The induced phase transition of a nonlinear oscillator interacting with a coherent electromagnetic field

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(Submitted 14 December 1995; resubmitted 12 April 1996)
Zh. Éksp. Teor. Fiz. **110**, 1186–1199 (October 1996)

We study the interaction of a nonlinear oscillator and a coherent electromagnetic field. The Goldstein–Primakoff representation is used to demonstrate that such an oscillator possesses $SU(2)$ symmetry if the anharmonicity parameter χ is negative and $SU(1,1)$ symmetry if this parameter is positive. We derive an equation for the density matrix and for the stationary case find its exact quantum solution, whose explicit form depends on the sign of χ. This dependence is caused by a new symmetry in the interaction of the nonlinear oscillator and the electromagnetic field for χ positive. Here the familiar induced phase transition of the number of excitations of the nonlinear oscillator is absent [P. D. Drummond and D. F. Walls, J. Phys. A **13**, 725 (1980)]. For negative χ the nonlinear oscillator behaves like a two-level system with a dipole–dipole interaction between the atoms. In this situation there can be an induced phase transition as a function of the field strength. We compare the quantum solution with the semiclassical one obtained on the assumption that there are no fluctuations in the number of excitations. [S1063-7761(96)00410-6]

1. INTRODUCTION

The interaction between a nonlinear oscillator and a coherent electromagnetic wave has been investigated by many researchers.[1–5] The interest in this problem is due to the possibility of building optical logical circuits on the basis of a phase transition induced by light in a nonlinear medium. The semiclassical and quantum approaches to this problem were examined in Refs. 1–3 (see also the references cited therein). In view of the existence of divergences at infinity in the complex plane in the expansion of the density matrix in the diagonal coherent states, Drummond *et al.*[1,2] were forced to discard the Glauber–Sudarshan P-representation. The convergence of the stationary solution for the density matrix was obtained via the off-diagonal P-representation. As noted by the researchers, using this representation involves certain difficulties of a fundamental nature (see, e.g., Ref. 2).

Such a problem has been solved for two-level media[6–9] whose atoms interact primarily via the dipole–dipole interaction. In such systems the square of the quasispin vector is conserved and the difficulties mentioned in Refs. 1 and 2 are absent. It would therefore be interesting to develop this method to solve the equation for the density matrix of a nonlinear oscillator interacting with a coherent laser field.

In this paper we study the interaction of a nonlinear oscillator with a coherent electromagnetic field in two cases: when the oscillator's anharmonicity parameter χ is negative and when it is positive. In the first case, $\chi<0$, we propose a new model Hamiltonian for a nonlinear oscillator interacting with a coherent electromagnetic field, a Hamiltonian that allows for higher excitation levels in comparison to the traditional Hamiltonian studied in Refs. 1 and 3. The Hamiltonian can be reduced to that of a two-level atomic system. The solutions for the stationary density matrix differ from the solutions given in Refs. 1 and 2; in our case the linear oscillator has a finite number of bound states and, as a result, the sum over the states is finite.

For $\chi>0$ the Hamiltonian of the nonlinear oscillator cannot be reduced to that of a two-level system because the number of bound states of the anharmonic oscillator is infinite. A characteristic feature of this case is a new spatial symmetry group, $SU(1,1)$ (see Refs. 10–14), for which we can introduce a conserved pseudovector by analogy with the angular momentum vector, as is done in the $SU(2)$ algebra. We suggest a model Hamiltonian for such a system and obtain an exact solution of the equation for the density matrix of a nonlinear oscillator interacting with a coherent electromagnetic field (in the stationary case). We show that this solution differs from the one with $\chi<0$, which makes it possible to consider the two cases together.

In our model of a nonlinear oscillator interacting with an electromagnetic field, unlike those developed in Refs. 1–5, the coherent laser field induces transitions between the states of the nonlinear oscillator. This difference manifests itself even in the semiclassical setting. As shown in Refs. 1–3, the behavior of the system in the presence of the laser field is bistable if the resonance detuning Δ satisfies the condition $|\Delta|>\sqrt{3}$. In our model this restriction on Δ is absent.

We show that for $\chi>0$ the quantum fluctuations in the system monotonically increase with the strength of the external electromagnetic field, which raises questions about the validity of the semiclassical approach.

2. THE SYSTEM HAMILTONIAN: THE SEMICLASSICAL APPROACH

Let us examine a one-dimensional quantum oscillator described by the Hamiltonian

$$\hat{H}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{m\omega^2x^2}{2}+\kappa x^4, \tag{1}$$

where κ is the nonlinearity parameter of the oscillator. We employ the well-known procedure and introduce the following operators:

$$\hat{a}=\frac{1}{\sqrt{2}}\left(y+\frac{\partial}{\partial y}\right),\quad \hat{a}^+=\frac{1}{\sqrt{2}}\left(y-\frac{\partial}{\partial y}\right), \tag{2}$$

with $y=\sqrt{m\omega/\hbar}x$. In terms of the operators (2) the Hamiltonian (1) in the rotating-wave approximation assumes the form

$$\hat{H}=\hbar\omega_r\hat{a}^+\hat{a}+\chi(\hat{a}^+)^2\hat{a}^2, \tag{3}$$

where

$$\omega_r=\omega\left(1+\frac{2\gamma}{\hbar\omega}\right),\quad \chi=\frac{3}{2}\frac{\hbar^2\kappa}{m^2\omega^2}.$$

In deriving the Hamiltonian (3) we have restricted our discussion to the situation in which the nonlinear term κx^4 is small compared to $m\omega^2x^2/2$. In terms of the operators (2) this assumption is equivalent to the requirement that the average number of excitations be small: $\langle\hat{a}^+\hat{a}\rangle<\hbar\omega_r/|\chi|$. Then the terms $\hat{a}^4(t)$, $[\hat{a}^+(t)]^4$, $\hat{a}^+(t)\hat{a}^3(t)$, and $[\hat{a}^+(t)]^3\hat{a}(t)$ dropped in (3) rapidly oscillate in comparison with the slowly varying terms of the form $[\hat{a}^+(t)]^2\hat{a}^2(t)$ [for instance, the term $\hat{a}(t)^+\hat{a}^3(t)\propto\exp\{-i2(\omega_r+\chi\hat{a}^+\hat{a}/\hbar)t\}\hat{a}^+\hat{a}^3$]. Since below we study the behavior of the system only for times that are long large compared to ω_r^{-1} in the Hamiltonian (1) we also drop the rapidly oscillating terms.

Using such a Hamiltonian in the single-mode approximation, we can describe many quasiparticles in a solid, say, the Wannier excitons at high excitation levels[3] and shallow impurity centers at high concentrations.

For convenience we use new quasiangular momentum operators in (3), whose explicit form and commutation relations depend on the sign of the parameter χ. If χ is negative, these operators can be chosen as follows:

$$\hat{J}^+=\frac{\hat{a}^+}{\sqrt{\sigma}}\sqrt{1-\sigma\hat{a}^+\hat{a}},\quad \hat{J}^-=\sqrt{1-\sigma\hat{a}^+\hat{a}}\frac{\hat{a}}{\sqrt{\sigma}},$$
$$\hat{J}^z=-\frac{1}{2\sigma}+\hat{a}^+\hat{a}. \tag{4}$$

They satisfy the well-known commutation relations for the $SU(2)$ algebra (see Refs. 6–10):

$$[\hat{J}^+,\hat{J}^-]=2\hat{J}^z,\quad [\hat{J}^z,\hat{J}^\pm]=\pm\hat{J}^\pm, \tag{5}$$

with $\sigma=|\chi|/\hbar\omega_r$. Expressing the boson operators $\hat{a}^+$ and $\hat{a}$ in terms of the operators (4), we arrive at the following expression for the total Hamiltonian of a nonlinear oscillator interacting with an electromagnetic field:

$$\hat{H}=\hat{H}_0+\hat{H}_{\text{int}}, \tag{6}$$

where

$$\hat{H}_0=\varepsilon\hat{J}^+\hat{J}^-+\sum_k\hbar\omega_k\hat{b}_k^+\hat{b}_k,$$

$$\hat{H}_{\text{int}}=i\hbar\sum_k\sqrt{\sigma}\varphi_k(\hat{J}^+\hat{b}_k-\hat{b}_k^+\hat{J}^-),$$

with $\varepsilon=\sigma\hbar\omega_r=|\chi|$. Here $\varphi_k\sqrt{\sigma}$ is the coupling constant of the nonlinear oscillator, and $\hat{b}_k$ and $\hat{b}_k^+$ are the annihilation and creation operators of a photon with energy $\hbar\omega_k$, polarization $\mathbf{e}_\lambda$, and momentum $\hbar\mathbf{k}$ ($k=\mathbf{k},\lambda$). When the nonlinear term in (3) is small, i.e., $\sigma\langle\hat{a}^+\hat{a}\rangle<1$, the operators $\hat{J}^+$ and $\hat{J}^-$ in $\hat{H}_{\text{int}}$ become $\hat{a}^+/\sqrt{\sigma}$ and $\hat{a}/\sqrt{\sigma}$, respectively. In this approximation the interaction Hamiltonian (6) is reduced to the well-known Hamiltonian of an oscillator interacting with an electromagnetic field:[1,2]

$$\tilde{\hat{H}}_{\text{int}}=i\hbar\sum_k\varphi_k(\hat{a}^+\hat{b}_k-\hat{b}_k^+\hat{a}). \tag{7}$$

The Hamiltonian (6) is similar to the Hamiltonian of a lumped atomic system. For this reason in the system we are considering here the angular momentum vector $\hat{J}^2$ is conserved, i.e.,

$$\hat{J}^2=(\hat{J}^z)^2+\frac{\hat{J}^+\hat{J}^-+\hat{J}^-\hat{J}^+}{2}=\text{const.} \tag{8}$$

For $\chi>0$ the operators $\hat{J}^+$ and $\hat{J}^-$ introduced above cease to be conjugate $[(\hat{J}^+)^+\neq\hat{J}^-]$. This leads to a non-Hermitian Hamiltonian. For this reason we introduce new conjugate operators

$$\hat{I}^+=i\hat{a}^+\sqrt{\frac{1+\sigma\hat{a}^+\hat{a}}{\sigma}},\quad \hat{I}^-=-i\sqrt{\frac{1+\sigma\hat{a}^+\hat{a}}{\sigma}}\hat{a},$$
$$\hat{I}^z=\frac{1}{2\sigma}+\hat{a}^+\hat{a}, \tag{9}$$

with the following commutation relations:

$$[\hat{I}^+,\hat{I}^-]=-2\hat{I}^z,\quad [\hat{I}^z,\hat{I}^\pm]=\pm\hat{I}^\pm. \tag{10}$$

In the given case the system Hamiltonian acquires the form

$$\hat{H}=\varepsilon\hat{I}^+\hat{I}^-+\sum_k\hbar\omega_k\hat{b}_k^+\hat{b}_k+\sum_k\hbar\sqrt{\sigma}\varphi_k(\hat{I}^+\hat{b}_k+\hat{I}^-\hat{b}_k). \tag{11}$$

Note that this Hamiltonian cannot be reduced to the Hamiltonian of a two level system. Hence, as we show below, it describes a system with a different symmetry, which obeys the $SU(1,1)$ algebra, and a different "spatial quantization" of the new conserved pseudovector,

$$\hat{I}^2=(\hat{I}^z)^2-\frac{\hat{I}^+\hat{I}^-+\hat{I}^-\hat{I}^+}{2}=\text{const.} \tag{12}$$

In examining the interaction with the electromagnetic field we assume that one mode of the field is in a coherent state and the rest are in the vacuum state. Let $\hat{Q}(t)$ be an operator of the nonlinear oscillator in the Heisenberg representation. Then the Heisenberg equation for this operator averaged over the initial state of the system has the form

$$\frac{\partial\langle\hat{Q}(t)\rangle}{\partial t}=\frac{i}{\hbar}\langle[\hat{H}_0(t),\hat{Q}(t)]\rangle-\sum_k\sqrt{\sigma}\varphi_k \times\langle[\hat{O}^+(t)\hat{b}(t)-\hat{b}^+(t)\hat{O}^-(t),\hat{Q}(t)]\rangle, \quad (13)$$

where

$$\hat{O}^+=\begin{cases}\hat{J}^+, & \chi<0,\\ -i\hat{J}^+, & \chi>0,\end{cases}$$

$$\hat{O}^-=(\hat{O}^+)^+, \quad (14)$$

$$\hat{O}^z=\begin{cases}\hat{J}^z, & \chi<0,\\ \hat{I}^z, & \chi>0.\end{cases}$$

We remove the boson operators of the electromagnetic field from Eq. (13) by a well-known procedure.[15–17] To this end we write the solution of the Heisenberg equation for the operators $\hat{b}_k(t)$ and $\hat{b}_k^+(t)$ in the form

$$\hat{b}_k(t)=\hat{b}_k(0)\exp(-i\omega_k t) -\sqrt{\sigma}\varphi_k\int_0^t d\tau\hat{O}^-(t-\tau)\exp(-i\omega_k\tau),$$

$$\hat{b}_k^+(t)=[\hat{b}_k(t)]^+ \quad (15)$$

and plug it into (13). After a cyclic permutation under the trace sign (see Ref. 8) we arrive at the following equation for the density matrix:

$$\frac{d\hat{\rho}(t)}{dt}=-\frac{i}{\hbar}[\varepsilon\hat{O}^+\hat{O}^--\hbar\omega_0\hat{O}^z-\hbar\Omega(\hat{O}^++\hat{O}^-),\hat{\rho}(t)] +g\{[\hat{O}^-,\hat{\rho}(t)\hat{O}^+]+[\hat{O}^-\hat{\rho}(t),\hat{O}^+]\}, \quad (16)$$

where ω_0 is the frequency of the external coherent laser field, $\Omega=\sqrt{\sigma}\varphi_{k_0}\alpha$ is the Rabi quasifrequency, $|\alpha|^2$ is the average number of photons in the coherent mode, and $g=\sigma\gamma$; $\gamma=\Sigma_k\pi\varphi_k^2\delta(\omega_r-\omega_k)$ is the linewidth of spontaneous emission of a photon by the nonlinear oscillator.

Using (16), we arrive at the equation of motion for the operators $\hat{O}^\pm$ and $\hat{O}^z$:

$$\frac{\partial\langle\hat{O}^+(t)\rangle}{\partial t}=-i\omega_0\langle\hat{O}^+(t)\rangle+2i\left[\frac{\varepsilon}{\hbar}+ig\ \mathrm{sgn}\ \chi\right] \times\langle\hat{O}^+(t)\hat{O}^z(t)\rangle-2i\Omega\ \mathrm{sgn}\ \chi\langle\hat{O}^z(t)\rangle, \quad (17a)$$

$$\frac{\partial\langle\hat{O}^z(t)\rangle}{\partial t}=i\Omega[\langle\hat{O}^+(t)\rangle-\langle\hat{O}^-(t)\rangle] -2g\ \mathrm{sgn}\ \chi\langle\hat{O}^+(t)\hat{O}^-(t)\rangle. \quad (17b)$$

Allowing for conservation of the pseudovectors (12) and (14), we can represent the correlators $\langle\hat{O}^+(t)\hat{O}^-(t)\rangle$ in the form

$$\langle\hat{O}^+(t)\hat{O}^-(t)\rangle=\mathrm{sgn}\ \chi[\langle\hat{O}^{z^2}(t)\rangle-j(j-\mathrm{sgn}\ \chi)], \quad (18)$$

with $j=1/2\sigma$. Here we have allowed for the fact that $I^2=j(j-1)$ and $J^2=j(j+1)$, which follows from the condition that initially (at $t=0$) we have $\langle\hat{a}^+\hat{a}\rangle=0$. Note that the fact that the norms of eigen-ket-vectors are positive (see Ref. 18) implies that the parameter j is either an integer or a half-integer. If $2j=1/\sigma$ is not an integer, we can slightly modify the Hamiltonian $\hat{H}_0$ in this situation by replacing the number $1/\sigma=\hbar\omega_r/|\chi|$ with its integral part $\hbar\tilde{\omega}_r/|\chi|$. The fractional remainder $\hbar(\omega_r-\tilde{\omega}_r)/|\chi|$ multiplied by $\hat{a}^+\hat{a}$ can be represented in terms of $\hat{J}^z$ or $\hat{I}^z$ [see Eqs. (8) or (9)]. Thus, the model Hamiltonian $\hat{H}_0$ generally has the form

$$\hat{H}_0=\tilde{\varepsilon}\hat{O}^+\hat{O}^-+\hbar\Delta\omega_r\hat{O}^z.$$

Here and in Eqs. (8) and (9) the parameters ε, σ, and ω_r are redefined in the following manner:

$$\tilde{\varepsilon}=\tilde{\sigma}\hbar\tilde{\omega}_r,\quad \tilde{\sigma}=|\chi|/\hbar\tilde{\omega}_r,\quad \Delta\omega_r=\omega_r-\tilde{\omega}_r,$$

with $\tilde{\sigma}^{-1}$ an integer. This modification somewhat renormalizes the parameter ω_0 in Eq. (6) by replacing it with $\tilde{\omega}_0=\omega_0+\Delta\omega_r$, where $\Delta\omega_r\ll\omega_0$. Since the above procedure of redefining the parameters ε, σ, and ω_r does not change the form of Eq. (16) for the density matrix but only changes the resonance detuning, below we drop the tilde above these parameters.

The chain of Eqs. (17) can be closed in the semiclassical approximation by ignoring the fluctuations of the operator $\hat{O}^z$, namely

$$\delta^2=\langle(\hat{O}^z)^2\rangle-\langle\hat{O}^z\rangle^2\ll(\langle\hat{O}^z\rangle-j)^2,$$

i.e.,

$$\langle\hat{O}^z(t)\hat{O}^+(t)\rangle\simeq\langle\hat{O}^z(t)\rangle\langle\hat{O}^+(t)\rangle,\quad \langle(\hat{O}^z)^2\rangle\simeq\langle\hat{O}^z\rangle^2. \quad (19)$$

Then in the stationary case we arrive at the following system of algebraic equations:

$$\begin{aligned}&ipU^++2i(q+i\ \mathrm{sgn}\ \chi)U^+Z-2iZf\ \mathrm{sgn}\ \chi=0,\\ &if(U^+-U^-)-2(Z^2-1)=0,\end{aligned} \quad (20)$$

where

$$U^+=(U^-)^+,\quad U^+=\frac{\langle\hat{O}^+\rangle}{j},\quad Z=\frac{\langle\hat{O}^z\rangle}{j},$$

$$p=\frac{2\omega_0}{\gamma},\quad q=\frac{2j\varepsilon}{\hbar\gamma},\quad f=\frac{2\Omega}{\gamma}.$$

In deriving the system (20) it was assumed that $j\gg1$. Eliminating U^+ and U^- from the system (20) yields the following equation:

$$f^2=\operatorname{sgn}\chi(Z^2-1)\left[1+\left(\frac{P}{2Z}-q\operatorname{sgn}\chi\right)^2\right], \tag{21}$$

which describes the nonlinear behavior of the average number of oscillator excitations, $\langle n\rangle=(Z-\operatorname{sgn}\chi)j$, as a function of the amplitude of the applied coherent laser field,

$$E_0=f\sqrt{\frac{2\pi\hbar\omega_0}{2V\gamma^2}},$$

where V is the quantization volume.

Let us investigate how the possibility of the function f being bistable depends on the value of the argument Z and the sign of the resonance detuning $\Delta=(\omega_0-\omega_r)/\gamma$ (or $\Delta=p/2-q$). Solving the equation $\partial(f^2)/\partial Z=0$ for $q\gg1$, which corresponds to $\omega_r\gg\gamma$, we obtain the following critical points of the f vs Z curve:

$$Z_1=\frac{P}{2q}\operatorname{sgn}\chi=\frac{\Delta+q}{q}\operatorname{sgn}\chi,\quad Z_2=\left(\frac{q+\Delta}{q}\right)^{1/3}\operatorname{sgn}\chi. \tag{22}$$

Analyzing the behavior of $f(Z)$, we see that the possibility of bistability existing in the system depends on the signs of the resonance detuning Δ and the nonlinearity parameter χ. As Eq. (21) implies, for $\chi>0$ bistability is possible only if $\Delta>0$. In the given situation the critical values of the function of the upper and lower branches of the hysteresis loop (see Fig. 1a) assume the following values:

$$f(Z_2)=\left[\left(\frac{\Delta}{q}+1\right)^{2/3}-1\right]^{1/2}\left[1+q^2\left(\left(\frac{\Delta}{q}+1\right)^{2/3}-1\right)^2\right]^{1/2},$$
$$f(Z_1)=\left[\left(\frac{\Delta}{q}+1\right)^{2}-1\right]^{1/2}. \tag{23}$$

When we have $\chi>0$ and $\Delta<0$, there is no bistability in the system. For $\chi\leqslant0$ and $\Delta<0$ there is again the possibility of bistability with the following critical values of the function f:

$$f(Z_2)=\left[1-\left(\frac{\Delta}{q}+1\right)^{2/3}\right]^{1/2}\left[1+q^2\left(\left(\frac{\Delta}{q}+1\right)^{2/3}-1\right)^2\right]^{1/2},$$
$$f(Z_1)=\left[1-\left(\frac{\Delta}{q}+1\right)^{2}\right]^{1/2}. \tag{24}$$

When $\Delta>0$ holds, there is no bistability.

A similar behavior of the Wannier-exciton number density as a function of the external field strength was obtained by Toyzawa[3] (see also Refs. 1 and 2). Expressing Z in terms of the exciton number density n_{ex}, we find that

$$Z=\operatorname{sgn}\chi+n_{ex}\beta,$$

where $\beta=\nu/\omega_{ex}$, $n_{ex}=\langle n\rangle/w$, $\nu=\chi w/\hbar$, $\hbar\omega_{ex}=\hbar\omega_r$ is the exciton energy, and w is the volume of the region where the light interacts with the excitons. The initial equation for the bistable behavior of the exciton number density as a function of the amplitude of the laser field,

$$\gamma^2f^2=2\beta n_{ex}[\gamma^2+(\Delta\gamma+n_{ex}|\nu|\operatorname{sgn}\nu)^2], \tag{21a}$$

can be obtained from (21) in the limit $n_{ex}\beta\ll1$ and $\Delta\ll1$. Without going into mathematical details, we note that the expressions for f and Z at critical points [see Eqs. (22)–(24)] coincide with similar expressions obtained in Refs. 1–3 for $|\Delta|\gg q$ and $|\Delta|\gg\sqrt{3}$. From Eq. (21a) and from the results of Refs. 1 and 3 it follows that bistability is absent for $|\Delta|\leqslant\sqrt{3}$. This result contradicts Eqs. (22), which show that when Δ and χ have the same sign, bistability is possible for arbitrary values of the resonance detuning Δ.

The lower limit in Δ is absent because the Hamiltonians (6) and (7) of the interaction with an electromagnetic field differ. Note that while the Hamiltonian (6) describes transitions between states of a nonlinear oscillator, the Hamiltonian (7) describes the interaction of an electromagnetic field with a linear oscillator.

Below we use the quantum approach to studying the problem and establish the effect of critical fluctuations on the average values of the excitation number density near the critical points.

3. QUANTUM TREATMENT

In closing the chain of Eqs. (17) in Sec. 2 we ignored the fluctuations of the operator $\hat{O}^z$. However, as noted by several authors (see, e.g., Refs. 1–9), these fluctuations become significant at the critical points of a hysteresis loop. To establish the role that these fluctuations play and hence the behavior of the system at the critical points of the hysteresis loop, below we give an exact solution of the equation for the density matrix in the stationary case and introduce the basis of eigenfunctions of the new operators of the nonlinear oscillator in the poorly studied case where $\chi>0$.

We assume that there is a ket vector $|I,p\rangle$ that is an eigen-ket-vector of the operators $\hat{I}^2$ and $\hat{I}^z$, i.e.,

$$\hat{I}^2|I,p\rangle=I^2|I,p\rangle,\quad \hat{I}^z|I,p\rangle=p|I,p\rangle. \tag{25}$$

Using this ket vector, we construct new ket vectors of the operators in such a way that $\hat{I}^{\pm}|I,p\rangle=|\pm\rangle$, where $|+\rangle$ and $|-\rangle$ are obtained after the action of the operators $\hat{I}^+$ and $\hat{I}^-$, respectively. We now employ the commutation relations (9) and find that the new ket vectors $|+\rangle$ and $|-\rangle$ satisfy the equations

$$\hat{I}^2|+\rangle=I^2|+\rangle,\quad \hat{I}^2|-\rangle=I^2|-\rangle, \tag{26a}$$

$$\hat{I}^z|+\rangle=(p+1)|+\rangle,\quad \hat{I}^z|-\rangle=(p-1)|-\rangle. \tag{26b}$$

The positivity of the norms of the ket vectors $|+\rangle$ and $|-\rangle$,

$$\langle+|+\rangle=p(p+1)-I^2\geqslant0,\quad \langle-|-\rangle=p(p-1)-I^2\geqslant0, \tag{27}$$

implies that either $p\geqslant\frac{1}{2}+(I^2+\frac{1}{4})^{1/2}$ or $p\leqslant-\frac{1}{2}-(I^2+\frac{1}{4})^{1/2}$. But the definition of the operator $\hat{I}^z=j+\hat{a}^+\hat{a}$ suggests that p cannot be negative, i.e., $p\geqslant\frac{1}{2}+(I^2+\frac{1}{4})^{1/2}$. On the other hand, the lowering of the eigenvalue of $\hat{I}^z$ must terminate at $p_{\min}=j$, which corresponds to the absence of excitation in the system ($\langle\hat{a}^+\hat{a}\rangle=0$). If for the given value $p_{\min}=j$ we require that the norm of the ket vector $|-\rangle^{\min}=\hat{I}^-|I,j\rangle$ be zero, the (27) yield $I^2=j(j-1)$. Hence it is convenient to denote the ket vector of the ground state as follows: $|I,j\rangle=|j,j\rangle$. Let us find the ket vectors of the other (excited) states of the nonlinear oscillator. To this end we successively apply the operator $\hat{I}^+$ to the ground-state ket vector $|j,j\rangle$. Thus, the ket vector of the state $|j,p\rangle$ can be obtained from that of the ground state by applying the vector $\hat{I}^+$ $p-j$ times, i.e.,

$$|j,p\rangle=C_{jp}(\hat{I}^+)^{p-j}|j,j\rangle. \tag{28}$$

Requiring that the ket vectors $|j,j\rangle$ and $|j,p\rangle$ have unit norms and using

$$(\hat{I}^-)^k(\hat{I}^+)^k=\prod_{s=1}^{k}[(\hat{I}^z-s+1)(\hat{I}^z-s)-\hat{I}^2],$$

we find

$$C_{jp}=\left(\frac{(2j-1)!}{(p-j)!(p+j-1)!}\right)^{1/2}, \tag{29}$$

where $p=j,\ j+1,\ j+2,\dots,j+n,\dots$. The result of the operators $\hat{I}^+$, $\hat{I}^-$, and $\hat{I}^z$ acting on the ket vector $|j,p\rangle$ is determined by the following relationships:

$$\hat{I}^+|j,p\rangle=\sqrt{(p-j+1)(p+j)}|j,p+1\rangle,$$

$$\hat{I}^-|j,p\rangle=\sqrt{(p+j-1)(p-j)}|j,p-1\rangle,$$

$$\hat{I}^z|j,p\rangle=p|j,p\rangle.$$

Since for $\chi<0$ the square of angular momentum is an integral of motion [Eq. (12)] and the commutation relations (5) coincide with those for the angular momentum components $\hat{J}^z$, $\hat{J}^+$, and $\hat{J}^-$, the normalized ket vectors of this operator have the following form:[10]

$$|j,m\rangle=\left[\frac{(j+m)!}{(2j)!(j-m)!}\right]^{1/2}(\hat{J}^-)^{j-m}|j,j\rangle,\quad -j\leqslant m\leqslant j. \tag{30}$$

The result of the operators $\hat{J}^+$, $\hat{J}^-$, and $\hat{J}^z$ acting on the ket vector (30) is determined by the relationships[10]

$$\hat{J}^+|j,m\rangle=\sqrt{(j-m)(j+m+1)}|j,m+1\rangle,$$

$$\hat{J}^-|j,m\rangle=\sqrt{(j+m)(j-m+1)}|j,m-1\rangle,$$

$$\hat{J}^z|j,m\rangle=m|j,m\rangle.$$

Note that for $\chi<0$ the requirement that the parameter j be a half-integer implies that the bound-state spectrum of the nonlinear oscillator is bounded, i.e., $|m|\leqslant j$, and coincides with the angular-momentum spectrum. But for $\chi>0$ the spectrum is not bounded above, i.e., $p\to\infty$. The stationary equation for the density matrix [see Eq. (16)] for $\chi>0$ has the following form:

$$-\frac{i}{\hbar}[\varepsilon\hat{I}^+\hat{I}^- -\hbar\omega_0\hat{I}^z+i\hbar\Omega(\hat{I}^+-\hat{I}^-),\hat{\rho}_s]$$

$$+g\{[\hat{I}^-\hat{\rho}_s,\hat{I}^+]+[\hat{I}^-,\hat{\rho}_s\hat{I}^+]\}=0.$$

Employing the methods developed in Refs. 6–9 to solve such equations, we seek the exact solution on the assumption that the stationary density matrix $\hat{\rho}_s$ can be written as[8]

$$\hat{\rho}_s=A^{-1}F(\hat{I}^-)F^+(\hat{I}^+),\quad A=\mathrm{Tr}\{F(\hat{I}^-)F^+(\hat{I}^+)\}.$$

Here $F(\hat{I}^-)$ and $F^+(\hat{I}^+)$ are functions of $\hat{I}^-$ and $\hat{I}^+$ defined in the following manner:

$$F(\hat{I}^-)=\sum_{n=0}^{\infty}C_n(\hat{I}^-)^n, \tag{31}$$

where the C_n are the coefficients of the expansion of the C-number function $F(x)$ in a Taylor series. Using the commutation relations (10), we can show that

$$\hat{I}^zF(\hat{I}^-)=-F(\hat{I}^-)\hat{I}^z-\left[\hat{I}^+,\int F(\hat{I}^-)d\hat{I}^-\right],$$

$$[\hat{I}^+\hat{I}^-,F(\hat{I}^-)F(\hat{I}^+)]=[\hat{I}^+,\hat{I}^-F(\hat{I}^-)]F(\hat{I}^+)-\text{H.c.}, \tag{32}$$

where, according to (31),

$$\int F(\hat{I}^-)d\hat{I}^-=\sum_{n=0}^{\infty}\frac{C_n(\hat{I}^-)^{n+1}}{n+1}.$$

Allowing for (32), we can write Eq. (25) in the form

$$[\hat{I}^+,G(\hat{I}^-)]F^+(\hat{I}^+)+\text{H.c.}=0, \tag{33}$$

where

$$G(\hat{I}^-)=\hat{I}^-F(\hat{I}^-)(-iq-1)-i\nu_0\int F(\hat{I}^-)d\hat{I}^-+vF(\hat{I}^-), \tag{34}$$

$$\nu_0=2j\omega_0/\gamma,\quad v=2j\Omega/\gamma.$$

Equation (33) implies that $i[\hat{I}^+,G(\hat{I}^-)]F^+(\hat{I}^+)$ is a Hermitian operator. This is true if $\hat{I}^+$ commutes with $G(\hat{I}^-)$ or $i[\hat{I}^+,G(\hat{I}^-)]=F(\hat{I}^-)$ holds to within a real factor, which in view of the commutation relations (9) cannot be true. Hence the operator $G(\hat{I}^-)$ must commute with $\hat{I}^+$ to within a constant factor found from the normalization condition

$$\hat{I}^-F(\hat{I}^-)(1+iq)+i\nu_0\int F(\hat{I}^-)d\hat{I}^- -vF(\hat{I}^-)=\text{const}. \tag{35}$$

Differentiating (35) with respect to $\hat{I}^-$ and solving the resulting equation, we find that

$$F(\hat{I}^-)=\frac{C}{(\hat{I}^- -id)^{1+\xi}}, \tag{36}$$

where

$$d=-\frac{iv}{1+iq},\quad \xi=\frac{i\nu_0}{1+iq}.$$

Then the stationary density matrix is

$$\hat{\rho}_1^s=|C|^2(\hat{I}^- -id)^{-(1+\xi)}(\hat{I}^+ +id^*)^{-(1+\xi^*)}$$

$$=\lim_{n_0\to\infty}D_1^{-1}\sum_{k=0}^{n_0}\sum_{l=0}^{n_0}i^{k-l}d^{-k}(d^*)^{-l}\frac{\Gamma(1+\xi+k)}{k!\Gamma(1+\xi)}$$

$$\times\frac{\Gamma(1+\xi^*+l)}{l!\Gamma(1+\xi^*)}(\hat{I}^-)^k(\hat{I}^+)^l, \tag{37}$$

where D_1 is a normalization factor such that $\mathrm{Tr}(\hat{\rho}^s)=1$, $\Gamma(z)$ is the ordinary gamma function, and

$$D_1=\sum_{l=0}^{n_0}|d|^{-2l}\frac{\Gamma(1+\xi+l)}{\Gamma(1+\xi)}\frac{\Gamma(1+\xi^*+l)}{(1+\xi^*)}C_l,$$

$$C_l=\sum_{m=0}^{n_0-l}\binom{m+l}{l}\binom{2j+m+l-1}{l}. \tag{38}$$

The number of excitations n_0 of the nonlinear oscillator is restricted by the condition that the number of discarded terms in the expression for the oscillator potential energy be small, i.e., $\langle\hat{a}^+\hat{a}\rangle<\hbar\omega_r/|\chi|$ [see Eq. (3)]. But because of the rapid convergence of the series (37) as a function of the applied external laser field [see Eqs. (42) and (43) below], formally the sum can be extended to infinity. Indeed, here we are interested in the system's behavior in the range of critical points corresponding to fairly low excitation levels $\langle n\rangle_i=\{Z_i-\operatorname{sgn}\chi\}<\hbar\omega_r/|\chi|$, with $i=1,2$ [see Eqs. (22)].

For $\chi<0$ the equation for the stationary density matrix acquires the form

$$0=-\frac{i}{\hbar}[\varepsilon\hat{J}^+\hat{J}^--\hbar\omega_0\hat{J}^z-\hbar\Omega(\hat{J}^++\hat{J}^-),\hat{\rho}_s]$$
$$+g\{[\hat{J}^-\hat{\rho}_s,\hat{J}^+]+[\hat{J}^-,\hat{\rho}_s\hat{J}^+]\}. \tag{39}$$

Applying the above method and the corresponding commutation relations (5), we arrive at the following expression for $\hat{\rho}_s$:

$$\hat{\rho}_2^s=|C|^2(\hat{J}^--d)^{-(1-\xi)}(\hat{J}^+-d^*)^{-(1-\xi^*)}$$
$$=D_2^{-1}\sum_{m=0}^{2j}\sum_{l=0}^{2j}(-1)^{m+l}d^{-m}(d^*)^{-l}$$
$$\times\frac{\Gamma(\xi)\Gamma(\xi^*)}{m!\Gamma(\xi-m)l!\Gamma(\xi^*-l)}(\hat{J}^-)^m(\hat{J}^+)^l, \tag{40}$$

where the normalization factor D_2 is specified by the following expression:

$$D_2=\sum_{l=0}^{2j}|d|^{-2l}\frac{\Gamma(\xi)\Gamma(\xi^*)}{\Gamma(\xi-l)\Gamma(\xi^*-l)}\frac{(2j+l+1)!}{(2j-l)!(2l+1)!}. \tag{41}$$

Let us calculate the average value $\langle n\rangle$ of the number of excitations. For $\chi>0$ the definition of $\hat{I}^z$ [see (9)] implies that $\langle n\rangle=\mathrm{Tr}\{\hat{r}_1^s\hat{I}^z\}-j$. Using the expression (37) for the density matrix and the derived system of functions, we get

$$\langle n\rangle=\lim_{n_0\to\infty}D_1^{-1}\sum_{l=0}^{n_0}|d|^{-2l}\frac{\Gamma(1+\xi+l)\Gamma(1+\xi^*+l)}{\Gamma(1+\xi)\Gamma(1+\xi^*)}\Phi_{l+1}, \tag{42}$$

where

$$\Phi_{l+k}=\sum_{m=0}^{n_0-(l+k)}\frac{(m+l+k)!(2j+m+l-1+k)!}{m!(2j+m-1+k)!},$$

with $k=1,2,3,\ldots$. To examine the role that quantum fluctuations play at critical points we must calculate the average value of the square of the number of excitations in the system, $\langle n\rangle^2=\langle(\hat{I}^z-j)^2\rangle$. We can easily see that

$$\langle n^2\rangle=\lim_{n_0\to\infty}D_1^{-1}\sum_{l=0}^{n_0}|d|^{-2l}\frac{\Gamma(1+\xi+l)\Gamma(1+\xi^*+l)}{\Gamma(1+\xi)\Gamma(1+\xi^*)(l!)^2}$$
$$\times\Phi_{l+2}+\langle n\rangle. \tag{43}$$

Using the expressions (43) and (42) for $\langle n^2\rangle$ and $\langle n\rangle$, we can calculate the square of the fluctuations of the number of excitations, $\delta^2=\langle n^2\rangle-\langle n\rangle^2$.

For $\chi<0$ the expressions for the average number of excitations, $\langle n\rangle$, and for $\langle n^2\rangle$, respectively, are

$$\langle n\rangle=D_2^{-1}\sum_{l=0}^{2j-1}d^{-2l}\frac{\Gamma(\xi)\Gamma(\xi^*)}{\Gamma(\xi-l)\Gamma(\xi^*-l)}\frac{(l+1)(2j+l+1)!}{(2j-l-1)!(2l+2)!}, \tag{44}$$

$$\langle n^2\rangle=D_2^{-1}\sum_{m=0}^{2j-2}d^{-2m}\frac{\Gamma(\xi)\Gamma(\xi^*)}{\Gamma(\xi-m)\Gamma(\xi^*-m)}$$
$$\times\frac{(m+2)(m+1)(2j+m+1)!}{(2j-m-2)!(2m+3)!}+\langle n\rangle. \tag{45}$$

Figures 1a and b depict the dependence of the number of excitations of the oscillator and the second-order coherence function $g^{(2)}=(\delta^2+\langle n\rangle^2)/\langle n\rangle^2$ on the strength of the applied coherent electromagnetic field when $\chi<0$. Clearly, instead of the bistability obtained in the semiclassical approximation, we have a quantum jump in the number of excitations of the oscillator as the strength of the coherent electromagnetic field increases (Fig. 1a, the solid curve), and the coherence function $g^{(2)}$ reaches its maximum value in the vicinity of the jump (and so does δ^2; see Fig. 1b). For a detailed explanation of this jump, we represent (44) in the form of a polynomial in powers of the strength of the applied field. To this end we multiply (44) and (41) by d^{4j} and perform the change of variables $m=2j-l$ in these sums. Then (44) and (41) become

$$\langle n\rangle=\tilde{D}_2^{-1}\sum_{m=1}^{2j}d^{2m}\frac{\Gamma(\xi)\Gamma(\xi^*)}{\Gamma(\xi-2j+m)\Gamma(\xi^*-2j+m)}$$
$$\times\frac{(2j-m+1)(4j-m+1)!}{(m-1)!(4j-2m+2)!}, \tag{44a}$$

$$\tilde{D}_2=\sum_{m=0}^{2j}|d|^{2m}\frac{\Gamma(\xi)\Gamma(\xi^*)}{\Gamma(\xi-2j+m)\Gamma(\xi^*-2j+m)}$$
$$\times\frac{(4j-m+1)!}{m!(4j-2m+1)!}. \tag{41a}$$

Using the definition of the function $\Gamma(x)$, we obtain

$$\frac{\Gamma(\xi)\Gamma(\xi^*)}{\Gamma(\xi-2j+m)\Gamma(\xi^*-2j+m)}$$
$$=(\xi-1)\cdots(\xi-2j+m)(\xi^*-1)\cdots(\xi^*-2j+m)$$
$$=[(\mathrm{Re}\,\xi-1)^2+(\mathrm{Im}\,\xi)^2][(\mathrm{Re}\,\xi-2)^2+(\mathrm{Im}\,\xi)^2]$$
$$\times\cdots[(\mathrm{Re}\,\xi-2j+m)^2+(\mathrm{Im}\,\xi)^2]. \tag{46}$$

Going back to the definition $\xi=i\nu_0/(1+iq)$, we find that for a $q\gg1$ whose value corresponds to $\omega_r>\gamma$ we have $\mathrm{Re}\,\xi\simeq\nu_0/q$, with the result that

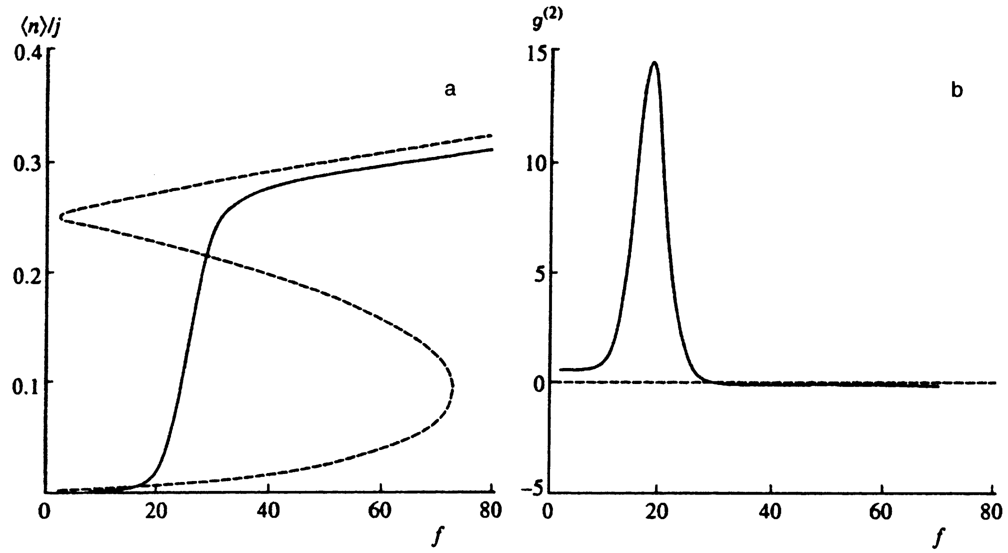

FIG. 1. (a) The dependence of the number of excitations in the system on the normalized external field strength f for $\chi<0$ at $j=12$, $P=1500$, and $q=1000$ in the classical (the dashed curve) and quantum (solid curve) settings. (b) The dependence of the second-order correlation function $g^{(2)}$ on the normalized external field strength f for $\chi<0$ at $j=12$, $P=1500$, and $q=1000$.

$$\mathrm{Re}\ \xi-2j=2j\frac{\omega_0-\omega_r}{\omega_r}=2j\frac{\Delta}{\omega_r/\gamma}. \tag{47}$$

As Eqs. (47) and (46) imply, when the resonance detuning is negative, or $\Delta\mathrm{Re}\xi-2j<0$, the power series in (46) can become truncated; the same is true of (44a) and (41a) at $m^*=-(\mathrm{Re}\ \xi-2j)$ (if $\mathrm{Re}\ \xi-2j$ is an integer). In this situation, with the field strength increasing, the last nonzero term in the expansion in powers of the field strength, i.e., the (m^*-1)st term, begins to dominate. Hence, the expression (44) tends to a finite number determined by the ratio of the coefficients of $d^{2(m^*-1)}$ in (44a) and (41a), i.e., $\langle n\rangle\simeq m^*-1=\mathrm{const}$, which is confirmed by numerical calculations (see Fig. 1a).

For $\chi>0$ varying the strength of the applied electromagnetic field leads to no jump in the number of excitations. This is due to the absence of truncation of the series analogous to (46) for any resonance detuning in (42) and (43). Clearly, under the previous assumptions, the real part of the argument contains the sum $2j\omega_0/\omega_r+l$, with the result that not a single resonance detuning at a difference analogous to (47) can exist. More than that, numerical calculations show (see Figs. 2a and b) that the square of the relative fluctuations of the number of excitations, $\delta^2/\langle n\rangle^2$, remains much larger than unity for low excitation numbers and high strengths of the external electromagnetic field. This suggests that in an external coherent field the oscillator becomes

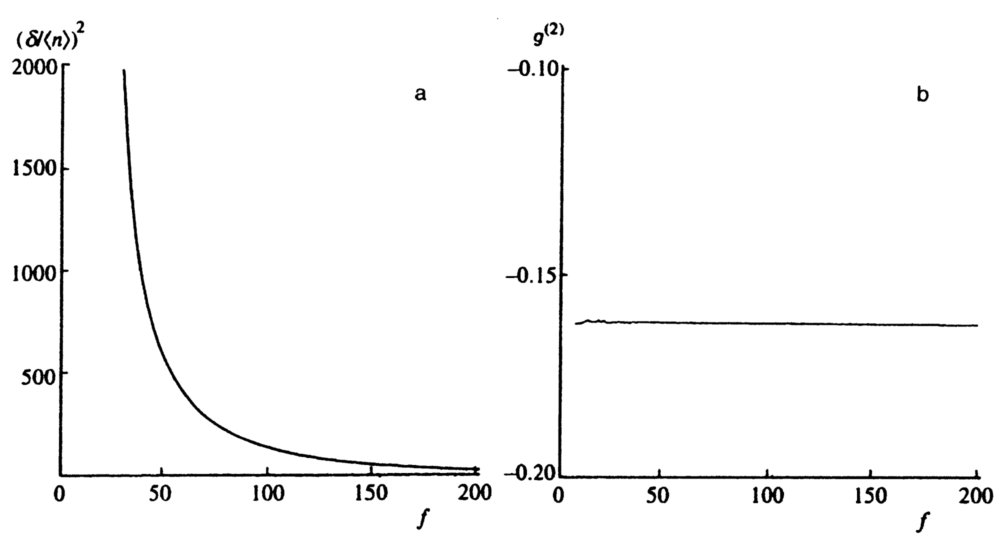

FIG. 2. (a) The dependence of the square of the relative fluctuations, $(\delta/\langle n\rangle)^2$, on the normalized external field strength f for $\chi>0$ at $j=9$, $n_0=12$, $P=2500$, and $q=1000$. (b) The dependence of the second-order correlation function $g^{(2)}$ on the normalized external field strength f for $\chi>0$ at $j=9$, $n_0=12$, $P=2500$, and $q=1000$.

weakly excited and that the fluctuations of the number of excitations remain much larger than unity, so that they cannot be ignored in the decoupling of Eqs. (17). As a result, the classical approach becomes invalid.

4. CONCLUSION

We see that the behavior of an anharmonic oscillator in a coherent electromagnetic field differs considerably for positive and negative values of the nonlinearity parameter χ. The difference is caused by the presence of a new symmetry in the interaction between the nonlinear oscillator and the electromagnetic field. As a result, there is no phase transition when $\chi > 0$. This case was neither examined or identified in Ref. 1.

When the anharmonicity is negative, as we move away from the phase transition point the "classical" behavior of the number of excitations of the oscillator as a function of the applied electromagnetic field almost coincided with the "quantum" behavior, and the quantum fluctuations of this quantity play an important role near the phase transition point. Such a description of the anharmonicity oscillator becomes invalid when the nonlinearity parameter becomes positive. In this situation the absolute fluctuations start to grow from the very beginning and do not reach a maximum for any value of the strength of the external field. Thus, the semiclassical decoupling of the correlators is invalid for any laser field strength.

Such nonlinear interactions can be detected in nonlinear media. When the excitation of the exciton subsystem is high,[4] a nonlinear excitonic mode with a positive or negative nonlinearity parameter χ can be realized in media with predominant repulsion or attraction between the excitons.

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Translated by Eugene Yankovsky