Phase resonance in the reflection of acoustic waves by a system of piezocrystalline layers separated by cladding layers with screening properties

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We discuss the reflection of a transverse elastic wave incident from a substrate onto a system consisting of a periodic system of piezocrystalline layers with identical material properties, one side of which is bounded by the substrate and the other by a mechanically free surface. The layers are separated by superconducting or metallized cladding layers, depending on whether the layers are made of piezomagnetic or piezoelectric material, respectively. Because these cladding layers screen the quasistatic magnetic or electric fields that accompany the elastic wave, they cause a reflection of the wave at the layer boundaries that is weak due to the smallness of the magneto- or electromechanical coupling parameters. The presence of the mechanically free surface ensures total reflection at any angle of incidence, so that the amplitude of the reflection coefficient is always unity. This allows us to study the effects of diffraction on the phase of the reflection coefficient for a wave reflected from the boundary of the substrate–layered structure against this constant (unity) amplitude background. We have analyzed the spectral dependence of the phase change of the reflected wave due to multiple reflection at the boundaries between layers, which adds to the path length of the wave. We show that this diffraction-induced phase shift has a characteristic resonance behavior, attaining a maximum value near Bragg-forbidden zones that is roughly twice the increase in the maximum background level. We discuss several possible wave effects based on the concept of phase resonance. [S1063-7761(96)01209-7]

1. INTRODUCTION

The authors of Ref. 1–3 investigated theoretically the properties of controllable Bragg diffraction of acoustic waves in a system made up of identical piezoelectric or piezomagnetic layers separated by infinitely thin cladding layers with metallic or superconducting properties, respectively (i.e., cladding layers whose thickness is small compared to the wavelength of the sound being used). It is obvious that the reflection of a piezoactive acoustic wave by such a metallized (or superconducting) cladding layer is solely a consequence of the latter's ability to screen the quasistatic electric (magnetic) field that accompanies the propagation of sound in the adjacent piezoelectric (piezomagnetic) layers. If we are discussing a multilayer structure made of such layers placed between two substrates, then the reflection coefficient r from each individual boundary is small, due to the smallness of the electro- or magnetomechanical coupling parameter q^2 (see Ref. 4). This determines such features of the acoustic diffraction as the enhanced sharpness of Bragg reflection resonances, their amplitude modulation, and the extinction effect. On the other hand, these properties can be controlled if we take advantage of the possibility of "shutting down" the reflection at a superconducting cladding layer by converting the latter to its normal state (i.e., "turning off" its screening action), or by regulating the linear electromechanical coupling of a paraelectric (centrosymmetric) layer by applying an external electric field.

In several previous papers[1–3] we investigated the properties of the amplitude of the reflection coefficient $|R|$ from a piezocrystalline multilayer structure. However, it is well known that when the conditions of Bragg resonance are satisfied, the diffracted wave also undergoes a rapid phase change (by π) in the spectral intervals corresponding to maxima of the quantity $|R|$. Note that the method of standing waves in x-ray optics is based on this phenomenon; for example, see Refs. 5–7. Obviously, such effects should also occur in a system of piezocrystalline layers of the sort we are discussing here when the latter are placed between two substrates.

However, it is more interesting to discuss the resonance properties of the phase of the reflection coefficient under conditions where there is no amplitude resonance at all. This very situation arises if one of the substrates is taken away, i.e., if the layered structure under discussion is bounded on one side by a substrate and on the other by a mechanically free surface. Of course, for a shear wave incident from the substrate onto the multilayer system the amplitude of the reflection coefficient equals unity at any frequency and angle of incidence (i.e., the boundary between the layered medium and vacuum plays the role of an "mirror" for sound). A fundamental question then arises: will the spectral dependence of the phase of the resulting diffracted wave have Bragg-like resonances in this case, i.e., can we create a "pure" phase resonance against a background of identically constant (unity) value of $|R|$?

Al'shits *et al.* showed in Ref. 8 that a transverse acoustic wave reflected from a mechanically free boundary of a piezocrystal acquires a phase shift due to excitation of a surface electric- or magnetic-field wave, and that in a geometry not too close to grazing, this phase shift is proportional to the parameter $q^2 \ll 1$. Analogously, we showed in Ref. 4 that a

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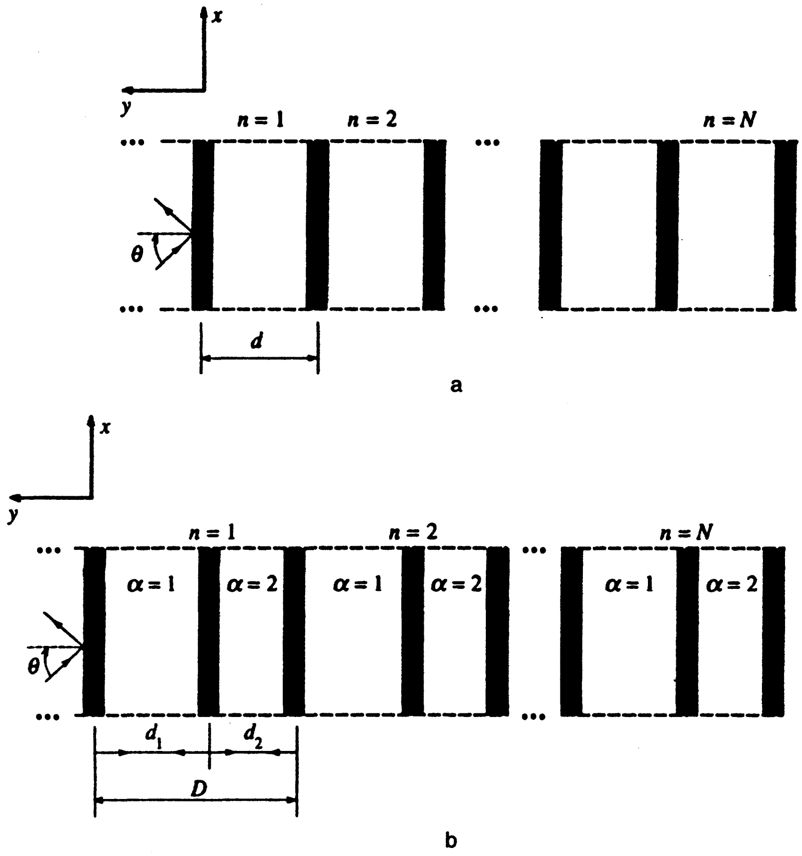

FIG. 1. Geometry of the problem: equidistant (a) and nonequidistant (b) cases.

wave passing through a single thin cladding layer of screening material between identical piezocrystalline media also acquires a small phase shift of order q^2.

If, however, a transverse wave is incident from an "input" substrate onto a system consisting of identical piezocrystalline layers and screening cladding layers with a mechanically free surface as its "output," the reflection coefficient obviously has the form $R=\exp(i\Phi)$, where Φ includes the "trivial" phase advance associated with the path length of the wave making a round trip through the layered system in the forward and backward directions, and an additional diffraction-induced phase shift $\Delta\Phi$ due to contributions from multiple reflections and refractions. We expect that the quantity $\Delta\Phi$, unlike the phase shift due to an individual cladding layer, will attain rather large values when the number of layers is large enough, and that its spectral dependence will exhibit resonant behavior. In this case, the option of "turning on" and "turning off" the reflection at boundaries between the layers mentioned above should make it possible to discontinuously change the phase Φ by the value of the diffraction contribution $\Delta\Phi$.

For definiteness, we illustrate this effect and derive various expressions for the case of piezomagnetic layers with no piezoelectric properties (the corresponding magnetic symmetry classes are listed in Ref. 9), and then discuss the changes that must be made in these expressions in order to treat the case of a piezoelectric layered structure.

2. STARTING RELATIONS

We consider a periodic structure consisting of N identical layers of piezomagnetic hexagonal crystal separated by thin superconducting cladding layers. On one side the layered system is adjacent to a semi-infinite substrate, while on the other side it is bounded by a mechanically free surface (i.e., a vacuum boundary). Coupled waves of elastic displacement $\mathbf{u}$ and quasistatic magnetic induction $\mathbf{B}$ and potential F are described by the standard equations

$$\rho \frac{\partial^2}{\partial t^2} \mathbf{u}=\nabla \hat{\sigma}, \quad \nabla \mathbf{B}=0, \tag{1}$$

where

$$\sigma_{ij}=c_{ijkl}\nabla_l u_k+m_{kij}\nabla_k F, \quad B_i=4\pi m_{ikl}\nabla_l u_k-\mu_{ij}\nabla_j F, \tag{2}$$

$\hat{\sigma}$ is the mechanical stress, ρ is the density, $\hat{c}$ are the elastic constants, $\hat{m}$ are the piezomagnetic moduli, and $\hat{\mu}$ is the magnetic permeability. The elastic displacement and force are continuous at the interlayer boundaries, while the normal component of the magnetic induction goes to zero at a cladding layer in the s-state by virtue of the Meissner effect (when the thickness of the superconducting film is much smaller than the wavelength but much larger than the London penetration depth).

We assume that an elastic wave incident from the substrate excites a shear wave in the layered system with $\mathbf{u}\|\mathbf{z}$, propagating in the xy plane (Fig. 1). In this case, according to (2), the elastic force $\mathbf{f}=\hat{\sigma}\mathbf{n}(\mathbf{n}\|\mathbf{y})$ is directed along z. The complete acoustomagnetic wave field in the nth layer consists of a superposition of incident and reflected waves, along with a pair of nonuniform modes whose parameters are given by Eqs. (1), (2):

$$\begin{pmatrix} u_z(y) \\ ik_x^{-1}f_z(y) \\ ik_x^{-1}B_y(y) \end{pmatrix}^{(n)} = b_i^{(n)}\begin{pmatrix} 1 \\ \kappa_i \\ 4\pi m_{14} \end{pmatrix}\exp[-ik_y(y+nd)] + b_r^{(n)}\begin{pmatrix} 1 \\ \kappa_r \\ 4\pi m_{14} \end{pmatrix}\exp[ik_y(y+nd)] + b_s^{(n)}\begin{pmatrix} 0 \\ \gamma \\ i\mu_{11} \end{pmatrix}\exp[-k_x(y+nd)] + b_{s'}^{(n)}\begin{pmatrix} 0 \\ \gamma^* \\ -i\mu_{11} \end{pmatrix}\exp[k_x(y+nd)], \tag{3}$$

where an overall factor of $\exp[i(k_x x-\omega t)]$ has been omitted; the labels i, τ and s, s' refer to the incident, reflected, and nonuniform modes, respectively, the asterisk denotes complex conjugation, d is the layer thickness, and

$$k_x=\omega\sqrt{\rho/c_{44}}\sin\theta, \quad k_y=k_x\cot\theta,$$
$$\gamma=-m_{14}-im_{15},$$
$$\kappa_i=\bar{c}_{44}\cot\theta-4\pi m_{14}m_{15}/\mu_{11},$$
$$\kappa_r=-\bar{c}_{44}\cot\theta-4\pi m_{14}m_{15}/\mu_{11},$$
$$\bar{c}_{44}=c_{44}+4\pi m_{15}^2/\mu_{11}. \tag{4}$$

The boundary conditions at $y=-nd$ that relate the wave fields in the nth and $(n+1)$th layers are written in the form

$$[u_z(-nd)]^{(n)}=[u_z(-nd)]^{(n+1)},$$
$$[f_z(-nd)]^{(n)}=[f_z(-nd)]^{(n+1)},$$
$$[B_y(-nd)]^{(n)}=0, \quad [B_y(-nd)]^{(n+1)}=0. \tag{5}$$

Furthermore, the free-surface condition holds at the boundary $y=-Nd$ of the layered system:

$$[f_z(-Nd)]^{(N)}=0, \tag{6}$$

along with an additional boundary condition for the magnetic field that depends on the phase state (superconducting or normal) of the free surface.

As shown in Ref. 1–3, by substituting Eq. (3) into (5) and eliminating the amplitudes of the surface modes by using the boundary conditions for the magnetic induction we can express the amplitudes of the incident and reflected waves in neighboring layers in terms of one another:

$$\begin{pmatrix} b_i \\ b_r \end{pmatrix}^{(n)} = \hat{W}\begin{pmatrix} b_i \\ b_r \end{pmatrix}^{(n+1)}, \tag{7}$$

where $\hat{W}$ is a matrix propagator with components

$$W_{11}=W_{22}^*=(1-p)e^{-ik_y d}, \quad W_{12}=W_{21}^*=p^*e^{ik_y d}, \tag{8}$$

$$p=\frac{iq_\theta^2(\cos k_y d-\cosh k_x d)}{\sinh k_x d+q_\theta^2\sin k_y d}, \tag{9}$$

$$q_\theta^2=q_M^2\tan\theta, \quad q_M^2=4\pi m_{14}^2/\mu_{11}\bar{c}_{44}. \tag{10}$$

In what follows, we will avoid tedious calculations by making the short-wavelength approximation, which is natural for the Bragg problem, corresponding to the condition

$$e^{k_x d}\gg 1. \tag{11}$$

In this case Eq. (9) simplifies:

$$p\approx -iq_\theta^2. \tag{12}$$

It is also convenient to assume that the substrate is made of the same piezomagnetic crystal as the layers (in the formulation we will use here, the sixfold axis is parallel to the z axis, and the boundary between the substrate and the layered system has superconducting properties). Then, according to Refs. 1–3, within the framework of the short-wavelength approximation (11) the transfer matrix from the semi-infinite substrate to the first layer precisely coincides with the propagator $\hat{W}$. Consequently, the amplitudes $b_i^{(0)}$ $b_r^{(0)}$ for the incident and reflected waves in the substrate are related to the amplitudes $b_i^{(N)}$ $b_r^{(N)}$ in the Nth layer by

$$\begin{pmatrix} b_i \\ b_r \end{pmatrix}^{(0)} = \hat{W}^N\begin{pmatrix} b_i \\ b_r \end{pmatrix}^{(N)}. \tag{13}$$

The matrix that equals the Nth power of the propagator $\hat{W}$ ($\det\hat{W}=1$) satisfies the standard relations

$$(\hat{W}^N)_{11}=(\hat{W}^N)_{22}^*=W_{11}\frac{\sin(NKd)}{\sin(Kd)}-\frac{\sin[(N-1)Kd]}{\sin(Kd)},$$
$$(\hat{W}^N)_{12}=(\hat{W}^N)_{21}^*=W_{12}\frac{\sin(NKd)}{\sin(Kd)}, \tag{14}$$

where K is the Bloch wave number, which specifies the eigenvalues $\exp(\pm iKd)$ of the matrix $\hat{W}$, so that $\cos(Kd)=(W_{11}+W_{11}^*)/2$. For this case, taking (8) into account we obtain

$$\cos(Kd)=\cos(k_y d)+\frac{q_\theta^2\sin(k_y d)[\cosh(k_x d)-\cos(k_y d)]}{\sinh(k_x d)+q_\theta^2\sin(k_y d)} \approx\cos(k_y d)+q_\theta^2\sin(k_y d), \tag{15}$$

where the approximate equality is valid under condition (11).

According to (3), near the points πl ($l=1,2,3$), when $|k_y d-\pi l|\sim q_\theta^2$ this equation reduces to:

$$(k_y d-\pi l)^2-2q_\theta^2(k_y d-\pi l)-(Kd-\pi l)^2=0. \tag{16}$$

Setting $Kd=\pi l$ in (16) determines the coordinates

$$(k_y d)^{(1)}=\pi l, \quad (k_y d)^{(2)}=\pi l+2q_\theta^2 \tag{17}$$

of the boundaries of the so-called "forbidden zones," within which the Bloch wave number $K=K(k_y)$ takes on complex values $Kd=\pi l+iK'$. Far from the forbidden zones, to first order in q_θ^2 the approximate equation (15) implies the simple relation

$$k_y d \simeq Kd + q_\theta^2. \tag{18}$$

3. RESULTS AND DISCUSSION

According to (13), (14), we can cast the reflection coefficient R at the "input" of the layered structure ($y=+0$, see (3)) in the form

$$R=\frac{b_r^{(0)}}{b_i^{(0)}}=\frac{e^{i\delta}(\hat{W}^N)_{11}^*+(\hat{W}^N)_{12}^*}{(\hat{W}^N)_{11}+e^{i\delta}(\hat{W}^N)_{12}}=e^{i\Phi}, \tag{19}$$

where $e^{i\delta}=b_r^{(N)}/b_i^{(N)}$ is the reflection coefficient from the free boundary of the layered system at $y=-Nd$ (see Fig. 1). Obviously, δ depends on the phase state (superconducting or normal) of this boundary. If a superconducting cladding layer is deposited on the latter ($\delta \equiv \delta_s$), i.e., the free-surface condition (6) is augmented by the requirement that the magnetic induction be screened, i.e., $[B_y(-Nd)]^{(N)}=0$. Then by substituting (3) into these boundary conditions and eliminating the surface-wave amplitudes, we obtain

$$e^{i\delta_s}=\frac{\sinh k_x d - iq_\theta^2[\cosh(k_x d)-e^{-ik_y d}]}{\sinh k_x d + iq_\theta^2[\cosh(k_x d)-e^{ik_y d}]}. \tag{20}$$

In the short-wavelength approximation (11), this gives

$$\delta_s \approx -2\arctan q_\theta^2, \tag{21}$$

which naturally coincides with the exact expression computed in Ref. 8 for the phase shift associated with reflection from the free superconducting surface of a piezomagnetic semi-infinite medium (i.e., not a layer).

Analogously, when the free boundary of the layered system is in its "free" state (i.e., Eq. (6) augmented by the condition of continuity of the magnetic field), we may set $\delta \equiv \delta_n$, i.e., in the short-wavelength approximation the phase shift is given by the expression obtained in Ref. 8 for a semi-infinite medium:

$$\delta_n = 2\arctan\left[\frac{q_\theta^2}{\mu_{11}+1}\left(\frac{m_{15}^2}{m_{14}^2}-\mu_{11}\right)\right]. \tag{22}$$

Note that

$$\delta_n \approx \delta_s \quad \text{for } \mu_{11} \gg 1,$$

$$\delta_n \approx \delta_s/2 \quad \text{for } m_{15}\equiv 0, \quad \mu_{11}\sim 1, \tag{23}$$

where the first condition is satisfied in sufficiently strong ferromagnets, the second in antiferromagnets belonging to the magnetic symmetry classes ∞/mm, $6/mmm$, $4/mmm$.[9]

Substituting (14) and (11) into (19) and using the dispersion relation (15) in the short-wavelength approximation, to first order in q_θ^2 we are led to the following expression for the phase of the reflection coefficient:

$$\Phi \approx 2\,\mathrm{Arctan}\frac{[\sin(k_y d)-2q_\theta^2\cos(k_y d)]\dfrac{\tan(NKd)}{\sin(Kd)}+\tan\dfrac{\delta}{2}}{1-\left(q_\theta^2+\tan\dfrac{\delta}{2}\right)\dfrac{\sin(k_y d)}{\sin(Kd)}\tan(NKd)} \tag{24}$$

(in accordance with (19), the phase is defined up to a multiple of 2π).

We assume that both the boundaries of the layer and the free boundary of the multilayer structure are in the superconducting state. Then Eq. (24) has the following form, taking (21) into account:

$$\Phi \approx 2\,\mathrm{Arctan}\left[[\sin(k_y d)-2q_\theta^2\cos(k_y d)]\frac{\tan(NKd)}{\sin(Kd)}-q_\theta^2\right]. \tag{25}$$

On the other hand, if in (24) we set $q_\theta^2=0$, and thus $k_y=K$, we obtain $\Phi=\Phi_0$ for the phase shift when all the superconducting cladding layers are in the n-state, i.e., all the boundaries between layers are "transparent":

$$\Phi_0=2k_y L+\delta_n, \tag{26}$$

where $L=Nd$ (i.e., $2k_y L$ is the length of the wave path along the y-axis; see Fig. 1); of course, the cladding layer of the free surface of the layered system is also in the normal state. Then we can introduce the value of the diffraction-induced phase shift

$$\Delta\Phi=\Phi-\Phi_0 \tag{27}$$

that discontinuously appears when all the layer boundaries whose reflections are "turned off" are converted to the s-state (including the free surface), and that conversely disappears when these boundaries are reconverted to the n-state. According to (25)–(27),

$$\Delta\Phi = 2\arctan\left[[\sin(k_y d)-2q_\theta^2\cos(k_y d)]\frac{\tan(NKd)}{\sin(Kd)}-q_\theta^2\right]+2\eta\left(KL+\frac{\pi}{2}\right)-2k_y L-\delta_n. \tag{28}$$

Here, in order to obtain an expression for $\Delta\Phi$ that is a continuous function of $k_y d$, we have taken the principal value of the arctangent and added a term containing η, which by definition equals the integer part of the argument. This ensures that when the piezomagnetic moduli are set to zero in (28), we obtain $\Delta\Phi\equiv 0$.

Note that in the case under discussion here, i.e., piezomagnetic layers, we always have $\Delta\Phi=\Phi-\Phi_0<0$. However, for a system of piezoelectric layers of the sort mentioned in the Introduction, which we will turn to shortly, we always have $\Delta\Phi>0$ (because there is a sign change in the phase shift at an isolated boundary due to the interchange (43), which takes us to the piezoelectric case; see below). For consistency we will work with the absolute value $|\Delta\Phi|$, so that in both cases we may speak of maxima and minima of the diffraction contribution $|\Delta\Phi|$, which is either subtracted from the "natural" phase advance Φ_0 (for piezomagnetic layers) or added to it (for piezoelectric layers).

Let us analyze the properties of the oscillatory function $\Delta\Phi(k_y d)$ given by Eq. (28) for a large number of layers $N \gg 1$. It is not difficult to verify that under condition (18), i.e., for $k_y d$ sufficiently far from the forbidden zones, the extrema of $|\Delta\Phi(k_y d)|$ are given by

$$(k_y d)_{\max} \approx \frac{\pi m}{N} + \pi l + q_\theta^2,$$

$$|\Delta\Phi|_{\max} \approx 2Nq_\theta^2 + (\delta_n - \delta_s), \tag{29}$$

$$(k_y d)_{\min} \approx \frac{\pi}{2N} + \frac{\pi(m-1)}{N} + \pi l + q_\theta^2,$$

$$|\Delta\Phi|_{\min} \approx 2Nq_\theta^2 + \delta_n \tag{30}$$

$(m=1,2,\ldots<N)$, i.e., under nonresonant conditions the diffraction-induced phase shift $\Delta\Phi$ is of order $(-2Nq_\theta^2)$.

This conclusion is entirely understandable in light of the following semiqualitative considerations. Note that the reflection coefficient (r) and transmission coefficient (t) of a wave incident on a superconducting boundary between layers with no multiple reflections (i.e., for $b_r^{(n+1)}=0$ in (7)) has, according to (7), (8), (12), the following form:[1)]

$$r = W_{21}/W_{11} \approx iq_\theta^2, \quad t = 1/W_{11} \approx e^{ik_y d}/(1+iq_\theta^2). \tag{31}$$

Far from the forbidden zones (i.e., from the Bragg resonances) the reflected waves are incoherent, and the diffracted wave at the input to the surface of the multilayer structure for $N \gg 1$ is primarily made up of waves transmitted by the $2N$ boundaries.

If under the additional assumption $Nq_\theta^2 \ll 1$ we limit ourselves to the approximation of a single pass through each boundary between layers, we can obviously write

$$R = e^{i\Phi} \sim r_s t^{2N} \sim \exp[i(2k_y L + \delta_s - 2Nq_\theta^2)], \tag{32}$$

where $r_s = e^{i\delta_s}$ is the reflection coefficient from the free surface of the structure (see (20), (21)). Hence, taking into account the condition $N \gg 1$ (i.e., $Nq^2 \gg |\delta_n - \delta_s| \sim q_\theta^2$), we obtain from (27) the desired estimate $\Delta\Phi \approx -2Nq_\theta^2$ for $\Delta\Phi$.

This result can also be interpreted in terms of the Bloch formalism. Note that according to (18) and (25), when $N \gg 1$ the condition $\Phi \approx 2KNd = 2KL$ holds for the phase Φ of the reflection coefficient outside the forbidden zones. Consequently, the diffraction contribution $\Delta\Phi$ (27) can be estimated by the difference between the length of the wave path along the y axis in units of the Bloch wave number K and its "ordinary" length $2k_y L$:

$$\Delta\Phi \approx 2(K - k_y)L, \tag{33}$$

whereupon, taking (18) into account, we once more obtain $\Delta\Phi \approx -2Nq_\theta^2$.

In and of itself, Eq. (33) implies that far from the forbidden zones the "physical" incident and reflected waves differ only slightly from the corresponding Bloch modes, which are continuous in the layered medium. Recall that according to the general theory (see, e.g., Ref. 10), the amplitude vectors $\binom{b_i}{b_r}$ corresponding to the incident and diffracted Bloch modes are proportional to the eigenvectors of the matrix propagator $\hat{W}$,

$$\begin{pmatrix} b_i \\ b_r \end{pmatrix}_I \sim \begin{pmatrix} W_{12} \\ e^{-iKd} - W_{11} \end{pmatrix}, \quad \begin{pmatrix} b_i \\ b_r \end{pmatrix}_R \sim \begin{pmatrix} W_{12} \\ e^{iKd} - W_{11} \end{pmatrix}, \tag{34}$$

where $W_{11} \approx \exp(-ik_y d)(1+iq_\theta^2)$ and $W_{12} \approx iq_\theta^2 \exp(-ik_y d)$ (see (8), (12)). Here we use the fact that the eigenvalue $\exp(-iKd)$ corresponds to an incident mode (labeled by I), while the eigenvalue $\exp(iKd)$ corresponds to a diffracted mode (labeled by R). Obviously, (18) implies that $\exp(-iKd) \approx \exp(-ik_y d)(1+iq_\theta^2)$. Therefore, far from the forbidden zones, to first approximation it follows from (33) that

$$\begin{pmatrix} b_i \\ b_r \end{pmatrix}_I \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} b_i \\ b_r \end{pmatrix}_R \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{35}$$

which confirms the correctness of the "Bloch" interpretation of our results.

We now discuss the case in which the value of $k_y d$ lies in the neighborhood of a forbidden zone, so that the wave numbers K and k_y are related by Eq. (16). By investigating the derivative of $\Delta\Phi$ with respect to $k_y d$, it is not difficult to verify that the phase shift $|\Delta\Phi|$ reaches its principal maximum exactly at the right-hand boundary of each forbidden zone:

$$(k_y d)_{\max} = \pi l + 2q_\theta^2, \quad |\Delta|\Phi|_{\max} \approx 4Nq_\theta^2 + \delta_n - \delta_s. \tag{36}$$

We can make sense of the value of the maximum (36) and establish the relationship between the phase resonance of the reflection coefficient and the Bragg condition by appealing to the following considerations. Under resonant conditions, in contrast to (32), we must include in the "total" reflected wave not only the waves transmitted at individual boundaries but also reflected waves as well, which become in-phase as $k_y d \approx \pi l$.

Assuming that $Nq_\theta^2 \ll 1$, let us discuss a single round trip through the layered structure in the forward and backward directions, with a single reflection at each boundary. It is obvious that in this case we must include in the total wave field reflected from the layered system a contribution from N waves reflected with coefficient $r = -iq_\theta^2$ (see (32)) in the direction $y>0$ (i.e., the incident wave propagates forward, see Fig. 1), and a contribution from N waves reflected in the direction $y<0$ and then specularly re-reflected (with coefficient $r_s \approx 1$ to first approximation; see (20)) from the free boundary of the layered system. Thus, for $Nq_\theta^2 \ll 1$ and $k_y d \approx \pi l$, we can write in place of (32)

$$R_{\text{res}} = e^{i\Phi} \sim r_s t^{2N} - 2iNq_\theta^2 = \exp[i(2k_y L + \delta_s - 4Nq_\theta^2)], \tag{37}$$

from which the desired estimate $\Delta\Phi \sim -4Nq_\theta^2$ follows. Note that this estimate is also in agreement with Eq. (33), which says that near the right-hand boundary of the forbidden zone we have $k_y d - Kd \approx 2q_\theta^2$.

On the other hand, the simple interpretation of Eq. (33) given above becomes incorrect near the forbidden zones. Moreover, right at the boundaries of the forbidden zones $(Kd = \pi l)$, the eigenvalues $\exp(\pm iKd)$ of the non-Hermitian matrix propagator $\hat{W}$ become degenerate, and a representation using Bloch waves as eigenvectors of the matrix $\hat{W}$ is in general unsuitable.[2)]

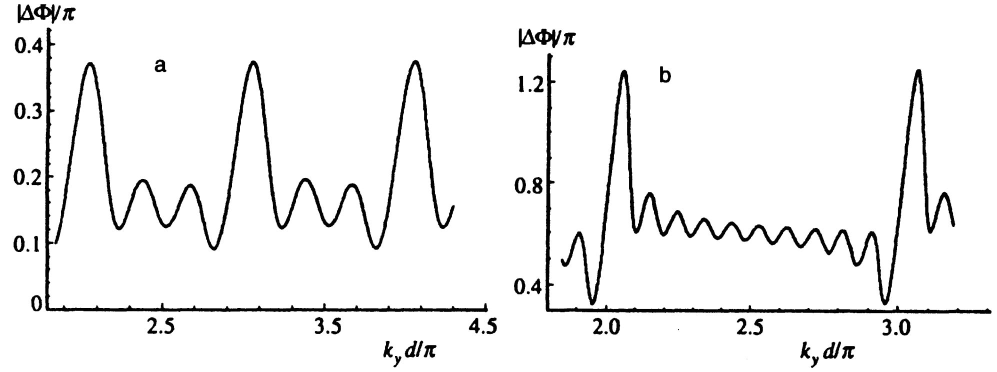

FIG. 2. Dependence of the diffraction phase shift of the reflection coefficient on the value of $k_y d/\pi$ for a system of identical layers when $q_\theta^2=0.1$, $\theta=\pi/4$, and $N=3$ (a) and $N=10$ (b).

Equations (29), (30), and (36) describe all extrema of the spectrum of the diffraction-induced phase shift $\Delta\Phi(k_y d)$ when $Nq_\theta^2 \ll 1$. In Fig. 2 we show the spectra calculated by using the exact expression (28) (for convenience we set $\delta_n \approx \delta_s$ in Figs. 2, 3; see (29)). It is clear from Fig. 2 that there are $N-1$ secondary maxima (29) between the principal maxima (36) with half the height of the latter; accordingly, these maxima are separated by N identical minima (30). Obviously, estimates (29), (36) are satisfactory even when $Nq_\theta^2 \lesssim 1$.

As Nq_θ^2 increases, the principal maximum clearly becomes narrower. In this case the height of the minimum immediately to the left of each principal maximum increases more slowly than the remaining minima, i.e., it eventually becomes the principal minimum (Fig. 2b). If the number of layers N is large enough that $Nq_\theta^2 \gg 1$, then in accordance with (16) and (28) this principal minimum, which lies near the left edge of the forbidden band, is approximately given by

$$(k_y)d_{\min} \approx \pi l - \frac{1}{2q_\theta^2}\left(\frac{\pi}{2N}\right)^2,$$

$$|\Delta\Phi|_{\min} \approx \pi - \frac{N}{q_\theta^2}\left(\frac{\pi}{2N}\right)^2, \tag{38}$$

where $\Delta\Phi \approx -\pi + 1/Nq_\theta^2$ at the left boundary of the zone $(Kd=\pi l)$ and $\Delta\Phi \approx 2(N-1)q_\theta^2$ at the center of the zone $(k_y d=\pi l+q_\theta^2,\ Kd=\pi l+iq_\theta^2)$.

4. GENERALIZING TO NONEQUIDISTANT STRUCTURES. PIEZOCRYSTALLINE ANALOG

The qualitative representation derived above for the effect of phase resonance allows us to predict the fundamental properties of this phenomenon for the more general case of a periodic structure made up of layers of hexagonal piezomagnetic material that differ in thickness—for example, with pairwise alternating thicknesses d_1, d_2. In Ref. 3, in our study of the Bragg diffraction of sound by a system of nonequidistant layers of this kind placed between two substrates, we showed that the intensity of the resonance peaks of the reflection coefficient is modulated by a factor $|\cos(\pi l d_1/D)|$, where $D=d_1+d_2$ is the period of the structure. If we now consider our phase-resonance scheme (i.e., removing one of the substrates) and use the approximate relations (32) and (37) for a system of N periodically repeating pairs of layers (i.e., $N\rightarrow 2N$ in (32) and (37)), we conclude that the background of the spectrum of $\Delta\Phi(k_y d)$ is of order $\sim -4Nq_\theta^2$ and Eq. (37) must be replaced by

$$R_{\text{res}}=e^{i\Phi}\sim r_s t^{2N}-4iNq_\theta^2|\cos(\pi l d_1/D)|, \tag{39}$$

from which, taking (26) and (27) into account, we have

$$|\Delta\Phi|_{\max}\sim 4Nq_\theta^2[1+|\cos(\pi l d_1/D)|].$$

We also calculated the phase Φ of the reflection coefficient $R=\exp(i\Phi)$ using the matrix propagator calculated in Refs. 1, 3, where this matrix propagator "translates" the wave field through a period $D=d_1+d_2$, and the boundary conditions for a free surface at the last $(2N+1)$th layer, by analogy with the equidistant case. The results of these calculations are

$$\Phi=2\ \text{Arctan}\left\{\frac{\{\sin(k_yD)-2q_\theta^2[\cos(k_yD)-\cos(k_yd_1)\cos(k_yd_2)]\}\tan(NKD)}{\sin(KD)+q_\theta^2\sin[k_y(d_1-d_2)]\tan(NKD)}-q_\theta^2\right\}. \tag{40}$$

When we examine the diffraction contribution $\Delta\Phi$ defined according to (27), we find that far from the forbidden zones the extrema of the function $|\Delta\Phi(k_y d)|$ are

$$|\Delta\Phi|_{\max}\approx 4Nq_\theta^2+\delta_n-\delta_s,\quad |\Delta\Phi|_{\min}\approx 4Nq_\theta^2+\delta_n, \tag{41}$$

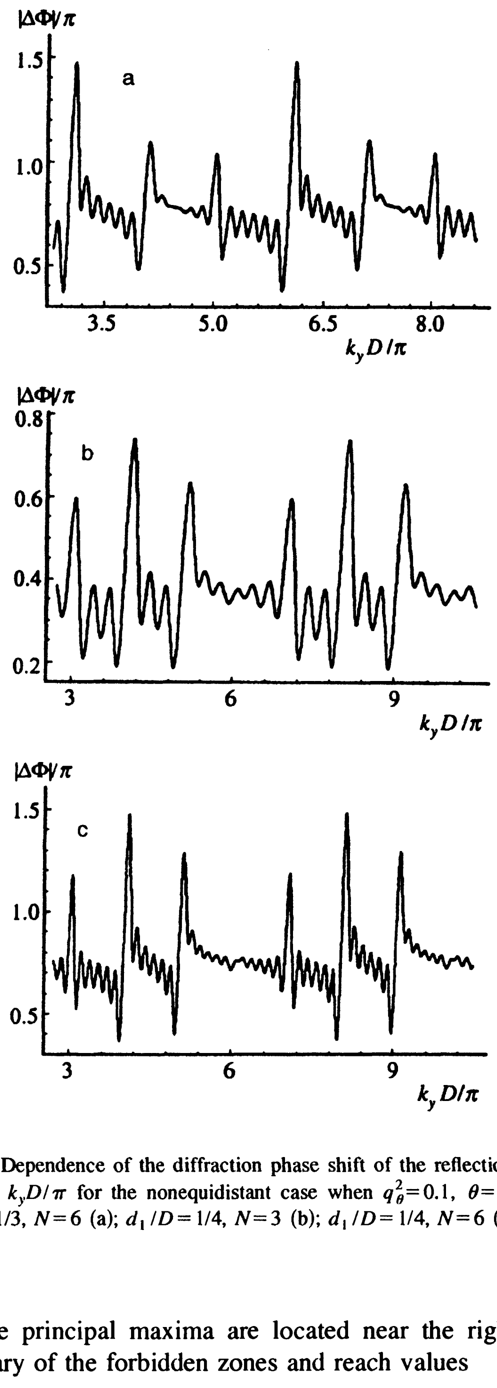

FIG. 3. Dependence of the diffraction phase shift of the reflection coefficient on k_yD/π for the nonequidistant case when $q_\theta^2=0.1$, $\theta=\pi/4$, and $d_1/D=1/3$, $N=6$ (a); $d_1/D=1/4$, $N=3$ (b); $d_1/D=1/4$, $N=6$ (c).

and the principal maxima are located near the right-hand boundary of the forbidden zones and reach values

$$|\Delta\Phi|_{\max}\approx 4Nq_\theta^2\,[1+|\cos(\pi l d_1/D)|]+\delta_n-\delta_s, \qquad (42)$$

which obviously confirms the qualitative estimate we made previously.

Thus, like the amplitude spectrum discussed in Ref. 3, the spectrum of $\Delta\Phi(k_yd)$ for a nonequidistant layered structure exhibits a modulation of the height of the resonance maxima, which depends on the ratio of thicknesses of adjacent layers; see Fig. 3 (the curves in Fig. 3 were calculated using the exact expression obtained from (40) by analogy with (28)). In particular, if the ratio d_1/D is a rational fraction S/T with an odd numerator S and an even denominator T, by analogy with Ref. 3 a so-called extinction effect will appear, in which each principal maximum of order $l=T/2+mT$, $m=0,1,2,\ldots$ is eliminated from the spectrum, i.e., weakened down to the background level $|\Delta\Phi|\sim 4Nq^2$ (the width of the corresponding forbidden zone vanishes in this case; see Figs. 3b, 3c).

To conclude this section we indicate how the results obtained above must be modified when we turn to a discussion of analogous effects in a system of layers made up of identical hexagonal piezoelectric crystals (nonmagnetic, with the sixfold axis orthogonal to the sagittal plane as before) separated by thin metallized cladding layers. Although the similarity between the original equations for piezomagnets and piezoelectrics is incomplete for the problem under discussion, according to Ref. 1 the entire matrix formalism developed for the piezomagnetic structure is preserved, and the case of piezoelectric layers is obtained by making the replacement

$$q_M^2\to -q_E^2, \qquad (43)$$

where $q_E^2=4\pi e_{15}^2/\tilde{c}_{44}\varepsilon_{11}$ is the electromechanical coupling parameter, $\tilde{c}_{44}=c_{44}+4\pi e_{15}^2/\varepsilon_{11}$, e_{15} is a component of the piezoelectric tensor, and ε_{11} is the dielectric constant. This implies that we can obtain expressions for the piezoelectric case from those we derived above by replacing the parameter $q_\theta^2=q_M^2\tan\theta$ with $-q_\theta^2$, where now $q_\theta^2=q_E^2\tan\theta$. Moreover, both δ_n and δ_s in the corresponding expressions become identically equal to δ, which is the phase change upon reflection from the free surface of a semi-infinite hexagonal piezoelectric medium. According to Ref. 8, this value is either

$$\delta=2\arctan(q_E^2\tan\theta),$$

or

$$\delta=2\arctan\left[\frac{q_E^2\tan\theta}{\varepsilon_{11}+1}\left(1-\frac{e_{14}^2\varepsilon_{11}}{e_{15}^2}\right)\right],$$

in the presence or absence of a metallized cladding layer at the surface of the medium, respectively.

5. CONCLUSION

There are various experimental schemes that allow us to observe and make use of the phase resonance exhibited by the reflection coefficient of an elastic wave from a system of piezocrystalline layers. It is clear that the possibility of "turning on" and "turning off" the diffraction-induced phase contribution, and the possibility of controlling its spectral dependence by converting individual superconducting cladding layers to the n-state (thereby changing the period d of the structure, realizing conditions for the extinction effect in a nonequidistant structure, etc.), allows us to treat this layered structure as an analog of a tunable acoustic phase filter.

Another concept of interest is based on the interference of reference and diffracted waves (similar to the method of standing waves in x-ray optics). Let us assume that the elastic displacement field at the "input" surface of the layered structure ($y=0$, Fig. 1) consists of a superposition of two fields: one from the incident wave $u_{zi}=\cos(k_xx-\omega t)$ and one from the reflected wave $u_{zr}=\cos(k_xx-\omega t+\Phi)$, where Φ is the phase of the reflection coefficient. The resulting wave

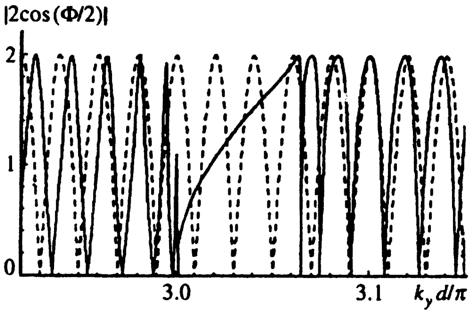

FIG. 4. Dependence of the amplitude of the interference wave field (44) on $k_y d/\pi$ for superconducting cladding layers in the s-state (continuous curve) and in the n-state (dashed curve).

field can obviously be written in the form (recall that the elastic displacement of the nonuniform modes equals zero; see (3)):

$$U=u_{zi}+u_{zr}=2\cos(\Phi/2)\cos(k_x x-\omega t+\Phi/2). \quad (44)$$

In Fig. 4 (for definiteness we examine a piezomagnetic structure), we compare the dependence of the amplitude modulus $|\cos(\Phi/2)|$ of the standing wave **U** (44) on the parameter $k_y d$ for "turned-on" reflections (i.e., with the superconducting cladding layer in the s-state and Φ given by Eq. (25)) and "turned-off" reflections (i.e., the cladding layer in the n-state, and $\Phi=\Phi_0=2k_yNd+\delta_n$; see (26)). For $q_\theta^2 N\ll 1$, diffraction displaces the zeros of the amplitude by a small quantity $\sim q_\theta^2\ll 1$. Obviously, the case of most interest is that in which the number of layers N is large. In the neighborhood of the forbidden zones, when $N\gg 1$ the value of the amplitude varies rapidly (obviously, for $k_y d=\pi l$ we have $\cos(\Phi_0/2)=\cos(\delta_n/2)\approx\cos q_\theta^2, \cos(\Phi/2)\approx\cos[\arctan(2q_\theta^2 N)]$). If the number of layers N is large enough that the parameter $q_\theta^2 N$ is not small, diffraction qualitatively changes the dependence of the amplitude of the interfering wave field (44) in the neighborhood of the forbidden zones, leading to quenching of the oscillations near its boundary and the appearance of a stationary segment (disappearance of the amplitude zeros) within the zone (see Fig. 4). Note that for $q_\theta^2 N\gg 1$, the phase shift of the diffracted wave relative to the incident wave changes from 0 to π over the width of the forbidden zone—exactly as in the x-ray case.

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[1)]Equation (31), which was obtained in the short-wavelength approximation (11), naturally coincides with the results of Ref. 4 for a superconducting boundary between two semi-infinite media (up to an additional phase factor $\exp(ik_y d)$ for the transmitted wave, arising from referencing the phase of the wave solutions according to (3)).

[2)]Note that the derivation of the important formula (14) for the Nth power of the matrix $\hat{W}$ given in the well-known Ref. 11, which is based on the possibility of diagonalizing the latter, is also incorrect at the boundaries of the forbidden zones ($Kd=\pi l$). In this case (14) can be proved by induction.

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