22}$$

For the transverse conductivity in the collisionless limit we obtain the classical conductivity of a collisionless plasma. The thermocurrent coefficient behaves similarly:

$$\sigma_\perp = \sqrt{\frac{\pi}{2}}\frac{e^2 n_0}{m_e}\frac{1}{kv_{Te}}, \quad \alpha_\perp = -\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{en_0}{m_e}\frac{1}{kv_{Te}},$$
$$k^2\lambda_{ei}^2 \gg 1. \tag{3.23}$$

The ionic convection transport coefficients corresponding to transverse flows behave like the corresponding coefficients for potential perturbations (see Fig. 10). They vanish in the hydrodynamic limit [cf. (3.20)]:

$$\beta_{\perp j} = 22k^2\lambda_{ei}^2, \quad \beta_{\perp q} = 88k^2\lambda_{ei}^2, \quad \beta_{\perp r} = 2.45k^2\lambda_{ei}^2; \tag{3.24}$$

in the opposite limit $k^2\lambda_{ei}^2 \gg 1$ the coefficients $\beta_{\perp j}$ and $\beta_{\perp r}$ tend to unity, while the coefficient $\beta_{\perp q}$ vanishes:

$$\beta_{\perp j} = 1 - \frac{2.95}{k\lambda_{ei}}, \quad \beta_{\perp q} = \frac{5.9}{k\lambda_{ei}}, \quad \beta_{\perp r} = 1 - \frac{2.76}{\sqrt{k\lambda_{ei}}}. \tag{3.25}$$

Note that the transverse short-wavelength asymptotic forms (3.23) and (3.25) do not contain logarithmic corrections, because electron–electron collisions do not contribute to the transverse transport coefficients. The above equations (3.15)–(3.25) qualitatively describe electron transport in the plasma for small perturbations.

3.4. Heat transfer in a non-current-carrying plasma

One of the more important applications of the hydrodynamic equations is the case of a plasma without a longitudinal electric current ($j_\parallel = 0$), which describes quasineutral plasma motions. The generalized Ohm's law in Eq. (3.3) can be used to eliminate the ambipolar electric field

$$\mathbf{E}_\parallel^* = -\frac{\alpha}{\sigma}i\mathbf{k}\delta T_e - \frac{\beta_j}{\sigma}en_0\mathbf{u}_\parallel$$

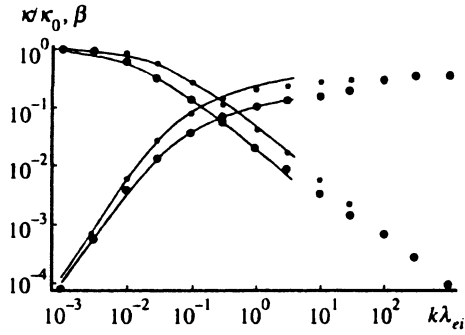from the expression for the electron heat flux by writing

FIG. 11. Thermal conductivity κ and ionic transport coefficient β for a plasma with zero longitudinal electric current and with $Z=8$ (small dots) and $Z=64$ (large dots). The solid curves correspond to the interpolation formulas (3.27).

FIG. 12. Deviation $\Delta\kappa/\kappa$ vs wavelength for a plasma with $Z=8$.

$$\mathbf{q}_\parallel = -\kappa i \mathbf{k} \delta T - \beta n_0 T_0 \mathbf{u}_\parallel, \qquad (3.26)$$

$$\kappa = \chi - \alpha^2 T_0 / \sigma, \qquad \beta = \beta_q - e\alpha\beta_j / \sigma.$$

The thermal conductivity κ and the coefficient β are shown in Fig. 11. They both depend significantly on the ionic charge. We note that for a not too strong inhomogeneity, $k\lambda_{ei} \lesssim 1$, the approximation formulas

$$\kappa = \frac{\kappa_0}{1 + (10\sqrt{Z}k\lambda_{ei})^{0.9}},$$

$$\beta = \frac{150k^2\lambda_{ei}^2}{1 + 22(1 + 12k\lambda_{ei})(\sqrt{Z}k\lambda_{ei})^{0.9}}, \qquad (3.27)$$

where $\kappa_0 = 128n_0 v_{Te}\lambda_{ei}/3\pi$ is the classical thermal conductivity, provide a good (within $\sim 20\%$ limits) description of nonlocal heat transfer.

The electron heat flux has two terms. One is associated with the temperature gradient, and the other is proportional to the ion velocity. The ion velocity contribution has not been considered explicitly in previous studies of nonlocal heat transfer, causing the thermal conductivity to be incorrectly defined, a major drawback in some cases. For example, the nonlocal electron thermal conductivity has been defined[18,19] as the ratio of the thermal flux of electrons to the temperature gradient for ion-acoustic perturbations. The incorrectness of this definition has been mentioned previously.[15] It follows from Eq. (3.26) that the definition of the thermal conductivity in Ref. 18 corresponds to $\kappa^* = \kappa + \beta n_0 T_0 u_i / ik\delta T$. The second term is insignificant in the strongly collisional limit, whereas both terms are comparable in order of magnitude in the intermediate and collisionless ranges. In Fig. 12 we show how the deviation $\Delta\kappa = \kappa^* - \kappa$ depends on the inhomogeneity scale k. The relative error attains 50% in the collisionless limit. It also follows from Eqs. (3.6) and (3.26) that the heat flux and temperature gradient vectors are noncollinear. This fact has been noted in numerical kinetic calculations under nonlocal heat-transfer conditions.[23] According to (3.5), $q_\perp$ arises simultaneously with the transverse component of the electron current, hence the onset of transverse heat transfer is accompanied by the generation of a magnetic field.
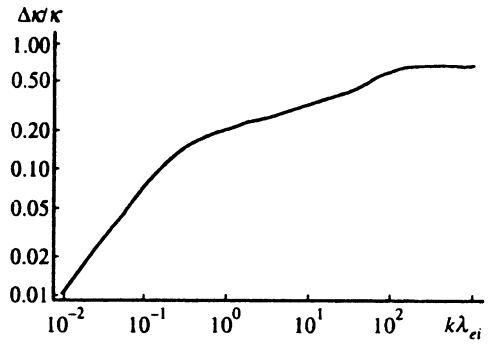
4. PLASMA DISPERSION PROPERTIES

The conservation laws (3.1) and the transport equations (3.3) and (3.4) together with the hydrodynamic equations for ions

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\mathbf{u}) = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} = Z\frac{e}{m_i}\mathbf{E} + \frac{1}{n_i m_i}\mathbf{R}_{ie}, \qquad (4.1)$$

where the friction force is defined by Eq. (3.11), can be used effectively in place of the more complicated kinetic equations. The transport coefficients are written in a form analogous to the classical collisional relations in the k-representation. They are also valid in the collisionless case. On the other hand, the conventional definitions of certain coefficients can differ in these two limits. For example, the hydrodynamic definition of the electrical resistance as the ratio of the current to the generalized electric field, when the temperature is not inhomogeneous and ions are not mobile, differs from the electrodynamic definition of the electrical resistance as the total plasma response to an applied electric field, even though the same notation is often used for both. We now look for the relationship between the hydrodynamic and electrodynamic definitions of the conductivity.

Inasmuch as our equations contain both a potential part and a nonpotential part, the total dielectric permittivity of the plasma

$$\epsilon_{ij} = \frac{k_i k_j}{k^2}\epsilon^l + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\epsilon^t \qquad (4.2)$$

is determined by the longitudinal (ϵ^l) and transverse (ϵ^t) components. Using the linearized hydrodynamic equations for electrons and ions (2.11), (3.4), and (4.1), we can express the perturbations of the ion velocity and the temperature gradient in terms of the electric field perturbations, writing the electric current in the form $j = \sigma j_e + \delta j_i$, and thus find the partial conductivities which are functionally related to the partial permittivities $\sigma_\alpha^{l,t}$ ($\alpha = e, i$) in the standard way. As a result, we obtain the following expressions for the permittivities of the plasma: in the case of cold ions and in the quasistatic limit $kv_{Ti} \ll \omega \ll k^2 v_{Te}\lambda_{ei}$

$$\epsilon^{l,t}(\omega,k) = 1 + \delta\epsilon_e^{l,t}(\omega,k) + \delta\epsilon_i^{l,t}(\omega,k), \qquad (4.3)$$

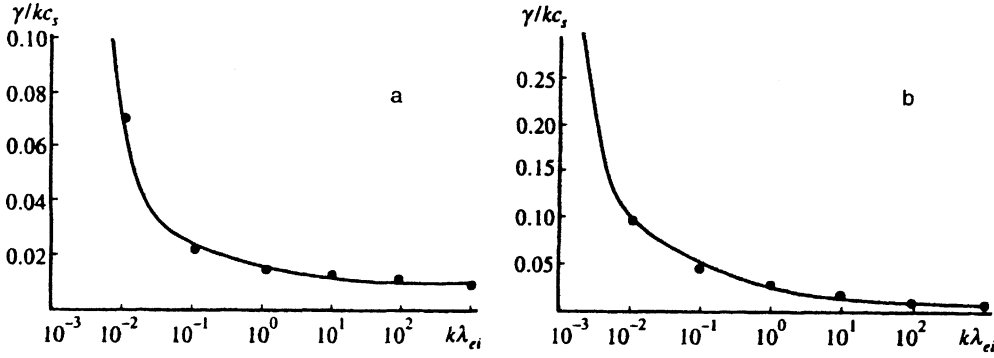where the longitudinal partial permittivities $\delta\epsilon_\alpha^l$ are given by the relations

FIG. 13. Ion-acoustic damping factor γ_s (solid curve) vs wavelength for a plasma with $Z=8$ (a) and $Z=64$ (b), compared with a numerical solution (dots) of the Fokker–Planck equation.[18]

$$\frac{1}{\delta\epsilon_e^l(\omega,k)}=k^2\lambda_{De}^2-i\frac{\omega}{4\pi\sigma}\beta_j-i\frac{\omega\lambda_{De}^2 n_0}{\kappa}$$

$$\times\left(1+\frac{e\alpha}{\sigma}\right)(1-\beta),\qquad(4.4)$$

$$\frac{1}{\delta\epsilon_i^l(\omega,k)}=-\frac{\omega^2}{\omega_{pi}^2}-i\frac{\omega\lambda_{De}^2}{v_{Te}\lambda_{ei}}\beta_r+i\frac{\omega}{4\pi\sigma}\beta_j(1-\beta_j)$$

$$+i\frac{\omega\lambda_{De}^2 n_0}{\kappa}\left(\frac{e\alpha}{\sigma}+\beta\right)(1-\beta).\qquad(4.5)$$

Here λ_{De} is the Debye radius of the electrons, and ω_{pi} is the ionic plasma frequency. Clearly, the nonlocal hydrodynamic transport coefficients contribute to the dissipative part of the dielectric susceptibility; in the collisionless limit this part corresponds to Landau damping. Ion–ion collisions are not included in relation (4.5). All the same, their contribution can be inserted additively using the results of Refs. 17 and 19.

The transverse permittivity is given by the expressions

$$\delta\epsilon_e^t=i\frac{4\pi\sigma_\perp}{\omega}+\frac{\omega_{pi}^2}{\omega^2}\frac{\beta_{\perp j}(\beta_{\perp j}-1)}{1+i\beta_{\perp r}Z(m_e/m_i)(v_{Te}/\lambda_{ei}\omega)},\qquad(4.6)$$

$$\delta\epsilon_i^t=-\frac{\omega_{pi}^2}{\omega^2}\frac{\beta_{\perp j}}{1+i\beta_{\perp r}Z(m_e/m_i)(v_{Te}/\lambda_{ei}\omega)},\qquad(4.7)$$

where $\sigma_\perp$ is the electron conductivity of the plasma (3.7), and the second term in $\delta\epsilon_e^t$ is associated with the inclusion of ion motion. It vanishes in the opposite limits of small and large values of the collisionality parameter, but in the intermediate range it is a small correction of order Zm_e/m_i. The quantity $\delta\epsilon_i^t$ decreases as k^2 in the strongly collisional limit, and in the collisionless range it leads to the usual ion dielectric permittivity $\delta\epsilon_i^t=-\omega_{pi}^2/\omega^2$.

4.1. Damping of ion-acoustic waves

As a possible application of the longitudinal dielectric permittivity (4.4), (4.5) we consider the decay rate of ion-acoustic waves for an arbitrary ratio of the ion-acoustic wavelength to the electron mean free path. The dispersion relation for an ion-acoustic mode in the quasineutral limit

$k\lambda_{De}\ll1$ is given by the approximate equation $1/\delta\epsilon_e^l+1/\delta\epsilon_i^l=0$, and its solution has the form $\omega_s=kc_s-i\gamma_s$, where

$$\frac{\gamma_s}{kc_s}=\frac{n_0 c_s}{2k}\left[\frac{(1-\beta)^2}{\kappa}+\frac{e^2\beta_j^2}{T_0\sigma}+\frac{\beta_r}{n_0 v_{Te}\lambda_{ei}}\right],\qquad(4.8)$$

and c_s is the ion-acoustic velocity. We note that all the nonlocal transport coefficients contribute to the damping factor γ_s. It corresponds exactly to the numerical solution of the Fokker–Planck equation[18] and the analytical solution in Ref. 17, but is written in a somewhat different form in terms of the transport coefficients. This form is more useful in that it gives a direct display of how the various transport coefficients influence the decay rate of ion-acoustic waves. The dependence of γ_s on k is shown in Fig. 13. The decay rate coincides with the hydrodynamic expression $\gamma_s/kc_s=3\pi c_s/256 v_{Te} k\lambda_{ei}$ in the long-wavelength limit $c_s/v_{Te}\ll k\lambda_{ei}\ll1$ and with the collisionless Landau damping rate of $\gamma_s/kc_s=\sqrt{\pi/8}c_s/v_{Te}$ in the short-wavelength range $k\lambda_{ei}\gg1$.

4.2. Surface impedance

The transverse dielectric permittivity (4.6) must be known in describing the penetration of an electromagnetic field into a plasma. We illustrate this need in the example of the incidence of an s-polarized (polarization vector perpendicular to the plane of incidence), low-frequency electromagnetic wave on a semi-infinite plasma. We choose a coordinate system with the x axis directed along the normal to the surface of the plasma, the z axis lying in the plane of wave incidence, and the y axis perpendicular to this plane. The dependence of the amplitude of the electric field E_y on the coordinate x can then be written in the form[24]

$$E_y(x)=-\frac{i}{\pi}\frac{\omega}{c}B_z(0)\int_{-\infty}^{\infty}\frac{e^{iqx}dq}{k_z^2+q^2-(\omega^2/c^2)\epsilon^t},\qquad(4.9)$$

where $k_z^2=(\omega^2/c^2)\sin^2\theta$, and θ is the angle of wave incidence. Since the contribution of ions to ϵ^t is proportional to the small quantity $\sim Zm_e/m_i$, we include only the electron contribution.
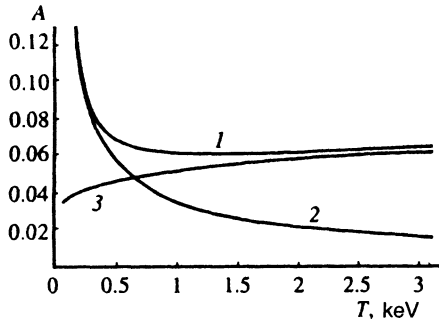
FIG. 14. Absorption coefficient of an electromagnetic field (wavelength $\lambda = 1~\mu$m) in a semi-infinite plasma ($n_0 = 10^{23}$ cm^{-3}, $Z = 10$) vs temperature, plotted using the dielectric permittivity (4.13), which leads to the normal skin effect (curve 2), the expression (4.14) corresponding to the anomalous skin effect (curve 3), and the exact equation (4.6) (curve 1).

The coefficients of absorption A and reflection R of electromagnetic radiation can be expressed in terms of the surface impedance, defined as the ratio of the electric and magnetic fields at the boundary:

$$Z_s = \frac{E_y(0)}{B_z(0)} = -\frac{2i}{\pi}\frac{\omega}{c}\int_0^\infty \frac{dq}{k_z^2 + q^2 - (\omega^2/c^2)\epsilon^t}, \qquad (4.10)$$

in accordance with the relation

$$A = 1 - R = 1 - \left|\frac{Z_s\cos\theta - 1}{Z_s\cos\theta + 1}\right|^2. \qquad (4.11)$$

In the case of interest involving a low-frequency field the absorption coefficient is small ($A \ll 1$) and, according to (4.11),

$$A \approx 4\cos\theta\cdot\mathrm{Re}\,Z_s, \qquad (4.12)$$

is proportional to the real part of the impedance $\mathrm{Re}\,Z_s$.

In the strongly collisional limit $\nu_{ei} \gg kv_{Te}$ the transverse permittivity goes over to the well-known expression

$$\epsilon^t = 1 + i\frac{32}{3\pi}\frac{\omega_{pe}^2\lambda_{ei}}{\omega v_{Te}}, \qquad (4.13)$$

which corresponds to the normal skin effect, and in the collisionless limit $\nu_{ei} \ll kv_{Te}$ the equation

$$\epsilon^t = 1 + i\sqrt{\frac{\pi}{2}}\frac{\omega_{pe}^2}{\omega kv_{Te}}, \qquad (4.14)$$

corresponds to the anomalous skin effect. Here ω_{pe} is the electron plasma frequency.

Equation (4.6) can be used to quantitatively describe the surface impedance of a semi-infinite plasma over a wide range of variation of the parameters from the strongly collisional limit (4.13) to the collisionless limit (4.14). This is what differentiates our description from the one proposed earlier[25,26] using interpolation formulas for ϵ^t.

Figure 14 shows the temperature dependence of the absorption coefficient (curve 1) in comparison with the corresponding dependences within the known limits of the normal (curve 2) and anomalous (curve 3) skin effects.

5. CONCLUSION

In summary, we have developed a systematic quantitative theory of nonlocal transport for small perturbations in a plasma whose ions have a high ionization state. The quasi-hydrodynamic equations (3.3)–(3.8) derived here to describe nonlocal hydrodynamics are fully equivalent to the kinetic description and are natural for investigating radiation transport over a wide range of the collisionality parameter: from the strongly collisional range of classical transport to the collisionless limit.

We have compared the analytical theory with numerical kinetic calculations[11] and found that it exhibits good qualitative agreement with the latter. We have thus demonstrated the advantages of the new theoretical nonlocal transport model, whose practical implementation, in contrast with kinetic calculations, does not require large computational resources. We have also demonstrated the incorrectness of the previously proposed definition of the thermal conductivity[18] for a weakly collisional plasma, where the ionic convection contribution to the heat flux is not expressed explicitly.

The foregoing theoretical study of nonlocal transport is based on perturbation theory. However, the domain of validity of the theory can be expected to extend beyond the implied limitations. In particular, kinetic calculations[27] have demonstrated specifically for the case of nonlocal transport the feasibility of extrapolating results obtained numerically in the small perturbation approach[11] to larger perturbations of the density and temperature by means of a certain procedure for averaging the transport coefficients over the electron mean free path. This possibility opens the door to practical applications of the present nonlocal transport equations.

The nonlocal hydrodynamic equations derived in the article can be used to investigate plasma instabilities for which transport phenomena are important. A significant result of the theory is the expression found for the dielectric permittivity of a plasma in the low-frequency range for an arbitrary collisionality parameter. We have demonstrated its usefulness in the example of calculating the ion-acoustic damping and surface impedance of a plasma.

APPENDIX A: THE ELECTRON–ELECTRON COLLISION INTEGRAL

We proceed from the standard Landau form of the ee collision integral[24]:

$$C_{ee}[f_e, f_e] = \frac{2\pi e^4\Lambda}{m_e^2}\frac{\partial}{\partial v_k}$$

$$\times\int dv'\frac{(v-v')^2\delta_{kj} - (v-v')_k(v-v')_j}{|v-v'|^3}$$

$$\times\left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial v_j'}\right)f_e(v)f_e(v'). \qquad (A1)$$

This equation is simplified by linearizing it with respect to a small deviation of the electron distribution function from a Maxwellian equilibrium distribution and isolating its isotropic part, since we include the contribution of ee collisions only in the equation for the isotropic part of the perturbation of the electron distribution function. The ee collision operator obtained in Eq. (2.4) as a result of these transformations has the form

$$C_{ee}[f_0] \equiv C_{ee}[f_0, F_0] + C_{ee}[F_0, f_0]$$

$$= \frac{2}{Z\sqrt{\pi}} \nu_{ei}(v)v \frac{d}{dv}\left(F_0(v)G\left[\frac{d\psi}{dv}\right]\right), \qquad (A2)$$

where the functional operator $G[h(v)]$ is written as follows:

$$G[h] = \frac{v_{Te}^2}{v} h(v) \gamma\left(\frac{3}{2}, \frac{v^2}{2v_{Te}^2}\right)$$

$$- \frac{2\pi^{3/2}}{3n_0}\left(\int_0^v dw w^3 h(w)F_0(w)\right.$$

$$\left. + v^3 \int_v^\infty dw h(w)F_0(w)\right), \qquad (A3)$$

and $\gamma(3/2, x) = \int_0^x dx \sqrt{x}\exp(-x)$ is a generalized incomplete gamma function.

APPENDIX B: APPROXIMATE EXPRESSIONS FOR THE MODIFIED ELECTRON–ION COLLISION FREQUENCIES

The functions $h_{lm}(kv/\nu_{ei}) \equiv \nu_{lm}/\nu_{ei}$ are determined from the recursion relation (2.8), which is not very convenient for practical application. We therefore use simple algebraic approximations for h_{10} and h_{11}, which work within error limits of no more than a few percent for $X = kv/\nu_{ei} \sim 1$ and have correct asymptotic representations for $X \gg 1$.

First of all, we note that both functions exhibit similar behavior in the strongly collisional limit ($X \ll 1$):

$$h_{10} \approx 1 + \frac{4}{45}X^2, \quad h_{11} \approx 1 + \frac{3}{15}X^2. \qquad (B1)$$

In the weakly collisional limit ($X \gg 1$) the first term on the right-hand side of Eq. (2.8) can be disregarded. Hence it follows that h_{lm} are linear functions of their argument: $h_{lm}(X) \approx s_{lm}X/2$, where the coefficients s_{lm} are given by the recursion relation

$$s_{lm} = \frac{(l+1)^2 - m^2}{4(l+1)^2 - 1}\frac{1}{s_{l+1,m}}. \qquad (B2)$$

Applying these relations in succession, we obtain the following expressions by mathematical induction:

$$s_{10} = \lim_{N \to \infty} \frac{4N+3}{3}\left[\frac{2^{2N}(N!)^2}{(2N+1)!}\right]^2,$$

$$s_{11} = \lim_{N \to \infty} \frac{(4N+3)(2N+1)}{3(N+1)}\left[\frac{(2N)!}{2^{2N}(N!)^2}\right]^2. \qquad (B3)$$

Calculating the corresponding limits by means of Stirling's formula, we obtain $s_{10} = \pi/3$ and $s_{11} = 8/3\pi$.

The approximation (2.9) seems reasonable in light of the quadratic behavior of the functions h_{10} and h_{11} for small values of the argument and their linear behavior for large values of the argument.

This approximation was proposed earlier[18] for the function h_{10}. The exact expressions for h_{10} and h_{11} and the approximate equations (2.9) are shown in Fig. 1. Despite the slight difference in the coefficients of X^2 in the strongly collisional range, these expressions approximate the true functions within 5% for $X \sim 1$. The accuracy of approximation is even better in the weakly collisional range.

APPENDIX C: SYMMETRY PROPERTIES OF THE KINETIC COEFFICIENTS

Here we show that the kinetic coefficients (2.17) form a symmetric matrix. This follows directly from Eq. (2.16). Let the functions ψ^A and ψ^B be two finite solutions of Eq. (2.16) corresponding to two distinct sources S_A and S_B. We multiply Eq. (2.16) for ψ^A by the function ψ^B and integrate the result with respect to the velocities from zero to infinity with weight $(4\pi/n_0)v^2F_0(v)$. In the resulting expression we then replace A by B and vice versa and subtract one of these equations from the other. The subtraction result expresses the difference between the kinetic coefficients in terms of the integral of the ee collision operator:

$$J_B^A - J_A^B = \frac{4\pi}{n_0}\int_0^\infty dv v^2[\psi^B C_{ee}(F_0\psi^A) - \psi^A C_{ee}(F_0\psi^B)]. \qquad (C1)$$

Using expressions (A2) and (A3) for the ee collision integral, we can integrate the left-hand side of Eq. (C1) by parts. This operation yields the expression

$$J_B^A - J_A^B = \frac{24v_{Te}^4}{Zn_0\lambda_{ei}}\int_0^\infty dv F_0(v)\left(\frac{d\psi^A}{dv}G\left[\frac{d\psi^B}{dv}\right] - \leftrightarrow B\right). \qquad (C2)$$

The first term in G (A3) is linear in ψ, hence it is symmetry in the indices A and B and does not contribute to Eq. (C2). The last two terms can be written as a double integral:

$$J_B^A - J_A^B = \frac{16\pi^{3/2}v_{Te}^4}{Zn_0^2\lambda_{ei}}\left(\int_0^\infty dv F_0(v)\frac{d\psi^B}{dv}\int_0^v dw w^3 F_0(w)\frac{d\psi^A}{dw}\right.$$

$$\left. + \int_0^\infty dv v^3 F_0(v)\frac{d\psi^B}{dv}\int_v^\infty dw F_0(w)\frac{d\psi^A}{dw} - \leftrightarrow B\right). \qquad (C3)$$

Changing the order of integration in one of the integrals in Eq. (C3), we see that both integrals are symmetric in A and B, so that the right-hand side of Eq. (C3) vanishes.

The symmetry of the kinetic coefficients J_A^B can be regarded as a generalization of Onsager symmetry to the intermediate range of values of the collisionality parameter.

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