Quantum variational measurements of force and compensation of the nonlinear backaction in an interferometric displacement transducer

S. P. Vyatchanin*) and A. B. Matsko

M. V. Lomonosov Moscow State University, 119899 Moscow, Russia
(Submitted 25 April 1996)
Zh. Éksp. Teor. Fiz. **110**, 1252–1265 (October 1996)

Quantum measurement of a small force with an error less than the standard quantum limit using a lossless optical interferometric sensor is discussed. The detector sensitivity has been investigated taking into account its nonlinearity. The case when the measurement time is shorter than the relaxation time of the optical cavity is considered. The conditions when the equations can be linearized in principle have been determined, and the limit sensitivity of the detector has been calculated with due account of the nonlinearity in the detection of the quadrature component of the reflected optical wave. Two methods for reducing the negative effect of nonlinearity have been proposed: (a) using nonlinear detection of the output optical wave (to measure the number of photons in the biased wave); (b) introducing additional nonlinearity to mechanical components of the device. Numerical estimates are given. [S1063-7761(96)00910-9]

1. INTRODUCTION

Quantum noise in an optical sensor of mechanical displacement is a key problem in designing a gravitational interferometric antenna (the LIGO project) and in other fundamental physical experiments. When the position is measured continuously, the backaction noise limits the detector sensitivity to the standard quantum limit.[1–4] In a simple optical transducer in which the position of the test mass is extracted from the reflected wave phase (Fig. 1a), the backaction is caused by fluctuations in the light-pressure force. As a result, amplitude fluctuations in the input wave amplitude are transformed to phase fluctuations of the reflected wave. The standard quantum limit is achieved at an optimal pumping power W_{SQL} at which the initial phase uncertainty equals the phase uncertainty due to the backaction noise. It is generally accepted that the standard quantum limit determines the minimum registered force. For example, if the force

$$F_S=F\cos(\omega_F t), \tag{1}$$

acts on a free mass m during its oscillation period $t_F=2\pi/\omega_F$, its standard quantum limit is

$$F_{SQL}\simeq\sqrt{m\hbar\omega_F^2/t_F},\quad W_{SQL}\simeq mc^2/\omega_0 t_F^2.$$

In this paper we discuss the measurement of a force which is a known function of time. We also assume that there is no intrinsic mechanical noise.

It is known that we can improve on the standard quantum limit even with coherent pumping, and this requires no special correlation of the device noise,[2,4] modulation of the input wave,[5] or nonclassical states (frequency anticorrelated,[6] squeezed,[7] etc.) if one measures not the phase, but a specially chosen quadrature component $B(\theta)$ (Fig. 2), which is squeezed.[8,9] It is the mechanical nonlinearity (in fact it is a mechanical version of the nonlinear susceptibility $\chi^{(3)}$) causing the backaction that leads to squeezing of reflected light. If the component $B(\theta)$ is measured, the backaction noise can disappear. An important consequence of squeezing the reflected wave is that, the angle θ is not constant, but depends on the spectral frequency, $\theta=\theta(\Omega)$. In order to measure the spectrum-dependent squeezing over a wide range (i.e., at a small measurement time $T\simeq t_F$, which is usually the case in experiments with gravitational waves), the phase of the local oscillator wave should be modulated in a specific manner in a balanced homodyne device throughout the measurement time. Then the error in the force measurement by the detector (Fig. 1a) is determined by radiation friction and can be reduced to

$$F_{\min}=\xi F_{SQL}\sqrt{\omega_F/\omega_0}. \tag{2}$$

Here ω_0 is the input optical frequency and ξ is a numerical factor of the order of unity. This sensitivity is achieved at the optimum input power $W_{opt}=mc^2/t_F$. In the power interval defined by the condition $W_{SQL}\ll W\ll W_{opt}$ we have[8]

$$F\simeq F_{SQL}\sqrt{W_{SQL}/W}.$$

It is worth emphasizing that this procedure is not a quantum nonperturbing measurement because no umperturbed variable of the mechanical oscillator is identifiable. In fact, the reflected wave carries little information about the position, the momentum, or their combinations because the measuring device strongly perturbs them. Only a variation in the position caused by the measured force is detected. Such measurements can be called quantum variation measurements. The detection of a signal action and quantum nonperturbing measurement are quite different problems, each of which requires a specific strategy that is, generally speaking, different from the other.

Equation (2) has been derived in the approximation linear in the parameter $2\omega_0 x/c$ (x is the deviation of the detecting mass from its equilibrium position). Only in this approximation can fluctuations in the reflected wave be described by a regular ellipse on the phase diagram (Fig. 2a). If terms of higher orders with respect to $\omega_0 x/c$ are included, the fluctuation ellipse is bent, whereupon the uncertainty in the quadrature amplitude increases (the segment CD in Fig. 2b

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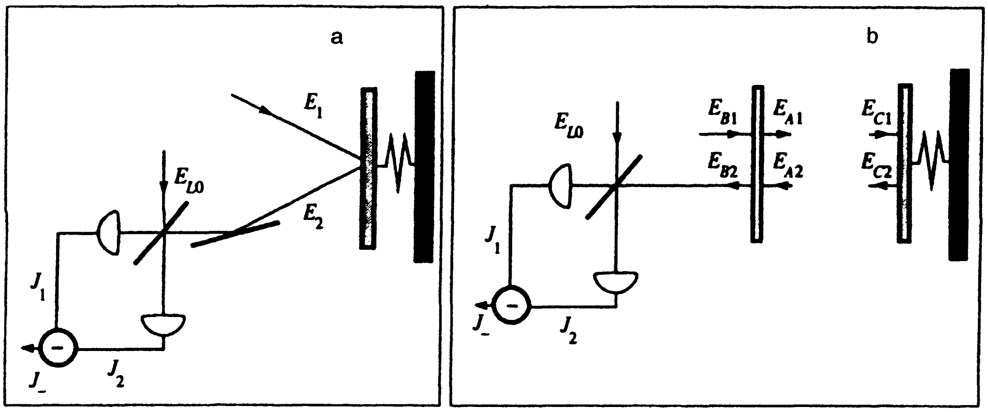

FIG. 1. (a) Simple and (b) interferometric optical displacement sensors. A change in the mobile mirror position due to the detected force leads to a phase shift in the reflected wave, which is measured by a balanced homodyne circuit.

is larger when the ellipse is bent). In our previous publication[10] we analyzed in detail the effect of nonlinear terms and initial conditions on measurements of the quadrature component and demonstrated that the sensitivity described by Eq. (2) cannot be achieved at realistic values of ω_0, m, and t_F. We also proposed a nonlinear scheme for measuring the number of photons in the shifted wave (nonlinear quantum variation measurement) in order to distinguish different bent ellipses. To do this, the reflected wave should be combined with an additional wave so that the bent ellipse describes the state of the resulting field squeezed with respect to its amplitude (photon number). This scheme does not fully compensate for the nonlinearity because the ellipse is not bent exactly to the circle with the center at the point O' (Fig. 2b) owing to fluctuating terms of the third and higher orders in the small parameter, but the negative effect of the nonlinearity on measurements is reduced. This, however, is sufficient to obtain the sensitivity defined by Eq. (2) at realistic parameters of experiment.

The sensor shown in Fig. 1a is convenient for theoretical analysis, but it cannot be used in real experiments because it demands very high input power $W_{\rm opt}$. From the viewpoint of a real experiment, such as LIGO, the interferometric sensor shown in Fig. 1b is more interesting. By linearizing its equations with respect to the parameter $\omega_0 x/cT_{FP}$, where T_{FP} is the transmittivity of the input mirror, one can easily prove that its sensitivity limit is also described by Eq. (2), but it can be achieved at an input power smaller by a factor $[T_{FP}^2/16+(\omega_F L_0 c)^2)]^{-1}$, where L_0 is the separation between the mirrors of the loaded optical cavity. The interferometric sensor, however, has a considerably higher nonlinearity since its power expansion parameter is $2\omega_0 x/cT_{FP}$ instead of $2\omega_0 x/c$ in the conventional detector.

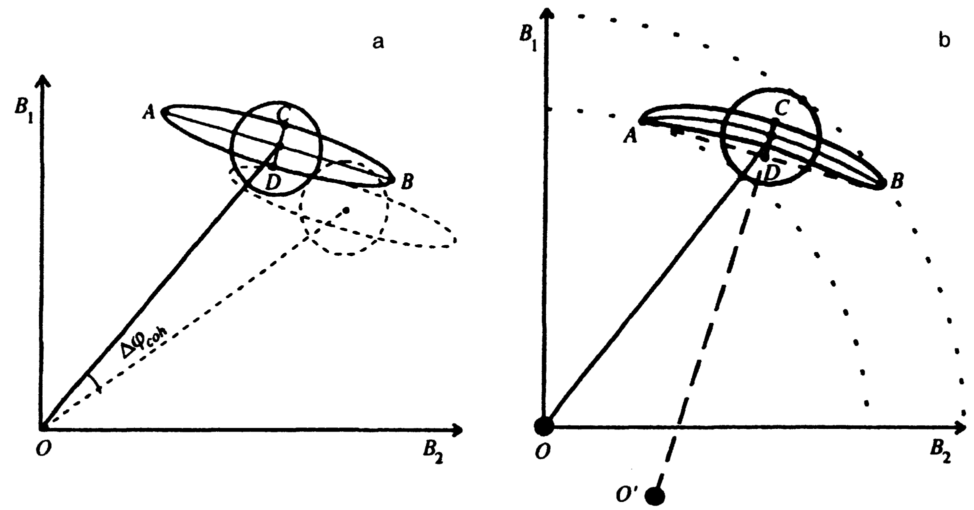

FIG. 2. Phase diagram of spectral amplitude of reflected light wave. (a) The mechanical nonlinearity due to the light pressure leads to a phase-amplitude correlation in the reflected wave, and the circle describing fluctuations in the input wave is transformed to an ellipse. Therefore one should measure not the phase, but a well defined quadrature component $B_{\rm opt}$. (b) Nonlinear corrections ''bend'' the fluctuation ellipse, and this bending is the higher, the higher the input power. The measurement error is increased because the projection of all the points on a selected vector (segment CD) is longer than in the linear approximation. In order to distinguish between ''bent ellipses,'' we propose to measure the number of photons in the biased wave. Adding a bias field with small fluctuations shifts, the origin of the phase plane to the point O', which is the center of the ''ellipse curvature,'' and the field amplitude is measured with respect to the new coordinate system.

This publication is a logical continuation of our previous study[10] and gives a detailed analysis of the interferometric sensor operation with due account of its nonlinearity. The conditions under which the equation describing its operation can be linearized have been determined, and the limit sensitivity has been calculated with due account of the nonlinearity in detecting the quadrature component of the reflected wave. We suggest two methods for improving the detector sensitivity: (a) nonlinear detection of the output optical wave (measurement of the number of photons in the biased wave); (b) introduction of mechanical nonlinearity to the device. We have demonstrated that the device sensitivity can be better than the standard quantum limit at a much lower input power than in the conventional detector, but the limit sensitivity of the interferometric detector is inferior to that of the conventional detector because even the combination of the two proposed techniques cannot totally eliminate the negative effect of nonlinearity, which is essentially stronger in the interferometric sensor.

To simplify our equations, we limit our discussion to the case which is most interesting for the LIGO experiment, when (a) the probe mass is effectively free, i.e., the time of force action is shorter than its inherent oscillation period; (b) the damping time of the optical cavity is longer than the time of force action: $\omega_F L_0/cT_{FP} \gg 1$; (c) the cavity is resonant with the input optical frequency: $\exp(iL_0\omega_0/c)=1$. In our numerical estimates we take the following parameters: $\omega_0=10^{15}\ \mathrm{s}^{-1}$, $\omega_F=10^3\ \mathrm{s}^{-1}$, $\omega_M=10\ \mathrm{s}^{-1}$, $m=10^3$ g, $T_{FP}=10^{-3}$, and $L_0=3\cdot 10^5$ cm.

2. BASIC EQUATIONS

Let us express the fields of the input and output waves as $E_{B1}\exp(-i\omega_0 t)+E_{B1}^+\exp(i\omega_0 t)$ and $E_{B2}\exp(-i\omega_0 t)+E_{B2}^+ \times\exp(i\omega_0 t)$, respectively. The field inside the interferometer can be expressed similarly (see notation in Fig. 1b). We represent the input wave as a sum of the average amplitude E_0 (taking for definiteness $E_0=E_0^*$) and the fluctuation $e_{B1}(t)$: $E_{B1}=E_0+e_{B1}(t)$. Then the average amplitudes of the output wave and the field inside the interferometer are given by the expressions $\langle E_{B2}\rangle=E_0$, $\langle E_{A1}\rangle\simeq 2E_0/\sqrt{T_{FP}}$, respectively. We assume that there are no losses in the mirrors and mechanical components.

Under a small perturbation of the mirror coordinate $(x(t)/\lambda_0 T_{FP}\ll 1)$ and slow damping of the field inside the cavity $(T_{FP}t/4\tau_0\ll 1)$ in the time $0\leqslant t\leqslant T$, where T is the measurement time, the fluctuations in the input and output waves are related by the following equation:

$$
\begin{aligned}
e_{B2}(t)\simeq{}&\int_0^t\left[e_{B1}(\tau)\,\frac{T_{FP}}{4\tau_0}-\dot{e}_{B1}(\tau)\right]\exp\left(\frac{T_{FP}}{4\tau_0}\,(\tau-t)\right)d\tau\\
&+e_{B2}(0)\exp\left(-\frac{T_{FP}}{4\tau_0}\,t\right)+\frac{2E_0}{\tau_0}\int_0^t\left[i\,\frac{x(\tau)}{\lambda_0}-\frac{\dot{x}(\tau)}{c}\right.\\
&\left.-\frac{x(\tau)}{\lambda_0}\int_\tau^t\frac{x(\tau_1)}{\lambda_0}\,\frac{d\tau_1}{\tau_0}-i\,\frac{x(\tau)}{\lambda_0}\left(\int_\tau^t\frac{x(\tau_1)}{\lambda_0}\,\frac{d\tau_1}{\tau_0}\right)^2\right]\\
&\times\exp\left(\frac{T_{FP}}{4\tau_0}\,(\tau-t)\right)d\tau,
\end{aligned}
$$

$$
e_{B1}(t)=\int_{-\omega_0}^{\infty}\sqrt{\frac{\hbar\omega_0}{Sc}}\,\lambda_+ b_1(\omega_0+\Omega)\exp(-i\Omega t)\,d\Omega, \tag{3}
$$

where $e_{B2}(0)$ is the fluctuating component of the output wave at the initial moment, $\tau_0=L_0/c$ is the time for light to travel between the cavity mirrors, $\lambda_0=\omega_0/c$, $x(t)$ is the mirror shift from its equilibrium position in the loaded cavity, S is the area of the movable mirror, $\lambda_\pm=\sqrt{1\pm\Omega/\omega_0}$, $b_1(\omega)$ is the annihilation operator (its commutator with the creation operator is $[b_1(\omega),b_1^+(\omega')]=\delta(\omega-\omega')$). Let us assume that the input wave is in a coherent state, i.e., $\langle b_1(\omega)b_1^+(\omega')\rangle=\delta(\omega-\omega')$.

Equation (3) must be supplemented with the equation of motion for a mechanical oscillator describing the mirror displacement:

$$
\begin{aligned}
\ddot{x}+\omega_M^2 x+{}&\frac{S}{\pi m}\,\frac{8E_0^2}{T_{FP}}\int_0^t\exp\left(\frac{T_{FP}}{4\tau_0}\,(\tau-t)\right)\left[\frac{\dot{x}(\tau)}{c}\right.\\
&\left.+\frac{x(\tau)}{\lambda_0}\int_\tau^t\frac{x(\tau_1)}{\lambda_0}\,\frac{d\tau_1}{\tau_0}\right]\frac{d\tau}{\tau_0}\\
={}&\frac{SE_0}{\pi m}\int_0^t(e_{B1}(\tau)+e_{B1}^+(\tau))\\
&\times\exp\left(\frac{T_{FP}}{4\tau_0}\,(\tau-t)\right)\frac{d\tau}{\tau_0}+\frac{S}{\pi m}\,\frac{2E_0}{T_{FP}}\\
&\times\exp\left(-\frac{T_{FP}}{4\tau_0}\,t\right)[(e_{B2}(0)+e_{B2}^+(0))\sqrt{1-T_{FP}}\\
&+e_{B1}(0)+e_{B1}^+(0)]+\frac{F_S(t)}{m}.
\end{aligned} \tag{4}
$$

Here $e_{B1}(0)$ and $e_{B2}(0)$ are the initial values of the fluctuating components of the input and output waves, ω_M is the natural frequency of mirror oscillations, m is the mirror mass, and F_S is the detected force. We assume that the free-mass approximation applies, i.e., $\omega_M t\ll 1$.

In deriving Eqs. (3) and (4) we introduced some approximations, which are discussed in detail in Appendix A.

3. LINEAR SCHEME OF THE QUADRATURE-COMPONENT MEASUREMENT

In this section we will discuss the interferometric detector sensitivity in the simplest linear approximation, when Eqs. (3) and (4) can be linearized and nonlinear terms can be ignored.

The difference photocurrent J_- in the device shown in Fig. 1b is proportional to the quadrature field component:

$$
B(\theta(t),t)=E_{B2}(t)\exp[-i\theta(t)]+E_{B2}^+(t)\exp[i\theta(t)], \tag{5}
$$

where the function $\theta(t)$ is determined by the phase modulation of the local oscillator wave $E_{LO}(t)$. If the detection time is limited, then the quadrature component is not measured, but rather the average

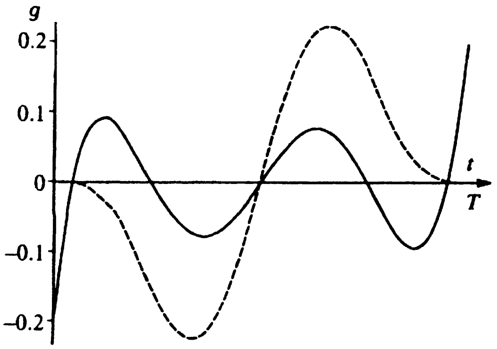

FIG. 3. Filtering function $g_S(t)$ (solid line) calculated by Eq. (10) and the function $g_C(t)$ (dashed line) for $|K(\omega_F)|=5\times 10^3$.

$$B_T=\int_0^T \tilde{\Phi}(t)B[\theta(t),t]dt. \tag{6}$$

The averaging function $\tilde{\Phi}(t)$ and the phase $\theta(t)$ must be chosen so that the parameter B_T should not contain information about the linear component of the backaction noise. Besides, the additional measurement error caused by (a) uncertainties of initial coordinate and momentum of the mirror and (b) uncertainties of initial fields inside the cavity and in the input wave must be eliminated. To this end, the following additional conditions must be satisfied (see Appendix B):

$$\frac{d^4 g_C(t)}{dt^4}=\frac{8\omega_0 W}{\tau_0^2 m c^2}\, g_S(t), \tag{7}$$

$$\frac{d^n g_C(0)}{dt^n}=\frac{d^n g_C(T)}{dt^n}=0, \quad n=0,\ 1,\ 2,\ 3. \tag{8}$$

Here $g_S(t)=\tilde{\Phi}(t)\sin\theta(t)$, $g_C(t)=\tilde{\Phi}(t)\cos\theta(t)$, and W is the average power of the input wave $E_{B1}(t)$. Then the parameter B_T defined by Eq. (6) can be expressed as

$$B_{T1}\simeq\int_0^T g_S(t)\left[\frac{4E_0}{\tau_0\lambda_0}\int_0^t\int_0^\tau \frac{F_S(\tau_1)}{m}(\tau-\tau_1)d\tau_1 + i(e_{B1}(t)+e_{B1}^+(t))\right]dt. \tag{9}$$

The function $g_S(t)$ is derived using the technique of optimum filtration, namely, the signal-to-noise ratio is maximized under the conditions defined by Eqs. (7) and (8). The condition (8) is introduced to get rid of the uncertainty of initial conditions.

Obviously, the additional conditions defined by Eq. (8) lead to a lower sensitivity, otherwise the function $g_S(t)$ would be simply proportional to the output signal [the first term in the brackets on the right-hand side of Eq. (9)]. Let us consider as an example the detected force $F_S(t)$ described by Eq. (1) that should be measured during the time of its action $T\simeq 2\pi/\omega_F$. The detector output signal is proportional to

$$y_S=2\pi t/T-\sin(2\pi t/T).$$

But with due account of Eqs. (7) and (8) we obtain (Fig. 3)

$$g_S(t)=\frac{5(2\pi^2-21)}{\pi^3}-\frac{2(45\pi^2-630+\pi^4)}{\pi^3}\frac{t}{T}+\frac{210(\pi^2-15)}{\pi^3}\left(\frac{t}{T}\right)^2-\frac{140(\pi^2-15)}{\pi^3}\left(\frac{t}{T}\right)^3+2\pi\frac{t}{T}-\sin\left(2\pi\frac{t}{T}\right). \tag{10}$$

Hence the minimum detectable force is

$$F_{\mathrm{Lin}}=\frac{\mu}{\sqrt{|K(\omega_F)|}}F_{\mathrm{SQL}}, \tag{11}$$

where $\mu\simeq 60$ is the numerical factor characterizing the degradation of the detector sensitivity owing to the additional conditions (7) and (8), and $|K(\omega_F)|$ is the absolute value of the squeezing factor of the output wave at the frequency ω_F:[8]

$$|K(\omega_F)|\simeq 4\omega_0 W/\tau_0^2\omega_F^4 m c^2.$$

We note at this point that Eq. (11) is valid only in the case when the detector sensitivity is considerably higher than the standard quantum limit, i.e., when the reflected wave is strongly squeezed: $|K(\omega_F)|\gg\mu^2$. If the expected gain in the sensitivity is small, it is quite unnecessary to satisfy the zero initial conditions (7) and (8) for $g_S(t)$ and its derivatives. If the expected sensitivity is within the standard quantum limit, there is no need whatever to eliminate the uncertainties in the initial conditions. For simplicity, we do not describe a full derivation of the optimization conditions for $g_S(t)$ in an arbitrary case, but limit our discussion to the case of strong squeezing.

Note that the output of the interferometric displacement sensor is controlled not by the instantaneous mirror displacement, but by its time integral [the first term in the brackets in Eq. (9)]. Therefore its sensitivity can be improved by increasing the detection time to the optical cavity damping time.

It follows from Eq. (11) that in the linear approximation the sensitivity of the device as a detector of classical force is unlimited. It increases with the input power proportionally to $\sqrt{W}$. A more accurate calculation, however, indicates that the sensitivity in this case is limited to that described by Eq. (2) owing to radiative friction.[8] In reality, the limitations due to the nonlinearity of the sensor are more essential.

4. LIMITATION OF THE LINEAR-SCHEME SENSITIVITY

It is significant that the nonlinear terms in Eqs. (3) and (4), describing the nonlinear backaction noise, increase with the input power. If the quadrature component of the output wave described by Eq. (5) is detected, the sensitivity is limited by the terms which are quadratic in $x(t)$. Suppose that the nonlinearity in the equation of mechanical motion is small, which is described by the inequality

$$4|K(\omega_F)|\frac{x(t)}{\lambda_0 T_{FP}}\ll 1. \tag{12}$$

Then $x(t)$ can be expressed as the sum $x(t)=x_{\rm Lin}+x_{\rm NLin}$, where $x_{\rm Lin}$ is a solution of the linear equation and

$$x_{N\,\rm Lin}\simeq\frac{8SE_0^2}{\pi mT_{FP}}\int_0^t(t-\tau)\int_0^\tau\frac{x_{\rm Lin}(\tau_1)}{\lambda_0}\times\int_{\tau_1}^\tau\frac{x_{\rm Lin}(\tau_2)}{\lambda_0}\frac{d\tau d\tau_1 d\tau_2}{\tau_0^2}. \tag{13}$$

Let us write the expression for B_T derived from Eq. (6) retaining to second order in $x(t)$:

$$B_{T\,\rm II}=B_{T\,\rm I}+\int_0^T\Bigg[-g_C(t)\zeta(t)+g_S(t)\frac{16W\omega_0}{mc^2\tau_0T_{FP}}\times\int_0^t\int_0^\tau(\tau-\tau_1)\zeta(\tau_1)d\tau d\tau_1\Bigg]dt,$$

$$\zeta(t)=\frac{2E_0}{\tau_0^2}\left(\int_0^t\frac{x_{\rm Lin}(\tau)}{\lambda_0}d\tau\right)^2. \tag{14}$$

Here the first term in the brackets is due to the nonlinearity of the optical system and the second is due to the quadratic stiffness characteristic of the mechanical components. Given that the damping time of the optical cavity is long, the second term is larger than the first by $\sim\omega_F\tau_0/T_{FP}$, i.e., the sensitivity the quadrature-component measurement is limited due to the dynamic nonlinearity in the mechanical oscillator.

From Eq. (14) we derive the minimum detectable force when the quadrature is measured:

$$F_{\rm LinCoh}\simeq\mu_1\left(\left(\frac{\omega_F\tau_0}{T_{FP}}\right)^2\tilde{k}\right)^{1/10}F_{\rm SQL},$$

$$\tilde{k}=\frac{\hbar}{m\omega_F}(\lambda_0\tau_0\omega_F)^{-2}. \tag{15}$$

With the parameters quoted in the Introduction, we have an estimate $F_{\rm LinCoh}\simeq2\times10^{-2}F_{\rm SQL}$. Note that the factor μ_1 resulting from the need to correct for the initial conditions is considerably smaller than that derived in the previous section because of the relatively small compression factor: $|K(\omega_F)|\leqslant\mu^2$ (see the previous section). Our estimates yield $\mu_1\simeq2$. The input power needed to achieve the sensitivity defined by Eq. (15) is determined by the equation

$$|K(\omega_F)|=((\omega_F\tau_0/T_{FP})^2\tilde{k})^{-1/5}.$$

Note that when a similar procedure is applied to the conventional detector,[10] its sensitivity is a factor of $\simeq T_{FP}^{-1/5}$ higher because its nonlinearity is weaker.

It follows from Eq. (15) that the nonlinearity of the interferometric sensor leads to an essential limitation of the minimum detectable force. In order to improve its sensitivity, one should linearize the detector by changing the characteristic of the mechanical components, i.e., by introducing a mechanical nonlinearity with a sign opposite to that of the dynamic nonlinearity. Suppose that the nonlinearity has been cancelled. Then in the case of linear detection (measurement of the quadrature component) the sensitivity, as in the case of the conventional detector, will be controlled by the optical nonlinearity, which leads to the bending of the fluctuation ellipse shown in Fig. 2b and is described by the first term in the brackets in Eq. (14). The minimum detectable force will be reduced by a factor $(\omega_F\tau_0/T_{FP})^{1/5}$ compared to that given by Eq. (15). In order to compensate for this optical nonlinearity, we propose a nonlinear measurement procedure which allows one to distinguish between bent ellipses.

5. NONLINEAR SCHEME FOR MEASURING ENERGY OF A BIASED WAVE

Suppose that the device shown in Fig. 1b measures only the current J_2, which is proportional to the energy of the biased wave (the current J_1 is ignored):

$$N_T=\int_0^T\Phi_1(t)(E_{B2}^+(t)+E_N^+(t))(E_{B2}(t)+E_N(t))dt, \tag{16}$$

where $\Phi(t)$ is the averaging function, $E_N=\sqrt{T_{\rm sp}}E_{\rm LO}$ is the field added to the field E_2 of the reflected wave, $E_{\rm LO}$ is the field of the local oscillator wave. We assume that the transparency of the beam splitter is small, $T_{\rm sp}\ll1$, in this case fluctuations in the wave E_N may be ignored. Suppose that the wave $E_N(t)$ is amplitude and phase modulated. We introduce the notation

$$E_0+E_N=\Phi_2(t)\exp[i\theta(t)],\quad \Phi_1(t)\Phi_2(t)=\Phi(t),$$

$$G_S(t)=\Phi(t)\sin\theta(t),\quad G_C(t)=\Phi(t)\cos\theta(t).$$

The backaction noise will be cancelled if the following conditions are satisfied:

$$\frac{d^4G_C(t)}{dt^4}=\frac{8\omega_0W}{\tau_0^2mc^2}G_S(t),$$
$$\frac{d^nG_C(0)}{dt^n}=\frac{d^nG_C(T)}{dt^n}=0,\quad n=0,1,2,3, \tag{17}$$

and

$$G_C(t)-2E_0\Phi(t)=f(t). \tag{18}$$

Here $f(t)$ is an arbitrary small function smooth on the segment $[0,T]$ and satisfying the condition $\sup|f(t)|\ll(\omega_0/c)\sup|x(t)|$. Equation (17) describes the cancellation of the linear terms proportional to $x_{\rm Lin}(t)$ and Eq. (18) that of the quadratic terms proportional to $x_{\rm Lin}^2(t)$. The function $G_S(t)$ is derived using the optimum filtration technique, just like $g_S(t)$. It is obvious that the condition (17) is equivalent to Eq. (7). In this case, the detection sensitivity is determined by the third-order backaction noise proportional to $x_{\rm Lin}^3(t)$ and the uncertainty of the initial phase of the input wave. The measured parameter can be expressed as

$$N_T\simeq B_{T\,\rm I}-\frac{4E_0}{\tau_0}\int_0^TG_S(t)\int_0^t\frac{x(\tau)}{\lambda_0}\times\left(\int_0^t\frac{x(\tau_1)}{\lambda_0}\frac{d\tau_1}{\tau_0}\right)^2d\tau dt, \tag{19}$$

which yields the equation for the minimum detectable force:

$$F_{N\,\rm LinCoh}\simeq\mu(2\tilde{k})^{1/4}F_{\rm SQL}\simeq4\times10^{-4}F_{\rm SQL}. \tag{20}$$

The input power needed to achieve this sensitivity is determined by the equation

$$|K(\omega_F)|=(2\tilde{k})^{-1/2}.$$

To conclude this section, let us discuss the problem of preparing the measuring system. In the linear approximation, the uncertainty in the initial conditions, $\Delta x_{\rm ini}$, does not affect the measurement accuracy if the condition (8) is satisfied. When the nonlinear corrections are taken into account, this uncertainty can be neglected if

$$\Delta x_{\rm ini} \ll \Delta x_{\rm dist}, \tag{21}$$

where $\Delta x_{\rm dist}$ is the coordinate perturbation due to fluctuations in the light pressure (backaction). At the measurement time $T \simeq t_F$, the parameter $\Delta x_{\rm dist}$ can be estimated as follows:

$$\Delta x_{\rm dist} \simeq \Delta x_{\rm SQL}\sqrt{|K(\omega_F)|}, \quad \Delta x_{\rm SQL}^2=\frac{\hbar}{m\omega_F}. \tag{22}$$

The uncertainty $\Delta x_{\rm ini}$ can be expressed in terms of initial uncertainties of the coordinate, $\Delta x(0)$, momentum, $\Delta p(0)$, and force acting on the mechanical oscillator, $\Delta F(0)$ (the uncertainty in the force is due to initial uncertainties of the fields inside the cavity, which are considered constant throughout the measurement time):

$$\Delta x_{\rm ini} \simeq \sqrt{\Delta x^2(0)+\left(\frac{\Delta p(0)T}{m}\right)^2+\left(\frac{\Delta F(0)T^2}{2m}\right)^2}. \tag{23}$$

Here our aim is not to propose an optimum design for the device, but to consider a procedure which can be easily followed in an experiment using the same input wave without amplitude and phase modulation. Let the input wave be in a coherent state for a long time prior to the measurement and its power be constant. Then the coordinate is strongly perturbed at large $K(\omega_F)$ and the condition (21) is not satisfied. Let us demonstrate that this perturbation can be measured, hence corrected for. This procedure can be used if sequential measurements of force are needed. Information about initial conditions can be derived from measurements of the reflected-wave phase during the time $\tau = T/|K(\omega_F)|^{1/4}$ at the end of the previous measurement since the phase is the measured parameter during the time τ at the end of the time interval $[0,T]$ if $G_S(t)$ is determined by Eq. (10). In this case the coordinate can be determined with an error

$$\Delta x(0) \simeq \Delta x_{\rm SQL}/|K(\omega_F)|^{1/8},$$

the momentum with an error

$$\Delta p(0) \simeq \Delta x_{\rm SQL} m |K(\omega_F)|^{1/8}/T,$$

and the force with the error

$$\Delta F(0) \simeq \Delta x_{\rm SQL} 2m |K(\omega_F)|^{3/8}/T^2.$$

This measurement accuracy corresponds to the standard quantum limit at the measurement time τ. Given these equations, one can easily find

$$\Delta x_{\rm ini} \simeq \Delta x_{\rm SQL}|K(\omega_F)|^{3/8}$$

and the condition (21) is satisfied.

6. CONCLUSION

We emphasize that the limitation on the interferometric detector sensitivity due to its nonlinearity can, in principle, be eliminated completely. To this end, in addition to introducing the delayed nonlinearity to mechanical components balancing the nonlinear dynamic backaction, one should perform a nonlinear transformation of the output wave to a state similar to that of the input wave without changing its phase shift, which carries interesting information. From the theoretical viewpoint, it may be transmission of the output wave through a nonlinear medium with a specific nonlinear susceptibility $\chi^{(3)}(\Omega)$. One may correct for the nonlinearity not completely, but only for the component which is responsible for bending the ellipse, for example, by introducing into the detector an additional nonlinearity with a specific delay characteristic. In this case, the detected force can be measured with an error given by Eq. (2). Unfortunately, we have no idea about how to produce this nonlinearity in experiment. The limitations discussed in this paper are caused only by the nonideal characteristics of the proposed facility measuring the energy of a biased wave, capable of only partially correcting for the nonlinearity.

The work was supported by the Russian Fund for Fundamental Research (Grant No. 94-02-04219) and USA National Science Foundation (Grant PHY-9503642).

APPENDIX A

A Fabry–Perot interferometer with a movable mirror (Fig. 1b) tuned to an optical resonance is described by the equation system

$$\begin{aligned}
&E_{A1}(t)=E_{B1}(t)\sqrt{T_{FP}}-E_{A2}(t)\sqrt{1-T_{FP}},\\
&E_{B2}(t)=-E_{B1}(t)\sqrt{1-T_{FP}}-E_{A2}(t)\sqrt{T_{FP}},\\
&E_{C2}(t)=-E_{C1}(t)(1-2\dot{x}/c)\exp(2ix(t)/\lambda_0),\\
&E_{C1}(t)=E_{A1}(t-\tau_0), \quad E_{C2}(t)=E_{A2}(t+\tau_0).
\end{aligned} \tag{A1}$$

Here E_{Aj}, $_{Bj}$, and E_{Cj} are slow amplitudes of plane electromagnetic waves. This system includes only terms of order $\dot{x}(t)/c$ [those proportional to $(\dot{x}(t)/c)^2$ are omitted]. Let us assume that the time for light to travel between the mirrors, τ_0, is smaller than all other characteristic times of the system. Then we can use the following power expansion:

$$E(t\pm\tau_0) \simeq E(t) \pm \tau_0\dot{E}(t)+O(\tau_0^2\ddot{E}(t)). \tag{A2}$$

Without specifying the function $x(t)$, one can derive from Eq. (A1) an approximate equation for the output wave:

$$\begin{aligned}
&\dot{E}_{B2}(t)+\frac{E_{B2}(t)}{\tau_0}g_1(t)=-\dot{E}_{B1}(t)g_2(t)-\frac{E_{B1}(t)}{\tau_0}g_3(t),\\
&g_1(t)=\frac{1-\sqrt{1-T_{FP}}(1-2\dot{x}/c)\exp(2ix/\lambda_0)}{1+\sqrt{1-T_{FP}}(1-2\dot{x}/c)\exp(2ix/\lambda_0)},\\
&g_2(t)=\frac{\sqrt{1-T_{FP}}+(1-2\dot{x}/c)\exp(2ix/\lambda_0)}{1+\sqrt{1-T_{FP}}(1-2\dot{x}/c)\exp(2ix/\lambda_0)},
\end{aligned} \tag{A3}$$

$$g_3(t)=\frac{\sqrt{1-T_{FP}}-(1-2\dot{x}/c)\exp(2ix/\lambda_0)}{1+\sqrt{1-T_{FP}}(1-2\dot{x}/c)\exp(2ix/\lambda_0)}.$$

Equation (A3) can be solved formally. Then its solution is expanded in powers of $x(t)$ and terms through third order are retained. As a result, one obtains Eq. (3).

Equation (A1) should be supplemented with the equation describing the motion of the mirror:

$$\ddot{X}+\omega_M^2X=\frac{S}{2\pi m}\,(E_{C1}^+(t)E_{C1}(t)+E_{C2}^+(t)E_{C2}(t))+\frac{F_S(t)}{m}. \tag{A4}$$

Then, using Eq. (A1) and expansion (A2) and representing the position as the sum, $X=X_0+x(t)$, where X_0 is the average shift of the mirror due to the light pressure and $x(t)$ is the fluctuating component, one obtains Eq. (4).

APPENDIX B

Let us write the linearized equations (3) and (4) as the system of equations

$$\begin{aligned}\dot{y}_2+y_2\,\frac{T_{FP}}{4\tau_0}&=-\frac{4E_0}{\tau_0 c}\,\dot{x}+y_1\,\frac{T_{FP}}{4\tau_0}-\dot{y}_1,\\ \dot{z}_2+z_2\,\frac{T_{FP}}{4\tau_0}&=\frac{4E_0}{\tau_0\lambda_0}\,x+z_1\,\frac{T_{FP}}{4\tau_0}-\dot{z}_1,\\ \ddot{x}+2\delta\dot{x}+\omega_M^2x&=\frac{2SE_0}{\pi m T_{FP}}\,(y_1+y_2)+\frac{F_S}{m}.\end{aligned} \tag{B1}$$

Here $y_j=e_{Bj}+e_{Bj}^+$ describes amplitude fluctuations of the field, $z_j=-i(e_{Bj}-e_{Bj}^+)$ describes phase fluctuations of the field, and $\delta=8W/mc^2T_{FP}$ is the damping parameter of the mechanical system due to radiation friction.

The system of equations (B1) is linear, so the fluctuations in the input and output power are independent of each other. Assuming a free mass and slow damping of field in the cavity, we obtain the output signal in the form

$$z_{2S}=\frac{4E_0}{\tau_0\lambda_0}\int_0^t\int_0^\tau\frac{F_S(\tau_1)}{m}\,(\tau-\tau_1)d\tau_1. \tag{B2}$$

Note that the signal is also carried by the output amplitude, but is very small ($z_{2S}/y_{2S}\sim\omega_0/\omega_F$). The following expressions for the fluctuations y_j and z_j can be derived from Eq. (B1):

$$\begin{aligned}y_2=&-\frac{2\tau_0}{\omega_0T_{FP}}\left[\frac{d^4z_\Sigma}{dt^4}\,\frac{\tau_0}{2\delta}+\frac{d^3z_\Sigma}{dt^3}\,\tau_0+\frac{d^2z_\Sigma}{dt^2}\right.\\ &\left.\times\left(1+\frac{\tau_0\omega_M^2}{2\delta}-\frac{T_{FP}^2}{32\delta\tau_0}\right)+\frac{dz_\Sigma}{dt}\,\frac{T_{FP}}{4\tau_0}-z_\Sigma\,\frac{T_{FP}^2\omega_M^2}{32\tau_0\delta}\right]\\ &+\frac{d^3z_1}{dt^3}\,\frac{\tau_0}{2\delta\omega_0}\left(1+O\left(\frac{T_{FP}}{4\tau_0\omega_F}\right)\right),\end{aligned} \tag{B3}$$

$$z_2=z_\Sigma-z_1.$$

The function z_Σ contains information about the backaction noise.

The homodyne technique measures an integral of the reflected-wave quadrature component:

$$B_T=\int_0^T(g_S(t)z_2(t)+g_C(t)y_2(t))dt. \tag{B4}$$

Let us take $g_S(t)$ in the form

$$g_S(t)=\beta_4\,\frac{d^4g_C}{dt^4}+\beta_3\,\frac{d^3g_C}{dt^3}+\beta_2\,\frac{d^2g_C}{dt^2}+\beta_1\,\frac{dg_C}{dt}+\beta_0g_C, \tag{B5}$$

where

$$\beta_4=\frac{\tau_0^2}{\omega_0\delta T_{FP}},\quad \beta_3=\frac{2\tau_0^2}{\omega_0T_{FP}},$$

$$\beta_2=\frac{2\tau_0}{\omega_0T_{FP}}\left(1+\frac{\tau_0\omega_M^2}{2\delta}-\frac{T_{FP}^2}{32\delta\tau_0}\right),$$

$$\beta_1=\frac{1}{2\omega_0},\quad \beta_0=-\frac{T_{FP}\omega_M^2}{16\delta\omega_0}.$$

Then for the boundary conditions given by Eq. (8) and in approximations described above, Eq. (B4) does not contain z_Σ, so it can be transformed to Eq. (9), and the condition (B5) to Eq. (8).

*)e-mail: vyat@mol330.phys.msu.su

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Translation was provided by the Russian Editorial office.