New foundation for the equations of the Wyld diagram technique

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A perturbation-theoretic interpretation that makes it possible to derive an equation describing the evolution of a pair correlation function in classical turbulent wave fields with an *a priori* assigned accuracy is proposed. The calculation of this equation is an intermediate step involved in devising a kinetic description of a wave field. The possibilities of the proposed interpretation are demonstrated in the derivation of the equations of the Wyld diagram technique without the usual practice of introducing a random external force. © *1996 American Institute of Physics.* [S1063-7761(96)02908-3]

1. INTRODUCTION

Classical turbulent wave fields are encountered in diverse situations as physically real objects. They are observed especially often in plasmas, both in space and in the laboratory. The parametric excitation of spin waves in ferromagnets can be cited as a second example of a situation with a clearly expressed turbulent wave field. There are other examples.

From the practical standpoint, the spectrum of a turbulent wave field, i.e., the distribution of the energy density of the wave motion with respect to the spatial scales of motion, is a perfectly adequate characteristic for describing effects associated with such fields. Under certain restrictions, i.e., for so-called weak turbulence, its evolution can be described by kinetic equations. In this case the temporal derivative of the spectral wave density is expressed in terms of the values of the instantaneous spectral density using the so-called collision integral. The collision integral is calculated by expanding the expression in one or several parameters that are proportional to the energy density of the field. The lowest order of the collision integral is usually obtained by classical perturbation theory after supplementing it with several hypotheses for expressing the higher correlation functions in terms of pair correlation functions (the "random phase approximation" in plasma physics; see, for example, the review in Ref. 1). When this approach is applied to the calculation of higher orders of the collision integral, disparities appear, which require certain renormalizations. In this case the calculation can be performed using a diagram technique like the Wyld diagram technique,[2] as was noted back in Ref. 3 and was clearly demonstrated for a very simple situation in Ref. 4.

The Wyld diagram technique was originally devised for hydrodynamic equations. Zakharov and L'vov subsequently rewrote it in terms of canonical variables[5] for the case of a wave field having a Hamiltonian.

Both in the technique proposed by Wyld and in the interpretation of the technique proposed by L'vov and Zakharov, the basic equations of the diagram technique are derived using a formal expansion with respect to an infinitesimal random external force (thermal noise), which is performed in an intermediate step. Equations relating a correlation function and a Green's function are obtained as a result, and in the case of a significantly superthermal wave field these equations are virtually independent of the external force. The latter circumstance raises the following question: how faithfully do the final equations of the Wyld technique describe the processes actually occurring in a turbulent field?

In this paper we demonstrate some principles for implementing perturbation theory, which make it possible, in particular, to derive the equations of the Wyld diagram technique without introducing a random external force.

2. BASIC EQUATIONS

To make the theoretical development specific, we focus on the case of a scalar, weakly turbulent wave field with a single branch of oscillations that does not interact with the medium. Such a field has a Hamiltonian, and its evolution is described in canonical variables by the equation

$$is\frac{\partial a^s(\mathbf{r},t)}{\partial t}=\frac{\delta\mathscr{H}}{\delta a^{-s}(\mathbf{r},t)}. \tag{1}$$

Here $\mathscr{H}$ is the Hamiltonian of the wave field. It depends on two field variables, viz., the amplitude of the waves $a(\mathbf{r},t)$ and its complex conjugate $a^*(\mathbf{r},t)$. The latter are labeled by the plus and minus signs in the superscript s.

The Hamiltonian of the field is proportional to the volume occupied by the medium, i.e., it formally diverges. However, even with consideration of the formal divergence of the Hamiltonian, Eq. (1) has a definite meaning, since its right-hand side is a sum of finite terms.

For simplicity, we assume that the expansion of the Hamiltonian contains only two terms. The first, which is quadratic in the wave amplitude, describes the linear dispersion of the waves,

$$\mathscr{H}_0=\frac{1}{2}\sum_{s,s'}\int d^3r_1d^3r_2U^{s,s'}(\mathbf{r}_2-\mathbf{r}_1)a^s(\mathbf{r}_1,t)a^{s'}(\mathbf{r}_2,t), \tag{2}$$

and the second, which is cubic, describes three-wave processes,

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$$\mathscr{H}_{\text{int}}=\frac{1}{6}\sum_{s_1,s_2,s_3}\int d^3r_1d^3r_2d^3r_3V^{s_1,s_2,s_3}(\mathbf{r}_2-\mathbf{r}_1,\mathbf{r}_3-\mathbf{r}_1)a^{s_1}(\mathbf{r}_1,t)a^{s_2}(\mathbf{r}_2,t)a^{s_3}(\mathbf{r}_3,t). \quad (3)$$

The coefficients U and V can be assumed symmetric with respect to interchange of the pairs $(s_i,\mathbf{r}_i)$. Reversal of the signs of all the s_i is equivalent to taking the complex conjugate:

$$V^{-s_1,-s_2,-s_3}(\mathbf{r}_1,\mathbf{r}_2)=[V^{s_1,s_2,s_3}(\mathbf{r}_1,\mathbf{r}_2)]^*.$$

The main object in our theory is the pair correlation function

$$N^{s,s'}(\mathbf{r},t,t')=\langle a^s(\mathbf{r}+\mathbf{R},t)a^{-s'}(\mathbf{R},t')\rangle_{\mathbf{R}}. \quad (4)$$

In its definition the subscript $\mathbf{R}$ on the right-hand side symbolizes averaging over the volume.

The pair correlation function has Hermitian self-conjugacy:

$$N^{s,s'}(\mathbf{r},t,t')=[N^{s',s}(-\mathbf{r},t',t)]^*.$$

When the temporal arguments of the pair function are equal, we obtain the matrix of the autocorrelation function $N^{s,s'}(\mathbf{r},t,t)$. The concept of a spectral wave density can be defined in terms of it. For example, the diagonal element $N_{\mathbf{k}}^{+,+}(t,t\rangle$ of the spatial Fourier transform of the latter, which is necessarily a positive real function, can be regarded as the spectral density. The evolution of the wave spectrum and the medium (the field influencing the medium in the general case) can be described in terms of this a spectral density, thereby stressing the significance of the pair correlation function as well.

The final purpose of all the ensuing manipulations is the derivation of an approximate equation describing the evolution of the pair correlation function in an *a priori* assigned range of accuracy of the expansion in the energy density of the wave field. We confine ourselves to the accuracy in which the cubic expansion terms are retained in the evolution equation. This approximation is fully illustrative and, at the same time, not excessive for describing the essence of the proposed approach.

To achieve the stated purpose, we also need the bare Green's function ${}^0G^{s,s'}(\mathbf{r},\tau)$, i.e., the Green's function of the linear problem. The latter is identically equal to zero at $\tau<0$, and at $\tau>0$ it is the solution of the homogeneous integrodifferential equation

$$is\frac{\partial}{\partial\tau}{}^0G^{s,s'}(\mathbf{r},\tau)=\sum_{s_1}\int d^3r_1U^{-s,s_1}(\mathbf{r}-\mathbf{r}_1){}^0G^{s_1,s'}(\mathbf{r}_1,\tau) \quad (5)$$

with the initial condition

$${}^0G^{s,s'}(\mathbf{r},0)=\delta^3(\mathbf{r}).$$

We shall henceforth regard this matrix function with its dependence on the time and coordinates as a representation of a certain operator ${}^0\hat{G}$.

Finally, the calculation is conveniently performed in a graphical formulation. The graphical symbols used here to represent the analytical calculations are the same as in Ref. 5. We denote the bare Green's function by a thin solid line, the correlation function matrix by a wavy line, and the matrix element of three-wave interactions by a vertex with three ends. A time t is assigned to the vertex, and the values of the indices s_i and the vectors $\mathbf{r}_i$ matching the corresponding arguments of the functions attached to these ends are assigned to each end of the vertex. (Because of the uniformity of the problem, to make the diagrams more readable, we subtract the vector $\mathbf{r}$ at the entrance to a diagram is subtracted from the other two arguments $\mathbf{r}_i$). The integrations are carried out in the final integral expressions over the corresponding radius vectors and the time. The problem of interpreting the final diagram relations will not be discussed in further detail. Where necessary, the reader can specify the interpretation principles independently, ignoring the rules for reading diagrams described in Ref. 4.

Before proceeding directly to a description of the calculation method, we once again precisely define the concept of "homogeneous turbulence." In this paper a turbulent field in which the average $\langle a^s(\mathbf{r}+\mathbf{R},t)a^{-s'}(\mathbf{R},t')\rangle_{\mathbf{R}}$ taken over a sufficiently large volume in the vicinity of an assigned point $\mathbf{r}$ is practically independent of the coordinates of the point $\mathbf{r}$ is considered homogeneous. This automatically means that the wave spectrum does not have a long-wavelength portion (with wavelengths equal to or exceeding the scale of the linear dimensions of the averaging volume). Homogeneity of a turbulent field in the sense indicated is realized only under such a condition. For simplicity, we assume that a nonlinear interaction between waves will likewise not lead to pumping of the long-wavelength portion of the spectrum.

Under this restriction the wave field has a finite correlation radius R_c, which is nothing but the characteristic decay length of the correlation function $\hat{N}(\mathbf{r},t,t')$ as $|\mathbf{r}|$ increases. The correlation radius is clearly determined by the dimensions of the region in the space of wave vectors occupied by the wave spectrum.

3. DESCRIPTION OF THE CALCULATION METHOD

Equation (1) for the pair correlation function can be used to obtain the equation

$$is\frac{\partial}{\partial\tau}N^{s,s'}(\mathbf{r},t,t')-\sum_{s_1}\int d^3r_1U^{-s,s_1}(\mathbf{r}-\mathbf{r}_1)N^{s_1,s'}(\mathbf{r}_1,t,t')$$
$$=\frac{1}{2}\sum_{s_1,s_2}\int d^3r_1d^3r_2V^{-s,s_1,s_2}(\mathbf{r}-\mathbf{r}_1,\mathbf{r}-\mathbf{r}_2)$$
$$\times\langle a^{s_1}(\mathbf{r}_1+\mathbf{R},t)a^{s_2}(\mathbf{r}_2+\mathbf{R},t)a^{s'}(\mathbf{R},t')\rangle_{\mathbf{R}}, \quad (6)$$

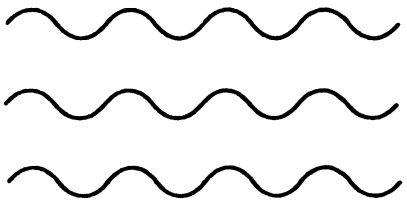

FIG. 1. "Sixfold" correlation function.

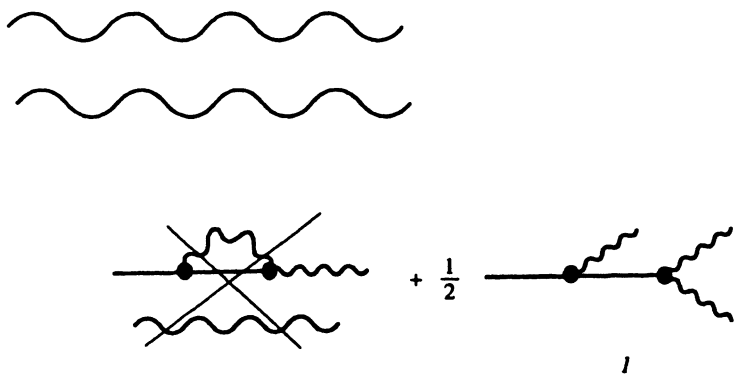

FIG. 3. Lowest order of the quadruple correlation function.

i.e., the evolution equation of the pair correlation function contains the triple correlation function

$$\langle a^s(\mathbf{r}+\mathbf{R},t)a^{s'}(\mathbf{r}'+\mathbf{R},t')a^{s''}(\mathbf{R},t'')\rangle_{\mathbf{R}}.$$

The evolution equation of the triple correlation function, in turn, contains a quadruple correlation function, etc. There is, thus, an infinite hierarchical sequence of evolution equations. In the case of weak turbulence this sequence can be truncated. In particular, in the situation under consideration we can stop at the "sixfold" correlation function and totally neglect the term containing the "sevenfold" correlation function. In fact, the sixfold correlation function itself is already cubic in the energy density of the wave field. It can be split into a product of three pair correlation functions (more precisely, into a sum of all the possible products of pair functions) with a sufficient degree of accuracy:

$$\langle a^s(\mathbf{r}+\mathbf{R},t)a^{s_1}(\mathbf{r}_1+\mathbf{R},t_1)a^{s_2}(\mathbf{r}_2+\mathbf{R},t_2)a^{s_3}(\mathbf{r}_3+\mathbf{R},t_3)$$
$$\times a^{s_4}(\mathbf{r}_4+\mathbf{R},t_4)a^{s_5}(\mathbf{R},t_5)\rangle_{\mathbf{R}}. \quad (7)$$

One of the terms is $N^{s,-s_4}(\mathbf{r}-\mathbf{r}_4,t,t_4)N^{s_5,-s_2}(-\mathbf{r}_2,t_5,t_2)$ $\times N^{s_3,-s_1}(\mathbf{r}_3-\mathbf{r}_1,t_3,t_1)$. To be convinced of its existence, it is sufficient to consider the sixfold correlation function (4) in the range of arguments

$$|\mathbf{r}-\mathbf{r}_4|,|\mathbf{r}_3-\mathbf{r}_1|,|\mathbf{r}_2|\lesssim R_c,$$
$$|\mathbf{r}-\mathbf{r}_3|,|\mathbf{r}_3-\mathbf{r}_2|,|\mathbf{r}-\mathbf{r}_2|\gg R_c.$$

The difference between the true correlation function and the sum of the threefold products of the pair functions is of the order of the fourth power of the energy density of the wave field and can consequently be neglected.

The sixfold correlation function is represented in the graphical formulation in Fig. 1 as the sum of all the possible products of the pair correlation functions. This figure contains three wavy lines, i.e., three independent pair correlation functions. They have six free ends, which are completely equivalent. Any of the spatial variables and any of the times can be assigned to each of these ends, and all these variables can be interchanged in all possible ways.

Attaching a straight line, i.e., a Green's function, to one end of a vertex and any two free ends of the wavy lines of the sixfold correlator (Fig. 1) to the other two ends, with consideration of the factor 1/2 for the vertex we obtain the expression for the "fivefold" correlation function (see Fig. 2). In the resulting graph all the free ends are once again equivalent. The only new restriction is that the straight line, i.e., the bare Green's function, must be read from its free end to the interior of the corresponding fragment.

Using the drawing of the quintuple correlation function, we can also express the quadruple correlation function in terms of pair correlation functions. It can clearly be represented in lowest order by the diagrams in Fig. 3. Joining any two free ends in Fig. 2 to exits of a vertex and attaching a solid line to its entrance, we obtain corrections to the quadruple function. They are shown in Fig. 4. The first and last terms in this figure have been purposely crossed out. The upper parts of these terms are fragments with two free ends, which, according to their meaning, cannot be anything but components of a pair correlation function. The corresponding terms have already been taken into account by the diagrams in Fig. 3 and should, therefore, be discarded. (Actually, these fragments are formally a complete integral representation of a pair correlation function with an accuracy specified by the lowest order of the kinetic equation.)

It is noteworthy that this figure contains two coefficients equal to 1/2, which are not shown in all the diagrams. The presence or absence of such coefficients in a specific diagram can be understood by studying the possible interchange of

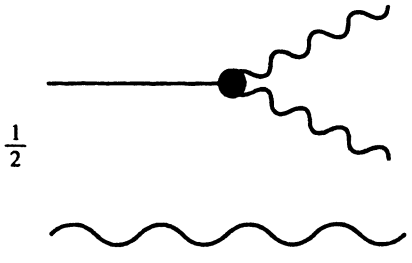

FIG. 2. Quintuple correlation function.

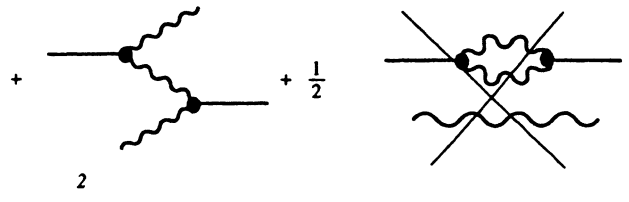

FIG. 4. Corrections to the quadruple correlation function.

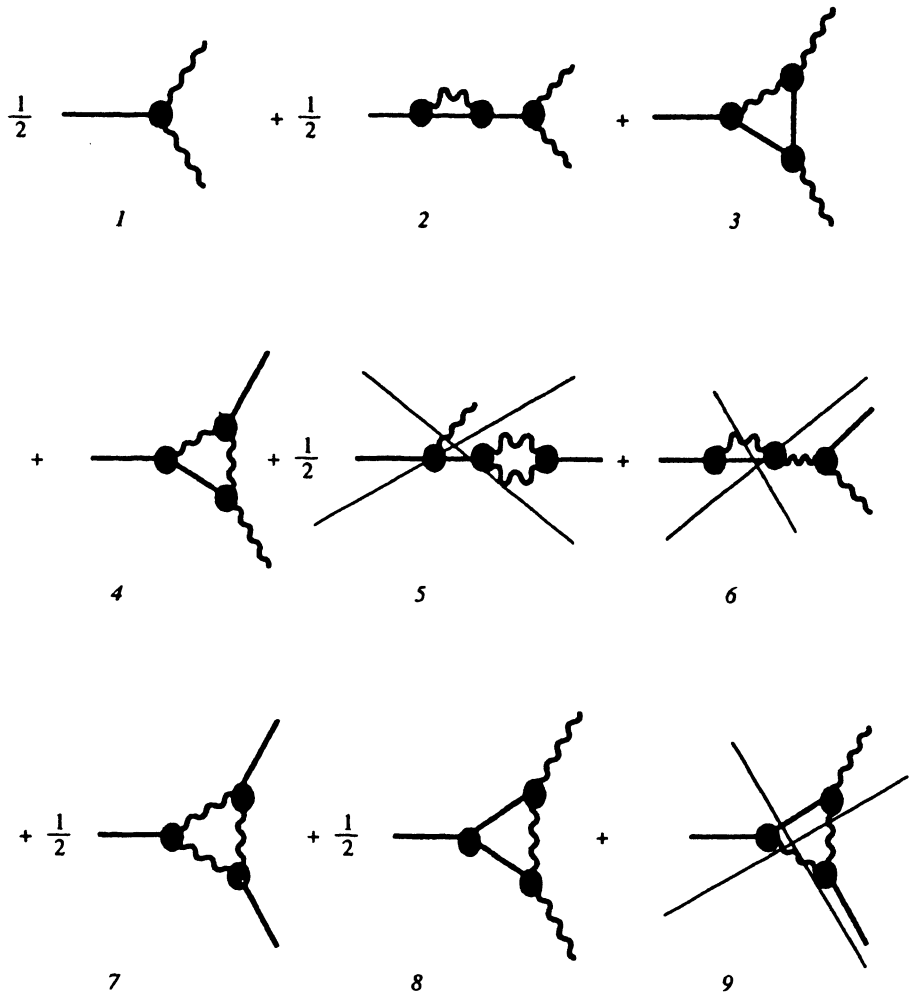

FIG. 5. Triple correlation function.

the pairs of times and spatial vectors $(t_i, \mathbf{r}_i)$ at the respective free ends of the diagrams. We shall not give a detailed description of the principles used to determine the power of the coefficient 1/2 corresponding to a particular diagram. The conclusion, however, is that the power of the coefficient 1/2 is determined by the number of elements in the symmetry group of the diagram when one of the Green's functions with a free entrance end is detached from it.

Beginning with the expressions represented in Figs. 3 and 4, we can obtain diagrams of the triple correlation function by splicing. These diagrams are shown in Fig. 5. For convenience, they are numbered from *1* to *9*. Diagram *1* is obtained from the lowest-order diagram of the quadruple correlation function (Fig. 3), diagrams *2–5* are obtained from diagram *1* in Fig. 4, and diagrams *6–9* are obtained from diagram *2* in Fig. 4. Diagrams *5*, *6*, and *9* have been crossed out: they should be discarded. Diagrams *5* and *6* correspond to corrections that have already been automatically adequately taken into account by diagram *1*. Correction *9* simply duplicates correction *4*. These two corrections differ only with respect to the order in which their Green's functions with free ends were attached to them. To some extent, an analogy to a second-order differential in higher mathematics would be appropriate here.

It should be specially stressed that the analytical analog of the diagram expression obtained for the triple correlation function is a fairly exact approximation of the true triple correlation function. During the calculation of this correlation function, we accurately calculated all the higher correlators needed, by adequately utilizing the corresponding evolution equations.

After substituting the expression obtained for the triple correlation function into the evolution equation (6), we determine the form of the latter in terms of the pair correlation function. It is represented in Fig. 6 in the graphical formulation. In it the left-hand side has the form of a diagram expression attached to the wavy line of the paired correlation function, and on the right-hand side one diagram expression is attached to another. (All three diagram expressions in the figure are enclosed in square brackets.) In addition, the self-evident graphical symbol for the operator ${}^0\hat{G}^{-1}$ has been introduced in the diagram expression on the left-hand side. We note that when we write the analytical analogs of the diagrams in this expression, as well as in the first diagram expression on the right-hand side, there is no integration over the temporal and spatial variables at the entrances to the diagrams.

The second expression on the right-hand side (in which each diagram should be read from right to left!) is nothing but a first corrected approximation of the renormalized

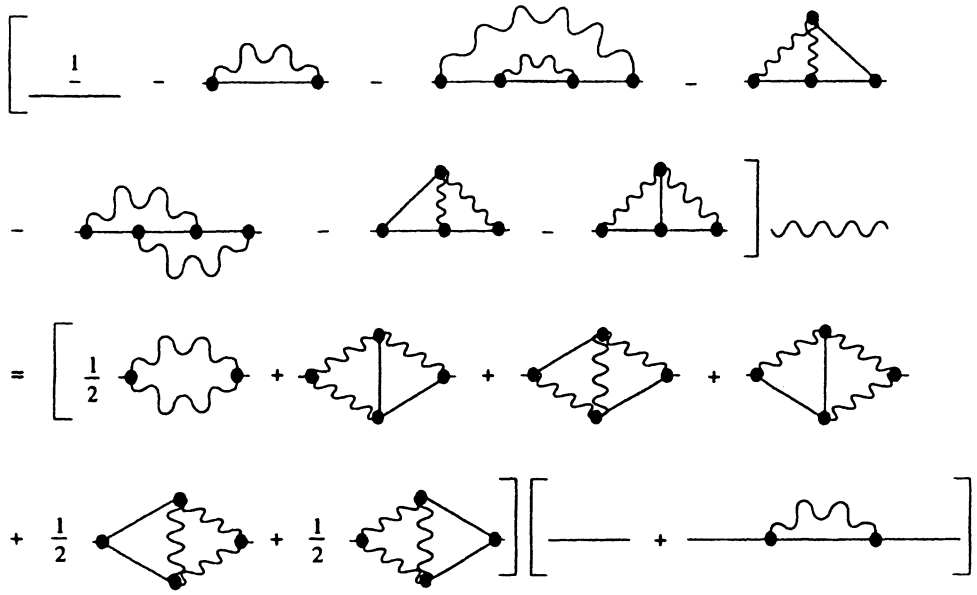

FIG. 6. Evolution equation of the pair correlation function.

Green's function of the Wyld diagram technique. This approximation has sufficient accuracy for calculating the cubic collision integral. We recall that we are dealing with a temporal, rather than a frequency, representation of all the functions; therefore, there is no problem with the singularity of the correction to the bare Green's function when we work with its evolution equation.

With consideration of the approximation of the renormalized Green's function indicated, the diagram expression on the left-hand side of the equation takes on the meaning of a graphical form of the inverse operator of the renormalized Green's function (see Ref. 4). In other words, the diagrams following the minus signs in it are the first terms of the expansion of the self-energy function $\hat{\Sigma}$ [see Eq. (16) in Ref. 5; the first two of our diagrams of this function have been combined in the first diagram of $\hat{\Sigma}$ in the reference just cited, particularly with consideration of the expansion of the renormalized Green's function].

The first diagram expression on the right-hand side of the evolution equation (Fig. 6) is an expansion of the compact part of the correlation function $\hat{\Phi}$ [compare the expansion (15) in Ref. 5].

Thus, the evolution equation obtained coincides to within the prescribed accuracy with the operator relation of the Wyld diagram technique:

$$\hat{G}^{-1}\hat{N}=\hat{\Phi}\hat{G}^{+}.$$

Here $\hat{G}$ is the renormalized Green's function, $\hat{N}$ is the pair correlation function, $\hat{\Phi}$ is the compact part of the correlation function, and the plus sign denotes Hermitian conjugation (see Ref. 4).

It is easy to understand that when calculations are performed with greater accuracy, the evolution equation of a pair correlation function that is the approximation corresponding to the last operator relation will ultimately always be obtained. On one hand, increasing the accuracy of the calculation (taking into account a larger number of equations in the hierarchy of evolution equations of the correlation functions) will give rise to new compact diagrams of the self-energy functions $\hat{\Phi}$ and $\hat{\Sigma}$ with correct weights. On the other hand, it will be manifested by the appearance of more and more new additions to the bare Green's function, which always combine with sufficient accuracy to form the appropriate approximation of the renormalized Green's function.

4. CONCLUSIONS

In this paper we have demonstrated some basic principles for devising a perturbation scheme that makes it possible to obtain the evolution equation of a pair correlation function with an *a priori* assigned accuracy of the expansion in the energy density of a turbulent field. The range of application of the approach illustrated here is not confined to the area of substantiating the Wyld diagram. The idea proposed in this paper can also be applied in more realistic problems with nonconservative classical fields, particularly, for verifying the soundness of the existing machinery of the kinetic description of a turbulent plasma.

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