On the theory of instabilities and oscillations in accelerated or gravitating layered structures

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Instabilities and waves in multilayered profiled shells for inertial confinement fusion and the atmospheres of stars and planets are considered. The paper gives analytical expressions for oscillation spectra and growth rates of dynamic instabilities for power-function profiles of shells, and for polytropic stars and atmospheres where the pressure is a power function of density ($P=\mathrm{const}_p\rho^{(N+1)/N}$). The polytropic description is widely used in astrophysics. One can prove that in this description the entropy distribution near the free surface of a star is given by a power function $S=\mathrm{const}_s(-\Delta r)^\theta$, where Δr is measured with respect to the star surface; in this paper the entropy is defined as P/ρ^γ, which is, in fact, a function of entropy. In high-energy applications the entropy is variable because of both initial conditions and effect of X-rays and penetrating fast particles. The polytropic index N, exponent θ, and adiabatic exponent γ are related through the equation $\theta=1-N(\gamma-1)$, and the density is proportional to $(-\Delta r)^N$. Previously only the simplest case of uniform density ($\theta=1$ and $N=0$), corresponding to Pekeris'well-known solution (see Ref. 5, Sec. 17.7 or Ref. 51, Sec. 76), was described by analytical expressions. The paper gives a more general solution valid at arbitrary values of N and θ. The problem proves to be similar to the Schrödinger equation for the Coulomb potential. The spectrum is separated into pairs of acoustic and entropic-rotational modes. Their relation means that one acoustic and one entropic-rotational mode correspond to each mode number n. The frequencies and growth rates of these modes are different, while the fields of pressure variations in Lagrangian particles are identical. [S1063-7761(96)01508-9]

1. INTRODUCTION

Instabilities, oscillations, and waves in stratified structures in a gravitational field[1–9] have been analyzed. In some cases the gravitation potential is generated by gravitating masses, in others the effective gravitation is due to acceleration of matter. Gravitating systems are of importance in the physics of the atmosphere and oceans, and in astrophysics. These are planetary atmospheres, the ocean thermocline, or stars. The dynamic behavior of the system near its hydrostatic equilibrium is interesting not only in studies of natural phenomena, but also for technological applications[7–9] related to the physics of high energy density.

In geo- and astrophysical applications, the stratification profile is derived from measurements of oscillation frequencies.[4–6] These measurements are related to variations in observed parameters, such as brightness and frequency Doppler shifts,[4,5] and the generation of forced oscillations due to tidal interaction between components of double stars, in simulations of star capture, and in calculations of corrections to Kepler orbits (apsidal rotation etc.)[10–14] is investigated. Oscillations and waves conduct energy from the deep layers compressed by the weight of the upper layers, which results in heating of the surface layers.[15–23] This is an exceptionally important physical effect closely related to formation of the chromosphere and, apparently, probably to enhancement of the solar wind.

In applications related to power generation, fuel must be compressed to the highest possible degree and heated, while the power consumption in the process should be minimized. To this end, one needs shells with a high aspect ratio (i.e., thin shells with a large radius-to-thickness ratio). Such shells, in turn, are very susceptible to the symmetry of the compression.[24–26] In order to upgrade their stability, first, shell materials are "smeared" and profiled structures are fabricated;[27–36] second, specialized techniques of sputtering or deposition of layers for fabricating very smooth, profiled, multilayered shell surfaces are used.[37,38] Techniques for manufacturing smooth, profiled, multilayered shells and characterizing their roughness in the nanometer range using atomic microscopy are described in these publications.

In profiled, multilayered shells, the density ρ usually increases with the depth from one layer to the next. The outer ablated layer is usually fabricated from a material of a very low density, such as foam (see Refs. 39–41).[1)] In such systems, the susceptibility to inhomogeneities of the compressing laser beam or particle (electron or ion) beam is lower. Besides, the "spread" or moderation of density gradients may lead to a lower instability growth rate. Since the instability of such systems is exponential, the effect of spread may also be strong, namely exponential. Thus, the problem is to investigate the stability as a function of the profiles of density ρ or entropy s.

In addition to the physics of high energy density, another very interesting application of the techniques described in the paper is helioseismology. This term is applied to the important field of present-day solar physics dedicated to solar oscillations. This branch of science has been making steady progress. This research is closely related to the theory of stellar oscillations. Abundant literature on this topic is available; several monographs[4–6] and reviews[42–45] have been published.

Solar oscillations were detected for the first time in the 1960s.[46] Progress in helioseismology is primarily driven by improvements in experimental techniques. Highly sensitive and efficient methods of detection are used in this field. Experimental data are accumulated for a long time and then processed by computers. This research often involves expensive and sophisticated programs, such as observations in Antarctic exploiting the long polar day or space experiments. Helioseismology allows us to refine our concepts about solar structure. Some problems, naturally, remain unsolved. They include the internal differential rotation and effect of magnetic field. There is also the problem of deep internal oscillations synchronized over many (tens of thousands) periods (160-min cycle, etc.).

Physical theories of solar structure and computer simulations of its spectra seem to be equally important in helioseismology. Radiation times for stars ($E/\dot{E}$, where E is the internal star energy and $\dot{E}$ is its luminosity) are larger than dynamic times by many orders of magnitudes. If the turbulent viscosity due to convection is neglected, the equations describing the dynamics of a nonrotating, nonmagnetic star take the form

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{1}$$

$$\rho \mathbf{v}_t + \rho(\mathbf{v}\nabla)\mathbf{v} + \operatorname{grad} p - \rho \mathbf{g} = 0, \tag{2}$$

$$s_t + (\mathbf{v} \operatorname{grad} s) = 0, \quad s = p/\rho^\gamma, \tag{3}$$

$$\Delta\varphi - 4\pi G\rho = 0, \quad \mathbf{g} = -\nabla\varphi, \tag{4}$$

where G is the gravitational constant. Let us linearize these equations about the hydrostatic point. The dependent variables are written in the form

$$\rho_0 + \rho, \ \mathbf{v}, \ P + p, \ \Phi + \varphi,$$

where the functions ρ_0, $\mathbf{V} \equiv 0$, P, and Φ describe the hydrostatic equilibrium, and ρ, $\mathbf{v}$, p, and φ are perturbations. The linearized equations are usually reduced to a system of four first-order equations, or to two second-order equations, or to one fourth-order equation.[4–6] Two derivatives d/dr come from Eqs. (1)–(3), and another two from Eq. (4). The equation system is supplemented with four boundary conditions.[4–6] Two of them are defined at the center, and two on the free star surface. The resulting equation system is usually integrated from the center to the surface. The square of the frequency is a real value. Using this frequency as a variable parameter, the solution is fitted to the condition on the surface by the shooting method. The points at which the discrepancy with the surface conditions passes across zero define the eigenmodes of the system.

Cowling's approximation[4–6] is often used in calculations because it provides fairly good accuracy when the spherical harmonic number l is not too large. This approximation ignores perturbations of the gravitation potential. In this case, the system of equations is second order with respect to d/dr and is supplemented with two boundary conditions. The spectrum is again calculated numerically by the shooting method. Thus a set of frequencies ω_n is determined, where $n = 0,1,2,\ldots$ is the number of nodes of the corresponding mode on the r axis. One cannot perform sufficiently accurate calculations for n higher than some threshold value determined by the approximation error.

Thus computer simulations use the entropy distribution $S(r)$ derived from a model of the Sun, and yield unperturbed functions ρ_0, P, T, c_0, and Φ, where T and c_0 are the unperturbed distributions of temperature and adiabatic speed of sound, respectively. The calculation is performed by integrating the hydrostatic and Poisson equations supplemented with the thermodynamic equation of state. Then the spectrum is calculated numerically by integrating the linearized system of equations and varying ω, and the calculations are compared to measurements. If a discrepancy between calculated and measured spectra is detected, parameters of the physical model are varied. Such procedures have been brought to perfection.[4–6] The procedure is somewhat inconvenient because it is not straightforward, so that the fitting of calculations to experimental spectra demands a lot of CPU time.

In these circumstances, it would be desirable, without doubt, to develop an acceptable analytical technique to streamline this complicated procedure. To this end, some approaches have been developed. One of them relies on the quasi-classical description, which is valid for $n \gg 1$, whereas the other uses exact solutions of the linearized equation system for some specific shapes of the function $S(r)$, which are irrelevant to real physical conditions. The cases of (a) a density jump, (b) constant unperturbed temperature, $T(r) \equiv$ const, and (c) constant unperturbed density, $\rho_0 \equiv$ const, have been studied. The density jump has been studied more substantially in the incompressible approximation.[7,8] This is the only distribution for which the analytical theory of nonlinear effects has been developed.[8,47] In an incompressible fluid, the jump may separate uniform half spaces. Under nondissipative conditions the problem has no length parameter. In the case of a compressible medium, there is a length parameter deriving from the weight compression, namely the scale height H of a homogeneous atmosphere. Therefore the result depends on the parameter kH (k is the wave number) or the Mach number, so short-range and long-wave asymptotics can be introduced.[48]

In qualitative interpretations and estimates, an isothermal problem is often considered.[2)] The linearized system, as will be demonstrated below, includes the functions $H_s(r)$, $H_\rho(r)$, and $c_0^2(r)$ derived from $S(r)$. The first two functions are given by two inverse logarithmic derivatives:

$$H_s = kdr/d \ln S(r), \quad H_\rho = kdr/d \ln \rho_0(r). \tag{5}$$

In the isothermal case, the distributions of S and ρ_0 are exponential, and the functions in Eq. (5) and c_0^2 are constant. Therefore we obtain a system with constant coefficients. The case (b) is popular with researchers partly because this system is so simple. This model demonstrates separation of acoustic and entropic-rotational modes, and a tendency of the entropic-rotational modes to the Brunt–Väisälä frequency or growth rate by the short-range limit. An exponential distribution of ρ_0 in an incompressible liquid was first introduced in a classic paper by Rayleigh.[49] Oscillations and instabilities in case (b) have been investigated many times both with and without compression under conditions of both stable and unstable stratification with graded and stepped profiles.[4–6,8]

The distribution in case (c) presents an interesting example. This problem was solved by Pekeris.[50] This statement of the problem was reconsidered several times (see Ref. 51, Sec. 76 and Ref. 5, Sec. 17.7). This interest is quite natural since analytical solutions of such problems are rare.

In this paper we consider power-law stratification profiles. They are often good approximations to real conditions. Oscillation frequencies and growth rates of instabilities have been determined in the case of an arbitrary polytropic distribution described by the equation $P=\text{const}_p\ \rho^{(N+1)/N}$. We assume in this paper that this function describes only the hydrostatic, i.e. unperturbed, states and derives from a prescribed entropy distribution. It is known that in the case of stars with such distributions the problem of hydrostatic distributions is reduced to the well-studied Emden equation, whose solutions for a spherical object with inherent gravitational field was tabulated long ago. This is why the polytropic model is so popular among the researchers. The thermodynamic functions included in the dynamic equations are related to one another through the equation of state and thermodynamic relations, and these relations are not directly affected by the polytropic equation. To complete the description of the polytropic model, note that sometimes the polytropic dependence is defined as a power relation between pressure and density with a coefficient and exponent which is constant throughout the studied system. This may be the case of an ideal gas with a uniform distribution of entropy, or the power-law approximation of a "cold" equation of state of a condensed matter, or a degenerate Fermi distribution. These cases are discussed separately in Sec. 4. They are simpler than the general case since the number of unknowns is fewer by one and the energy equation can be omitted.

Near a vacuum interface, the gas-dynamic approach applies outside a thin layer whose depth is determined by the mean free path l_f. Thus the solutions given in the paper apply only to the dynamics of perturbations with long waves, $\lambda \gg l_f$.

The paper is organized as follows. Section 2 describes the problem statement and approximations used in different cases. It is demonstrated that they yield good accuracy when the harmonic numbers l are not too small.

In Sec. 3 we will discuss the description of pressure perturbations $\hat{p}$ in Lagrangian variables. Important isobaric solutions,[3)] in which the pressure in Lagrangian particles remains unchanged during motion, and the transformation of the Rayleigh equation to the equation for $\hat{p}$ through the inversion of density ρ_0 indicate the importance of this description.

In the case $\theta=0$, $N=1/(\gamma-1)$, with entropy uniform distribution, the problem can be formulated in the enthalpy-and-potential variables. This topic is discussed in Sec. 4.

In Sec. 5 equations describing the general polytropic case are derived and solved.

In Sec. 6 the solutions are applied to a description of instabilities in multilayered shells.

2. APPROXIMATIONS USED IN SOLVING THE PROBLEM

Perturbations of the gravitational potential φ are of minor importance as compared to density perturbations. The functions φ are spread over the space, since they describe a response to a density perturbation averaged over the mass localization region:

$$\varphi(\mathbf{r})=G\int d^3r_1\rho(\mathbf{r}_1)/|\mathbf{r}-\mathbf{r}_1|,$$

where the integral is taken over all of space and ρ is the density perturbation. As a result of integration, spatial fluctuations of φ are smoothed. It is reasonable, therefore, to ignore perturbations of the potential. This approximation is called Cowling's approximation.[4–6] It rapidly approaches the exact solution as l and/or n increases. The accuracy of this approximation is better in the case of a star with a larger fraction of its mass concentrated around its center.

In this approximation the system of equations describing the perturbation dynamics is of second order. It is usually assumed that the basic mathematical properties of the spectral problem are not affected by the approximation (see Ref. 51, Sec. 79). The approximation is often used in oceanology, meteorology, and astrophysics. In this paper we will consider only solutions obtained in this approximation. Furthermore, since Cowling's approximation applies only when l is sufficiently large, it is natural to consider the planar problem. Another natural simplification is the omission of the contribution to the gravitational field from the peripheral layer. It seems justified in Cowling's approximation because, as was stated above, the approximation is more accurate in modelling stars with a larger fraction of mass concentrated around the center. Therefore the acceleration g may be approximately considered as independent of r near the free surface of the star.

3. RAYLEIGH EQUATION

This section is dedicated to the case of an incompressible fluid, which is easier to analyze than that of a compressible fluid. It seems reasonable to start our study with this case. In the model described above, we obtain instead of the equation system (1)–(4) the following equations:

$$\rho_t+(\mathbf{v}\ \text{grad}\ \rho)=0, \tag{6}$$

$$\rho\mathbf{v}_t+\rho(\mathbf{v}\nabla)\mathbf{v}+\ \text{grad}\ p-\rho\mathbf{g}=0, \tag{7}$$

$$\text{div}\ \mathbf{v}=0. \tag{8}$$

In the unperturbed state the matter is at rest, i.e., all functions are constant with time. Under these conditions, Eqs. (6)–(8) are reduced to the equation of static balance of forces:

$$dP/dr=-\rho_0\,g,\quad g\equiv|\mathbf{g}|. \tag{9}$$

Hereinafter we denote the coordinates as x and y (the y-axis is taken instead of the radius r). The corresponding components of velocity are u and v. The unknown perturbations ρ, u, v, and p are functions of y. The dependence on x and t is described by the factor $\exp(i\omega t+ikx)$.

Let us expand the equation system (6)–(8) in small perturbations about the equilibrium state. After standard transformations, we obtain a linearized system of the form

$$i\omega\rho+\rho_0'v=0,\quad i\omega\rho_0 u+ikp=0,$$

$$i\omega\rho_0 v+p'+g\rho=0,$$

$$iku+v'=0,$$

where the prime means differentiation with respect to y. Eliminating the unknown functions ρ and u, whose derivatives are not included in the equations, we obtain

$$v'=-\frac{k^2}{i\omega\rho_0}p,\quad p'=\left(i\omega\rho_0+\frac{\rho_0'g}{i\omega}\right)v. \tag{10}$$

After eliminating v from Eq. (10), we obtain

$$\left\{p'\Bigg/\left[-i\omega\rho_0+\frac{\rho_0'g}{i\omega}\right]\right\}'=-\frac{k^2}{i\omega\rho_0}p. \tag{11}$$

Equation (11) is inconvenient because it contains the second derivative of ρ_0 but if we eliminate p, we obtain the classical Rayleigh equation[49]

$$(\rho_0 v')'=k^2\left(\rho_0+\frac{g\rho_0'}{\omega^2}\right)v,$$

$$v''_{\xi\xi}+v'_\xi/H_\rho-\left[1+\frac{1}{\Omega^2 H_\rho}\right]v=0 \tag{12}$$

written in dimensional and dimensionless forms. Here we have written $\xi=ky$, $\Omega=\omega/\sqrt{gk}$, and the function $H_\rho(\xi)$ is derived from Eq. (5).

Later we will need the pressure $P(a,b,t)$ expressed in terms of the Lagrangian coordinates a and b. We expand this pressure in a small perturbation:

$$P\left(x+\int^t u\,dt',y+\int^t v\,dt',t\right)$$

$$=P_0(y)+\frac{dP_0(y)}{dy}v(y)\frac{\exp(i\omega t+ikx)}{i\omega}$$

$$+p(y)\exp(i\omega t+ikx), \tag{13}$$

where $\int^t \mathbf{v}(a,b,t')dt'$ is the Lagrangian particle displacement. We assume that this particle with current coordinates $x(t)=a$ and $y(t)=b$ was at the point with coordinates x and y at $t=0$. To first order in a small perturbation around the state of rest, omitting the terms of the second and higher orders with respect to the perturbation amplitude, we have

$$\int^t \mathbf{v}(a,b,t')dt'=\int^t \mathbf{v}(x,y,t')dt',$$

and also $p(a,b,t)=p(x,y,t)$ in the same approximation. Similar relations are also valid for other physical parameters. Hence the $P(a,b,t)-P_0(y)$ perturbation of the pressure in a Lagrangian particle after $\exp(i\omega t+ikx)$ is fractured out is

$$\hat{p}(y)=-\rho_0 g v(y)/i\omega+p(y). \tag{14}$$

In deriving Eq. (14) we used Eq. (9). Let us replace p with $\hat{p}$ in the system of two equations (10) using Eq. (14). Then the system of equations takes the form

$$v'_\xi-\frac{v}{\Omega^2}=\frac{ik\hat{p}}{\omega\rho_0},\quad \frac{v'_\xi}{\Omega^2}-v=-\frac{ik\hat{p}'_\xi}{\omega\rho_0}. \tag{15}$$

After solving Eqs. (15) with respect to v and v'_ξ we obtain

$$\left(\Omega^2-\frac{1}{\Omega^2}\right)v=\frac{ik}{\omega}\frac{\hat{p}+\Omega^2\hat{p}'_\xi}{\rho_0}, \tag{16}$$

$$\left(\Omega^2-\frac{1}{\Omega^2}\right)v'_\xi=\frac{ik}{\omega}\frac{\Omega^2\hat{p}+\hat{p}'_\xi}{\rho_0}. \tag{17}$$

We differentiate Eq. (16) with respect to ξ and substitute the resulting derivative v'_ξ into Eq. (17). After collecting and canceling out similar terms, we obtain a second-order equation for $\hat{p}$ in the form

$$\hat{p}''_{\xi\xi}-\frac{\hat{p}'_\xi}{H_\rho}-\left[1+\frac{1}{\Omega^2 H_\rho}\right]\hat{p}=0. \tag{18}$$

It is clear that, like Eq. (12) and unlike Eq. (11), it does not contain the second derivative of ρ_0 and differs from the classical equation (12) only in the sign of $\hat{p}'_\xi$.

It turns out that Eqs. (12) and (18) are closely related (it will be demonstrated below from the inversion transformation). Specifically, let us consider Eq. (12). Assume that the density profile ρ_0 is such that the function $\rho_0(\xi)$ has constant asymptotic limits $\rho_0(+\infty)$ and $\rho_0(-\infty)$ as $\xi\to\pm\infty$. Equation (12) includes the derivative $(\ln\rho_0)'_\xi$ which tends to zero as $\xi\to\pm\infty$. Under these conditions, a physically acceptable spectrum can be found only if the perturbations decay at infinity:

$$v(-\infty)=0,\quad v(\infty)=0. \tag{19}$$

It follows from the Sturm–Liouville spectral theorem, which applies to Eqs. (12) and (18),[4)] that (a) the spectrum of the problems are defined by (12) and (19) is discrete; (b) it has an infinite, countable set of eigenvalues in the case of continuous or piecewise continuous function ρ_0; (c) the eigenvalues Ω^2 are real and their signs depend on the sign of the derivative ρ_0', namely, if there are sections of both increasing and decreasing ρ_0, there are both positive and negative infinite subspectra; if ρ_0 is a monotonic function, the values Ω^2 are either positive or negative, depending on the sign of ρ_0'; (d) density jumps generate isolated modes; (e) the values Ω^2 are bounded, namely, there are the maximum positive values $(\Omega^2)_{\max}$ and minimum negative $(\Omega^2)_{\min}$ of this parameter; there exist strict bounds for these limits: $(\Omega^2)_{\max}$ cannot be larger than unity, and $(\Omega^2)_{\min}$ cannot be smaller than minus unity; the specific values of these limits depend on the variations in $\rho_0(\xi)$ on intervals of ξ of the order of unity; $(\Omega^2)_{\max}$, $(\Omega^2)_{\min}\to 0$ in the transition from a distribution with a wide spread to a flat distribution; (f) the spectrum of Ω^2 has only one accumulation point, namely $\Omega^2=0$.

Now let us consider the transformation which inverts the density:

$$\rho_0(\xi)\to\rho_0^{\rm inv}(\xi)=\frac{1}{\rho_0(-\xi)}. \qquad (20)$$

The spectral problem defined by (12) and (19) for the inverted profile has the form

$$(v^{\rm inv})''_{\xi\xi}+(\ln\rho_0^{\rm inv})'_\xi(v^{\rm inv})'_\xi-\left[1+\frac{(\ln\rho_0^{\rm inv})'_\xi}{(\Omega^{\rm inv})^2}\right]v_\xi^{\rm inv}=0,$$

$$v^{\rm inv}(-\infty)=v^{\rm inv}(\infty)=0.$$

The cases with other interesting boundary conditions were discussed in a previous publication.[30] Let us perform the transformation described by Eq. (20). We should replace $\rho_0\to 1/\rho_0$, then $\xi\to-\xi$, and again denote the new coordinate as ξ. As a result, we have

$$(v^{\rm inv})''_{\xi\xi}-(\ln\rho_0)'_\xi(v^{\rm inv})'_\xi-\left[1+\frac{(\ln\rho_0)'_\xi}{(\Omega^{\rm inv})^2}\right]v_\xi^{\rm inv}=0,$$

$$v^{\rm inv}(-\infty)=v^{\rm inv}(\infty)=0. \qquad (21)$$

Let us compare the spectral problems (12), (19), and (21). It is surprising that, given an arbitrary distribution ρ_0, the spectra of these two problems are identical:[5)]

$$\{(\Omega^2)\}=\{(\Omega^2)^{\rm inv}\}. \qquad (22)$$

This hidden point symmetry is an example of isospectral deformation of the distribution of ρ_0 described by Eq. (12). It is, apparently, similar to the Miura transformation and Bäcklund transformations in the case of the Schrödinger equation. It is possible that, as in the case of the Backlund transformations, there is a countable set of various isospectral deformations, and Eq. (22) presents one simple example of them.

The rigorous proof of the theorem expressed by Eq. (22) was given in Ref. 52. It is based on cluster expansions and transformations of 3-by-3 matrices, and is extremely complicated and lengthy. Another proof was given in Ref. 8, Sec. 6.2.3. The property (22) was noted by Mikaelian,[28] who performed numerical calculations of spectra for stepped distributions of ρ_0. In addition, he gave explicit characteristic equations for the cases when the number of steps m was small ($m=1$, 2, and 3). The hypothesis about the invariance described by Eq. (22) was based on these particular characteristic equations. In connection with the problem of hydrodynamic stability for a laser thermonuclear fusion, Mikaelian[28] considered the optimization problem of finding the distribution with the largest $(\Omega^2)_{\rm min}$ under certain limitations on the shape of ρ_0 such that ρ_0 cannot be reduced to a constant.[6)] This problem is based on a particular inverse problem of deriving ρ_0 from properties of the spectrum. The property (22) was discovered in the process of solving the optimization and inverse problems.

Equation (18) for the Lagrangian pressure yields a very simple and physically clear proof of the existence of hidden symmetry which differ from earlier versions.[8,52]

Let us formulate the spectral problem in terms of $\hat p$. We derive boundary conditions for Eq. (18). From the definition of $\hat p$ [Eq. (14)], the first equation of the system (10), and the definition of the variable ξ, we have

$$\hat p=-\rho_0\left[\frac{gv}{i\omega}+\frac{i\omega v'_\xi}{k}\right]=-\frac{i\omega}{k}\rho_0\left(v'_\xi-\frac{v}{\Omega^2}\right). \qquad (23)$$

Consider the boundary conditions given by Eq. (19). For $\rho_0\to$ const$_\pm$ and $\xi\to\pm\infty$ the function $v(\xi)$ decays exponentially as $\exp(\pm\xi)$. Therefore $v'_\xi(\pm\infty)=0$. Hence we obtain, with due account of Eqs. (19) and (23), the boundary conditions for the Lagrangigan pressure:

$$\hat p(\infty)=\hat p(-\infty)=0. \qquad (24)$$

Let us compare the problems (12), (19), (21) and (18), (24). The spectrum of the problem (8), (12), and (19) coincides with that of (18), (24). This is obvious because we consider the same problem formulated in terms of the different variables v and $\hat p$. The problems defined by (12) and (19) and by (21) are related to one other through the inversion (20). The problems defined by (21) and by (18) and (24) are identical, so these spectra are also identical, q.e.d.

The eigenfunctions $v^{\rm inv}$ coincide with those of the perturbation $\hat p$ of the Lagrange pressure for the direct problem and vice versa. This is an interesting corollary of the new proof to the theorem.

We must note another property closely related to the Lagrange pressure. Using a direct substitution, one can prove that the classical equation (12) has solutions which satisfy one of the boundary conditions in (19) and have the form

$$\Omega^2=1,\quad v=\exp\xi,\quad \Omega^2=-1,\quad v=\exp(-\xi). \qquad (25)$$

These solutions occupy a prominent position in the dynamics of heavy liquids and gases. The first of them describes a gravitational wave, and the second the Rayleigh–Taylor instability.[27,29,30] The solutions (25) are isobaric. This means that during motion the pressure in Lagrangian particles is constant ($\hat p\equiv 0$). The latter result is derived from Eqs. (23) and (25). The gravitational wave $\Omega^2=1$ corresponds to the fundamental mode in Cowling's classification.

If follows from the isobaric condition that these solutions can also be generalized to compressible liquid. In fact, the pressure in Lagrange particles is constant, so in the adiabatic approximation, when the particles are thermally insulated, the particle volume is also constant, which implies that the solutions (25) exist for an arbitrary distribution of density in an incompressible medium or of entropy in a compressible one. Moreover, the equations of state may be arbitrary and different for different Lagrangian particles. Another consequence is that div $\mathbf{v}=0$ holds for these solutions. The first solution in (25) is a linear limit of nonlinear trochoidal waves.[27,29] The trochoidal waves are rotational and can be generalized to the case of cylindrical symmetry.[27,29,53] A linear gravitational wave on a plane interface [Eq. (25)] can be generalized for constant g to the case of spherical geometry. This is the Kelvin wave with the spectrum

$$\omega^2=[2l(l-1)/(2l+1)][g/R],$$

where $g=GM/R^2$, M and R are the mass and radius of the star, respectively (Ref. 5, p. 237). An important point is that the density is uniform and the acceleration is a linear function of radius. In the cylindrical geometry, this dependence between the acceleration and radius is related to rotation of trochoidal waves.

It is clear that the solutions (25) do not belong to those of Eq. (21) because the inverted equation is transformed to that for $\hat{p}$, and for this problem statement the solutions (25) vanish.

The above reasoning demonstrates the importance of stating the problem in terms of the Lagrangian pressure.

4. THE CASE OF A POLYTROPE COINCIDENT WITH THE ADIABAT

Let the polytrope exponent $(N+1)/N$ in the equation $P \propto \rho_0^{(N+1)/N}$ coincide with the adiabatic exponent γ. In this case the entropy distribution over the matter of the star or shell is uniform. There are no free parameters. The unperturbed distributions of the thermodynamic parameters is uniquely determined. Owing to the isentropic condition, the problem of the perturbation dynamics has one thermodynamic variable. Any thermodynamic variable may be considered as the independent variable. The equations take a simpler form if enthalpy is selected as an independent variable. In Eq. (2), we have $(\text{grad } p)/\rho = \text{grad } H$, where H is the enthalpy. Since the derivable forces in Eq. (2) are from a potential, the circulation's conserved and potential theory applies. The continuity equation (1) and the Bernoulli equation, which is an integral of Eq. (2), are transformed to

$$H_t + c^2\Delta\varphi + (\text{ grad } H \text{ grad } \varphi) = 0,$$

$$\varphi_t + (\text{grad } \varphi)^2/2 + H + gy = 0,$$

where $\mathbf{v} = \text{ grad } \varphi$, and $H = c^2/(\gamma-1)$. By eliminating H from this system of equations, we obtain the following equation for the potential:

$$\varphi_{tt} + 2\varphi_x\varphi_{xt} + 2\varphi_y\varphi_{yt} + (\gamma-1)\varphi_t\Delta\varphi + \left(\frac{\gamma+1}{2}\varphi_x^2 + \frac{\gamma-1}{2}\varphi_y^2\right)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + \left(\frac{\gamma-1}{2}\varphi_x^2 + \frac{\gamma+1}{2}\varphi_y^2\right)\varphi_{yy} + (\gamma-1)gy\Delta\varphi + g\varphi_y = 0.$$

The linearized equation system has the form

$$i\omega h + c_0^2(\varphi'' - k^2\varphi) + g\varphi' = 0, \quad i\omega\varphi + h = 0, \tag{26}$$

where c_0^2 corresponds to the unperturbed distribution, $c_0^2 = (\gamma-1)(-y)g$, $H_0 = (-y)g$, and h and φ are the perturbations of the enthalpy and potential, respectively. Since the unperturbed potential is trivial (uniform), the same symbol denotes both unperturbed and perturbed potentials. By eliminating h from the Eqs. (26) and linearizing the equation for the potential, we obtain

$$(\gamma-1)y\varphi'' + \varphi' - [\omega^2/g + (\gamma-1)k^2y]\varphi = 0. \tag{27}$$

Let us transform to dimensionless variables and substitute $\varphi(\xi) = \exp(-\xi)f(\xi)$. Then Eq. (27) takes the form

$$\xi f''_{\xi\xi} + [1/(\gamma-1) - 2\xi]f'_\xi - (\Omega^2+1)f/(\gamma-1) = 0. \tag{28}$$

The solutions of Eq. (28) are expressed in terms of confluent hypergeometric functions. A brief description of these functions is given in the book by Landau and Lifshits[54] in connection with solutions of the Schrödinger equation in the Coulomb field. The general solution with two arbitrary constants is a sum of a partial solution regular at the origin and a singular one. It has the form

$$f = c_{\text{reg}}F\left(\frac{\Omega^2+1}{2(\gamma-1)}, \frac{1}{\gamma-1}, 2\xi\right) + c_{\text{sing}} \times (-\xi)^{(\gamma-2)/(\gamma-1)}F\left(\frac{\Omega^2-1}{2(\gamma-1)} + 1, \frac{2\gamma-3}{\gamma-1}, 2\xi\right), \tag{29}$$

where $F(\alpha,\gamma,x)$ is the confluent hypergeometric function,

$$F(\alpha,\gamma,x) = 1 + \frac{\alpha}{\gamma}\frac{x^1}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \dots .$$

As usual, the spectrum is determined by two boundary conditions. One condition determines the ratio of the constants $c_{\text{reg}}/c_{\text{sing}}$ (a solution of a linear equation is arbitrary to within a constant factor), and the second condition can be satisfied only by introducing a specific relation (dispersion relation) between the frequency and wave number. In this problem one condition is defined at $(-y)=0$ (on the free surface of the star), the other at a large depth as $(-y)\to\infty$. If we compare our problem to the Schrödinger equation for the Coulomb potential, the free surface corresponds to the atomic center, $r=0$.

Let us analyze the boundary conditions on the free surface of the star. The familiar kinematic and dynamic boundary conditions have the form

$$s_t - \varphi_y|_s + s_x\varphi_x|_s = 0, \tag{30}$$

$$H|_s = \frac{g\varepsilon}{k}, \quad -\varphi_t|_s - \frac{1}{2}(\text{ grad } \varphi)^2\Big|_s - gs = H|_s, \tag{31}$$

where the function

$$y = s(x,t) = \eta_0 + \eta \exp(i\omega t + ikx)$$

defines the perturbed free surface, $\eta_0 = -\varepsilon/k$ defines the unperturbed position of the surface, and η is the small perturbation. If the star has a boundary with vacuum, then $\varepsilon=0$. But it is more convenient to shift the boundary and consider the solution in the case when the star has a boundary with a region of nonzero constant pressure equal to $[(\gamma-1)g/\gamma kS_0^{1/\gamma}]^{\gamma/(\gamma-1)}\varepsilon^{\gamma/(\gamma-1)}$, then to let this pressure go to zero concurrently with ε. Since the dynamic condition for the total pressure $(P+p)|_s = \text{ const}$ yields $H|_s = \text{ const}$, $H = H(P+p)$, the condition $H|_s = g\varepsilon$ means that the total pressure is constant on the boundary $y = s(x,t)$. The value $H|_s$ equals the unperturbed enthalpy on the boundary.

Note that on the isobaric boundary we have $\hat{p}=0$, irrespective of the full pressure on this isobaric surface. If the pressure on this surface is zero and $\rho_0(\eta_0)=0$ holds, as in the case discussed here, when $\rho_0(y) = \text{ const}(-y)^{1/(\gamma-1)}$ holds, then the Eulerian perturbation of the pressure $p = \hat{p} + \rho_0 g\eta$ [Eq. (14)] is zero on the boundary, which is also the zero-pressure surface. We will demonstrate below that the perturbations φ, v, and η are finite everywhere in the bounded region, including the boundary with the vacuum.

Let us linearize the conditions (30) and (31). We obtain

$$i\omega\eta-\varphi'|_{-\varepsilon}=0,\quad i\omega\varphi|_{-\varepsilon}+g\eta=0. \tag{32}$$

Eliminating η from Eqs. (32), we obtain

$$\varphi'_{\xi}|_{-\varepsilon}-\Omega^2\varphi|_{-\varepsilon}=0. \tag{33}$$

We substitute the general solution described by Eq. (29) into Eq. (33) and determine the ratio of the constants. We have

$$\frac{c_{\rm sing}}{c_{\rm reg}}\left\{-\frac{\gamma-2}{\gamma-1}\frac{1}{\varepsilon^{1/(\gamma-1)}}e^{\varepsilon}F_2+\varepsilon^{(\gamma-2)/(\gamma-1)}\right.$$
$$\left.\times[(e^{-\xi}F_2)'_{\xi}|_{\xi=-\varepsilon}-\Omega^2e^{\varepsilon}F_2]\right\}+(e^{-\xi}F_1)'_{\xi}|_{\xi=-\varepsilon}$$
$$-\Omega^2e^{\varepsilon}F_1=0. \tag{34}$$

For brevity, the confluent hypergeometric functions of the first and second independent solutions in Eq. (29) are denoted as F_1 and F_2, respectively. The values of the functions $\exp(-\xi)$, F_1, F_2, $(F_1)'_{\xi}$ and $(F_2)'_{\xi}$ at the point $\xi=-\varepsilon$, where $\varepsilon\ll1$, are finite. As $\varepsilon\to0$, the expression in braces in Eq. (34) tends to $[-(\gamma-2)/(\gamma-1)]\varepsilon^{-1/(\gamma-1)}$. The terms of Eq. (34) including F_1 not included in the braces yield $\exp(\varepsilon)[(F_1)'_{\xi}-(\Omega^2+1)F_1]$. We rewrite the latter expression taking into account the relation

$$F'_x(\alpha,\gamma,x)=\frac{\alpha}{\gamma}F(\alpha+1,\gamma+1,x).$$

As a result, it takes the form $[(\gamma-1)\gamma](\Omega^4-1)\varepsilon$, which retains only the lowest order term of the ε-expansion. Finally we have

$$\frac{c_{\rm sing}}{c_{\rm reg}}=\frac{(\gamma-1)^2}{\gamma(\gamma-2)}(\Omega^4-1)\varepsilon^{\gamma/(\gamma-1)}. \tag{35}$$

Note that the factor Ω^4-1 is related to the isobaric solutions given by Eq. (25).

It follows from Eq. (35) that the singular solution in Eq. (29) should be rejected in order to satisfy the condition on the boundary with the vacuum. Thus the solution satisfying the upper boundary condition is

$$\varphi(\xi)=e^{-\xi}F\left(\frac{\Omega^2+1}{2(\gamma-1)},\frac{1}{\gamma-1},2\xi\right). \tag{36}$$

Now let us try to satisfy the lower boundary condition defined at a large depth inside the star. We fix the wavelength λ. For $(-y)\gg1/k$ the relative variation of the function $c_0^2(y)$ in Eq. (26) over a length order the wavelength is small. Equation (27) is approximated by $\varphi''_{\xi\xi}-\varphi=0$ at $(-\xi)\gg1$. Its general solution (to within algebraic factors multiplying the exponential functions) consists of growing exponentials and damping and has the form

$$\varphi(\xi)=a_{\rm decr}\exp(\xi)+a_{\rm incr}\exp(-\xi). \tag{37}$$

It is clear that the perturbation described by Eq. (37) should decay with the depth, which implies

$$a_{\rm incr}=0. \tag{38}$$

The condition (38) determines the spectrum of the problem.

In order to find the spectrum, let us calculate the coefficients $a_{\rm incr}$ and $a_{\rm decr}$ the linearly independent solutions in Eq. (37). These functions vary more slowly than exponentials. We need the asymptotic expansions of the confluent hypergeometric function F (see Ref. 54):

$$F(\alpha,\gamma,x)=\frac{\Gamma(\gamma)G(\alpha,\alpha-\gamma+1,-x)}{\Gamma(\gamma-\alpha)(-x)^{\alpha}}$$
$$+\frac{\Gamma(\gamma)\exp(x)G(x)(\gamma-\alpha,1-\alpha,x)}{\Gamma(\alpha)x^{\gamma-\alpha}}, \tag{39}$$

where $\Gamma(\alpha)$ is the gamma-function. In evaluating the exponents, the absolute values of both x and $-x$ must be taken as small as possible. The asymptotic series G is

$$G(\alpha,\gamma,x)=1+\alpha\gamma/(1!x^1)$$
$$+\alpha(\alpha+1)\gamma(\gamma+1)/(2!x^2)+\dots\,.$$

Substituting the expansion (39) into Eq. (36), we have

$$\frac{\varphi(\xi)}{\Gamma(2b)}=\frac{G\{[(\Omega^2+1)b],[(\Omega^2-1)b+1],(-2\xi)\}}{\Gamma[(1-\Omega^2)b](-2\xi)^{(\Omega^2+1)b}}e^{-\xi}$$
$$+\frac{G\{[(1-\Omega^2)b],[1-(\Omega^2+1)b],(2\xi)\}}{\Gamma[(\Omega^2+1)b](2\xi)^{(1-\Omega^2)b}}e^{\xi}, \tag{40}$$

where $b=1/[2(\gamma-1)]$.

The component that grows in the limit $\xi\to-\infty$ in Eq. (40) drops out if

$$(1-\Omega^2)b=-m, \tag{41}$$

where $m=0,1,2,\dots$, or in dimensional variables if

$$\omega^2=[1+2m(\gamma-1)]\sqrt{gk}, \tag{42}$$

since the gamma-function $\Gamma[(1-\Omega^2)b]$ in the denominator of the first term on the right-hand side of Eq. (40) tends to infinity in this case. Equations (41) and (42) define the desired eigenvalue spectrum.

Let us write expressions for the eigenfunctions. Under the condition (41), Eq. (36) takes the form

$$\varphi(\xi)=\exp(-\xi)F\left[\frac{1}{\gamma-1}+m,\quad\frac{1}{\gamma-1},2\xi\right].$$

We employ the useful relation

$$F(\alpha,\gamma,x)=\exp(x)F(\gamma-\alpha,\gamma,-x),$$

derived in the theory of confluent hypergeometric functions. The expression for φ can be transformed using this relation to

$$\varphi(\xi)=\exp(\xi)F\left[-m,\frac{1}{\gamma-1},-2\xi\right].$$

The series which defines F is finite for negative integer values of the first argument [see the expansion of F in Eq. (29)]. In this case, it is a polynomial of power m. Given the formulas for such polynomials,[54] we can express the eigenfunctions corresponding to the interesting values of ξ ($\xi<0$) as

$$\varphi(\xi)=\frac{(-1)^m}{\beta(\beta+1)\cdots(\beta+m-1)}(-\xi)^{1-\beta}e^{-\xi}\frac{d^m}{d\xi^m}\times[(-\xi)^{\beta+m-1}e^{2\xi}], \tag{43}$$

where $\beta=1/(\gamma-1)$. The polynomial multiplying the exponential function in Eq. (43) is proportional to the generalized Laguerre polynomial $L_m^{(\beta-1)}(-2\xi)$ since

$$F(-m,\beta,x)=m!L_m^{(\beta-1)}(x)/[\beta(\beta+1)\ldots(\beta+m-1)],$$

(see Ref. 55, p. 806). In the case of the hydrogen atom, the upper indices of the Laguerre polynomial incorporated in the radial functions are integers depending on the orbital quantum moment l. The functions in Eq. (43) are normalized so that the perturbation amplitude is unity on the star surface $[\varphi(0)=1]$.

5. DESCRIPTION OF AN ARBITRARY POLYTROPE

Let the polytropic exponent $(N+1)/N$ be a parameter independent of the adiabatic exponent γ. In this case we have a nonzero index θ of the entropy distribution compensating for the difference between $(N+1)/N$ and γ. Then the force in the Euler equation (2) cannot be expressed as the gradient of some potential, and the expression $\mathbf{v}=\nabla\varphi$ is not valid, in contrast to the case discussed in the previous section.

Let us rewrite the equations of continuity, momentum and energy conservation (1)–(3) in the form

$$\frac{d\rho}{dt}=-\rho\,\mathrm{div}\,\mathbf{v},\quad \frac{d}{dt}=\frac{\partial}{\partial t}+(\mathbf{v}\nabla), \tag{44}$$

$$\rho\mathbf{v}_t+\rho(\mathbf{v}\cdot\nabla)\mathbf{v}=-\,\mathrm{grad}\,p+\rho\mathbf{g}, \tag{45}$$

$$\frac{dp}{dt}=c^2\frac{d\rho}{dt}. \tag{46}$$

Let us find the unperturbed solution. To this end, distributions of thermodynamic functions in hydrostatic equilibrium must be calculated. We integrate the hydrostatic equation (9) with $g=|\mathbf{g}|$, $\mathbf{g}=(0,-g)$ together with the polytrope $P=\mathrm{const}_p\rho_0^{(N+1)/N}$ and the adiabat $P=S\rho_0^\gamma$. As a result, we have

$$\rho_0=\left[\frac{-gy}{(N+1)\,\mathrm{const}_p}\right]^N,\quad P=(\mathrm{const}_p)^{-N}\left[\frac{-gy}{N+1}\right]^{N+1}, \tag{47}$$

$$S=(\mathrm{const}_p)^{N(\gamma-1)}\left[\frac{-gy}{N+1}\right]^\theta,\quad c_0^2=\frac{-\gamma gy}{N+1},$$

$$\theta=1-N(\gamma-1). \tag{48}$$

The origin on the vertical y-axis is selected so that the plane $y=0$ coincides with the unperturbed boundary with the vacuum. In this section we consider the dynamics of perturbations of the states described by Eqs. (47) and (48). The solution of the respective spectral problem should include only powers of the distributions of ρ_0 and S, and the factor multiplying $(-gy)$ in the equation for c_0^2.

Let us start by linearizing the system of equations (44)–(46). Following the standard procedure, we obtain

$$i\omega\rho+\rho_0'v=-\rho_0\,\mathrm{div}\,\mathbf{v}, \tag{49}$$

$$i\omega\rho_0u=-ikp, \tag{50}$$

$$i\omega\rho_0v=-p'-\rho g, \tag{51}$$

$$i\omega p-\rho_0gv=-\rho_0c_0^2\,\mathrm{div}\,\mathbf{v}, \tag{52}$$

where the perturbations are denoted, as in the previous sections, by ρ, u, v, and p. In deriving Eq. (52) we have used Eq. (9).

In the derivation of Eqs. (13) and (14), the perturbed pressure in a Lagrangian particle was defined by Eq. (14). Since the condition of incompressibility was not used in this derivation, Eq. (14) is valid in both incompressible and compressible cases.

Following Sec. 3, is it interesting to derive a spectral equation with $\hat{p}$ as an unknown parameter. To this end, let us supplement the equation system (49)–(52) with the equation defining $\hat{p}$. As a result, we have a system consisting of the five equations (49)–(52) and (14) for the five unknown functions, ρ, u, v, p, and $\hat{p}$. This system has the form

$$i\omega\rho+\rho_0'v=(i\omega/c_0^2)\hat{p}, \tag{53}$$

$$i\omega\rho_0u=-ikp, \tag{54}$$

$$i\omega\rho_0v=-p'-g\rho, \tag{55}$$

$$i\omega p-\rho_0gv=i\omega\hat{p}, \tag{56}$$

$$i\omega\hat{p}=-\rho_0c_0^2(iku+v'). \tag{57}$$

Equation (53) is the mass conservation equation (49) with the right-hand side transformed according to Eqs. (14) and (52). Equations (54) and (55) describe conservation of the momentum components. Equation (56) is identical to Eq. (14), and Eq. (57) is a combination of Eqs. (14) and (52). If we disregard the physical meaning of the parameter $\hat{p}$, Eq. (56) can be considered as a formal definition of the additional unknown $\hat{p}$.

Let us start eliminating the unknowns. First of all, we eliminate u and ρ, which are included in the equation system in algebraic form. We can express p in all the equations in terms of $\hat{p}$ through Eq. (56), which is algebraic, i.e., does not include derivatives. After these transformations, we obtain a system of two equations for the unknowns v and $\hat{p}$ similar to Eq. (15) for incompressible media

$$v_\xi'-\frac{v}{\Omega^2}=\left(1-\frac{\Omega^2g}{kc_0^2}\right)\frac{ik}{\omega\rho_0}\hat{p},$$

$$\frac{v_\xi'}{\Omega^2}-v=-\frac{ik}{\omega\rho_0}\left(\hat{p}_\xi'+\frac{g}{kc_0^2}\hat{p}\right). \tag{58}$$

By solving the linear system (58) for the unknowns v and v', we obtain expressions for v and v' in terms of $\hat{p}$ and $\hat{p}'$ similar into Eqs. (16) and (17). After substituting the expression for v into that for v', we obtain a second-order equation with respect to $\hat{p}$ similar to Eq. (18), whose coefficient includes only first derivatives of ρ_0 and S with respect to y or $\xi=ky$. After some manipulation, this equation assumes the form

$$\hat{p}''_{\xi\xi}-\frac{\hat{p}'_{\xi}}{H_\rho}-\left\{1+\frac{1}{\Omega^2 H_\rho}+\left(\Omega^2-\frac{1}{\Omega^2}\right)\left[\frac{1}{\gamma H_s}+\frac{1}{H_\rho}\right]\right\}\hat{p}=0, \tag{59}$$

where the local relative scale parameters H_s and H_ρ are

$$H_s=k\left(\frac{d\ln S}{dy}\right)^{-1},\quad H_\rho=k\left(\frac{d\ln\rho_0}{dy}\right)^{-1},$$

as was stated above.

Let us substitute the expressions for H_s and H_ρ associated with the equilibrium polytropic distribution described by Eqs. (47) and (48) into Eq. (59). After simple calculations, we obtain

$$\xi\hat{p}''_{\xi\xi}-N\hat{p}'_{\xi}-\left[\xi+\frac{N}{\Omega^2}+(N+1)\left(\Omega^2-\frac{1}{\Omega^2}\right)\Big/\gamma\right]\hat{p}=0. \tag{60}$$

We compare this with the equation for the confluent hypergeometric function[54]

$$zF''_{zz}+(\kappa-z)F'_z-\alpha F=0. \tag{61}$$

One can see that Eq. (60) can be transformed into Eq. (61) if the ξ-dependence of the factor multiplying $\hat{p}$ is eliminated. In order to cancel out the term $\xi\hat{p}$ and transform Eq. (60) into Eq. (61), we introduce the variables, $\hat{p}=\exp(\xi)f$ and $\xi=-z/2$. The equation for f takes the form of Eq. (61) with the coefficients

$$\kappa=-N,\quad \alpha=-\frac{1}{2}\left[N+\frac{N}{\Omega^2}+(N+1)\frac{\Omega^2-\Omega^{-2}}{\gamma}\right]. \tag{62}$$

The general solution of Eq. (61) is[54]

$$c_1F(\alpha,\kappa,z)+c_2z^{1-\kappa}F(\alpha-\kappa+1,2-\kappa,z).$$

Therefore the desired general solution of Eq. (60) has the form

$$\hat{p}(\xi)=c_1e^{\xi}F(\alpha,-N,-2\xi)+c_2(-\xi)^{N+1}e^{\xi}F(\alpha+N+1,N+2,-2\xi), \tag{63}$$

where α is defined by Eq. (62).

The dispersion relation is derived from the boundary conditions. Let us proceed to the analysis of these conditions. We start with the boundary condition on the free boundary. Assume that this is the boundary with vacuum. Then the unperturbed boundary coincides with the plane $\xi=0$, as follows from Eqs. (47) and (48). The perturbed pressure in Lagrangian particles vanishes on the free boundary because it is an isobaric surface. Since the free boundary is at $\xi=0$, we should impose the condition

$$\hat{p}(0)=0. \tag{64}$$

It is clear that this condition (64) implies that the parameter of the general solution (63) satisfies

$$c_1=0. \tag{65}$$

In fact, we have $F(\alpha,\kappa,0)=1$, and $(-\xi)^{N+1}$ vanishes at $(-\xi)=0$ since, as will be demonstrated below, $N+1>0$.

Let us investigate the internal boundary condition. Inside the system, the solution given by Eq. (63), as in the case considered in Sec. 4 [Eqs. (37) and (38)], is a sum of the growing and decaying exponentials because the speed of sound increases with the depth, the medium becomes asymptotically incompressible, and relative changes in the coefficients of Eq. (60) over a length of order λ are small far from the vacuum boundary. The necessary condition is that the solution should decay as $(-\xi)\to\infty$.

Let us find the factor at the exponent that increases with $(-\xi)$. We derive the asymptotic form of the solution (63) with the condition (65) using Eq. (39). After this substitution we have

$$\hat{p}\propto(-\xi)^{N+1}\Gamma(N-2)\left\{e^{\xi}\frac{G[(\alpha+N+1),(\alpha+4),2\xi]}{\Gamma(-\alpha-3)(2\xi)^{\alpha+N+1}}+e^{-\xi}\frac{G[(-\alpha-3),(-\alpha-N),(-2\xi)]}{\Gamma(\alpha+N+1)(-2\xi)^{-\alpha-3}}\right\}. \tag{66}$$

The factor multiplying $e^{-\xi}$ must vanish so the function $\Gamma(\alpha+N+1)$ must be infinite. This implies the condition

$$\alpha+N+1=-m,\quad m=0,1,2,\dots. \tag{67}$$

After substituting the expression for α in Eq. (62) into Eq. (67), we have the desired dispersion relation:

$$\Omega^4-\frac{N+2m+2}{N+1}\gamma\Omega^2+\frac{N\gamma-N-1}{N+1}=0,\quad m=0,1,2,\dots. \tag{68}$$

The solution of Eq. (68) is

$$\Omega_{1,2}^2=\frac{1}{2}\frac{N+2m+2}{N+1}\gamma\pm\sqrt{\frac{1}{4}\left(\gamma\frac{N+2m+2}{N+1}\right)^2-\frac{N\gamma-N-1}{N+1}},$$

$$m=0,1,2,\dots. \tag{69}$$

This spectrum should be supplemented with the fundamental mode $\Omega_2^2=1$ (see below).

The physical meaning of the roots Ω_1^2 and Ω_2^2 in Eq. (69) is clear if we consider the asymptotic forms in the limit $m\to\infty$. The expansion of the right-hand side of Eq. (69) at large m yields

$$\Omega_1^2=\frac{2\gamma m}{N+1}\to\infty,\quad \Omega_2^2=\frac{N\gamma-N-1}{2m\gamma}=-\frac{\theta}{2\gamma m}\to 0. \tag{70}$$

It follows from the expansion (70) that the first root corresponding to high frequencies is associated with acoustic waves, and the second with entropic-rotational modes. The frequencies of the latter go to zero in the isentropic case, when $\theta=0$. For $\theta<0$ (stability) we have $\Omega_2^2>0$ in Eq. (69) and for $\theta>0$ we have $\Omega_2^2<0$, which corresponds to the unstable condition. We see that the spectrum splits into pairs of acoustic and entropic-rotational or gravitational modes. According to the terminology suggested by Cowling, the former are called p-modes and the latter g-modes (p means pressure and g gravitation). These pairs have identical eigenfunctions

$$\hat{p}(\xi)=c_2(-\xi)^{N+1}e^{\xi}F(-m,N+2,-2\xi). \tag{71}$$

The expressions for the other unknown function described by Eqs. (53)–(57) contain frequency, so the functions describing acoustic and entropic-rotational modes are different.

Let us express the functions in Eq. (71) in terms of the Laguerre polynomials:

$$\hat{p}(\xi)=c_2\frac{(-1)^m e^{-\xi}(d^m/d\xi^m)[(-\xi)^{N+m+1}e^{2\xi}]}{(N+2)(N+3)\dots(N+m+1)}$$

$$=c_2\frac{(-\xi)^{N+1}e^{\xi}m!L_m^{(N+1)}(-2\xi)}{(N+2)(N+3)\dots(N+m+1)}.$$

To compare the spectrum defined by Eq. (69) with that derived in Sec. 4, we take $\theta=1-N(\gamma-1)=0$. One can easily check that in this case the spectrum of acoustic or p-modes Ω_1^2 [Eq. (69)] is identical to that determined in Sec. 4 [Eq. (42)]. The only difference is the presence of the fundamental or f-mode. According to Cowling's classification,[4–6] p- and g-modes are separated by the f-mode. The spectrum defined by Eq. (42) consists of a countable set of acoustic modes with $m\geqslant 1$ and the first gravitational mode. It corresponds to $m=0$ in Eq. (42). This mode is called fundamental. The frequency of the acoustic modes becomes infinite in the incompressible limit $\gamma\to\infty$ [Eqs. (42) and (69)], while the frequencies of the gravitational modes remain finite.

The fundamental mode is not included in the spectrum defined by Eq. (69) because this spectrum was derived from Eq. (59) for the perturbation of $\hat{p}$ in Lagrangian particles, whereas in the case of the f-mode this pressure is identically equal to zero because the isobaric surfaces are "frozen."[27] Therefore the respective eigenfunction cannot be derived from Eq. (59).

Let us check our results using measurements of solar oscillations. The comparison is shown in Fig. 1. The solid curves of $\Omega(l)$ were plotted by converting experimental data from Refs. 56 and 57. We assumed $\Omega=\omega/\sqrt{gk}$, where g is the acceleration on the surface, $k=\sqrt{l(l+1)}/R$, and R is the star radius. The variables Ω and l are more convenient than ω and l because the functions $\Omega(l)$ are nearly flat. The calculations are shown in Fig. 1 by lines of circles. Figure 1 shows calculations in the isentropic case, $\theta=0$. Varying θ by ± 0.1 has little effect on the spectrum of Ω.

We can see the satisfactory agreement between the theory and experiment. The fundamental mode is not susceptible to the thermodynamic state of the Lagrangian particles[27] because the isobars are "frozen." Therefore its frequency Ω equals unity with good accuracy throughout the range of

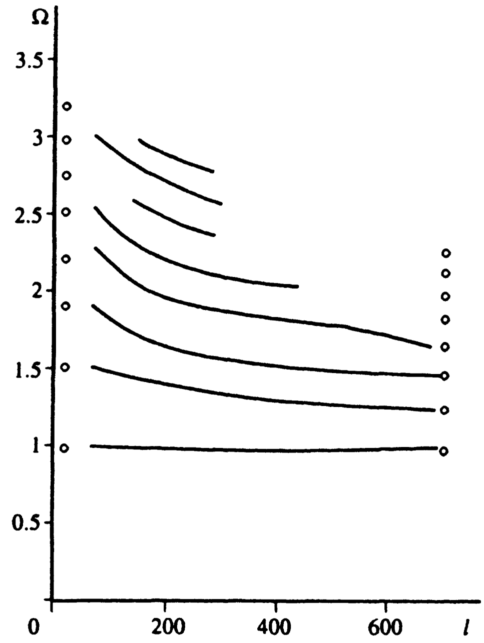

FIG. 1. Comparison of calculations (circles) with experimental data on solar oscillations (solid curves). Curves of Ω versus the harmonic number l are given for the f-mode and the first seven p-modes. The left line of circles corresponds to $\gamma=5/3$, the right line to $\gamma=1.3$.

l. The frequency Ω of the p-modes decreases with l in the case of the mode concentrated near the surface, probably owing to a drop in the effective adiabatic exponent due to ionization, which is important near the surface.

6. INSTABILITY OF SHELLS WITH POWER-LAW PROFILES

It is well known that in the case of shells used in inertial confinement experiments the artificial gravitation is due to the acceleration of the reference frame connected to the shell material. Typical curves of the power of a compressing beam versus time, $P_c(t)$, power released due to fusion, $P_b(t)$, and shell radius, $R(t)$, are given in Fig. 2. The duration and sometimes the shape of the compressing pulse are predetermined by the target hydrodynamics. The short pulse of released power is centered around the point of maximum shell compression.

The process includes a brief stage *1–2* (Fig. 2b) in which the shell is set in motion and the stage of shell acceleration towards the center, when its velocity increases. There is an

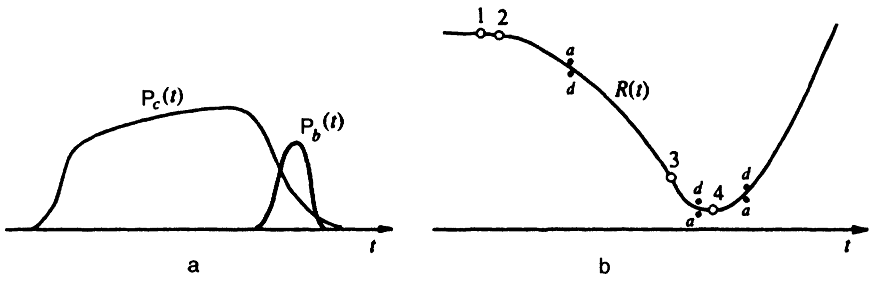

FIG. 2. (a) Power of the compressing pulse, $P_c(t)$, and released power $P_b(t)$; (b) radial coordinate $R(t)$ of the shell driven by the compressing pulse.

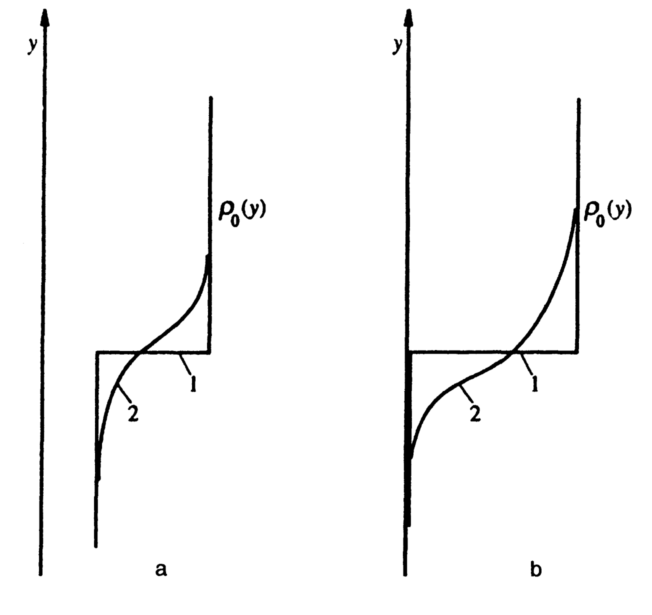

FIG. 3. Comparison between (*1*) narrow and (*2*) spread-out profiles for the case (a) $\rho(-\infty) \neq 0$ and (b) $\rho(-\infty)=0$. The latter corresponds to a free lower boundary.

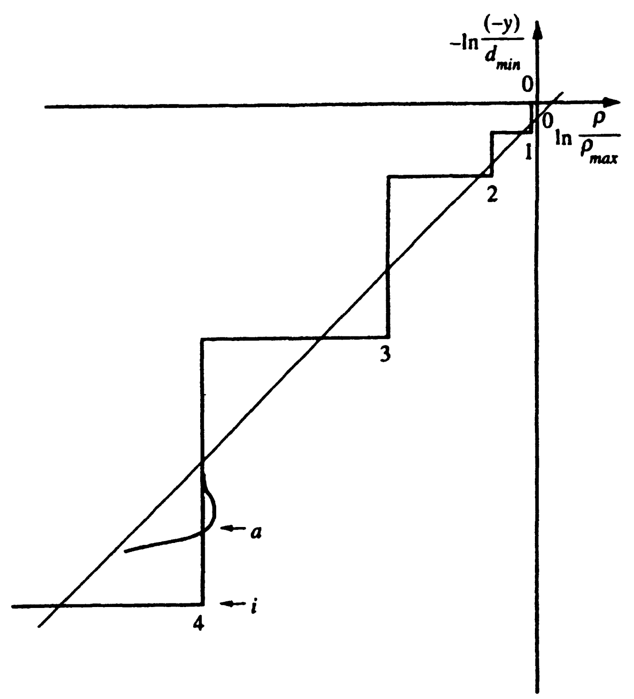

FIG. 4. Stepped distribution of density in a multilayered shell and its continuous approximation.

inflection point *3* at which the acceleration changes its sign. Then the shell is decelerated by the thrust generated by the fuel (stage *3–4*), stopped (point *4*), and flies apart.

In the acceleration stage the external pressure is higher than the internal pressure. Therefore the points a and d to which the accelerating p_a and decelerating p_d pressures are applied are located outside and inside the shell, respectively (Fig. 2b). In the deceleration stage these points exchange their locations. The instability develops in the outside layers of the shell in the acceleration stage and in the outside layers in the deceleration stage. In the stage *2–3* the instability can disrupt the shell, which ends the compression. In the deceleration stage instability causes mixing of the hot fuel with the cold shell material, which results in a lower efficiency of the power generation.

Let us discuss the instability in the acceleration stage. Is it worthwhile to coat the shell with layers of gradually decreasing density on the outside surface in order to reduce the growth rate of unstable modes? On one hand, it seems that a decrease in the gradients of ρ_0 or S should lead to a decrease in the instability growth rate at fixed λ. But on the other hand, if an arbitrarily stratified shell is bounded below by a free surface, then an additional isobaric mode with the growth rate $\Gamma^2=-\Omega^2=-1$ emerges.[27,29] It has the maximum possible growth rate.[7)] It seems that spreading out the density profile is useless, at least in the case of laser compression, when the boundary conditions on the ablation front are close to those on a free boundary. Given this reasoning, investigators usually analyze the instability damping taking stepped profiles with a finite ratio $\ln[\rho_0(\infty)/\rho_0(-\infty)]\simeq 1$ as shown in Fig. 3a, when there is no unstable f-mode and the effect of profiling is self-evident.

But the situation may be quite different. Consider a deep, spread-out distribution of the density ρ_0. It may be either stepped or continuous (Fig. 4). The steps *1*, *2*, ... correspond to the shell compressing the fuel (step *1*) the shell *2* protecting the fuel from preheat by superthermal electrons and x-rays, and the transitional layer *3* between the ablation layer *4* and the high-density layers *1* and *2*. The function shown in Fig. 4 is normalized to the minimum length—the thickness $d_{\min}$ of layer *1*—and its density $\rho_{\max}$ (the subscript 0 of the density function indicating that the function describes the unperturbed hydrostatic distribution is omitted here and below). We consider a model of a real multilayered shell fairly adequate from the dynamical viewpoint. The model distribution is composed of one material, whose entropy is discontinuous on the boundaries between layers. The masses of the layers are comparable, and their structure is similar to that of typical multilayered shells designed for the National Ignition Facility.[37,38] The ratios $\ln(\rho_1/\rho_n)$ and $\ln(d_n/d_1)$ are much larger than unity (here $d_1, d_2, \ldots, d_n$ and $\rho_1, \rho_2, \ldots, \rho_n$ are the thicknesses and densities of consecutive layers, respectively). This distribution has been defined above as spread-out and strongly profiled.

The stepped distribution can be approximated by a continuous one. If the approximating curve is approximately linear in the $\ln(-y)$, $\ln(\rho)$ plane, as shown in Fig. 4, the corresponding profile is approximately described by a power law.

The arrows i and a in Fig. 4 indicate the initial position of the outside boundary of the shell and the current position of the ablation front, which propagates through the ablated material. The point $(-y)=0$ is on the inside surface of the shell. Let us denote the current coordinate of the ablation front by $(-y_a)$. It was said above that $(-y_a)$ is notably larger than the thicknesses of first high-density shells. In the acceleration stage the modes with wavelengths smaller than

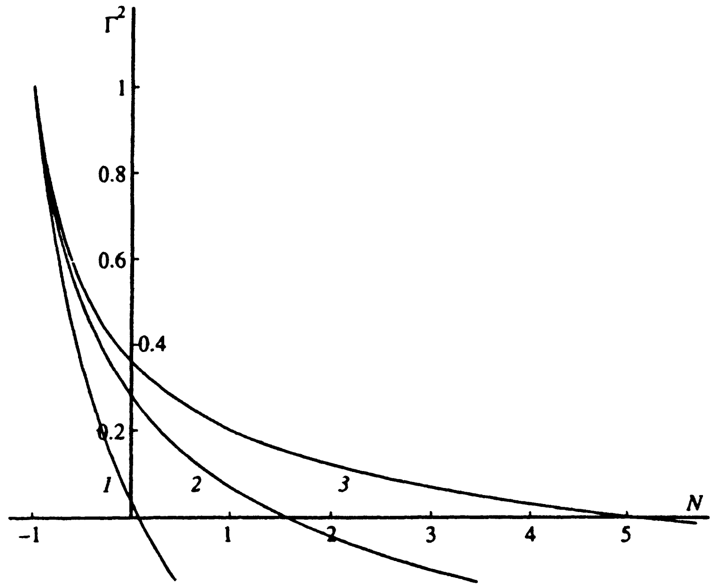

FIG. 5. Damping of instability in profiled shells with masses of the outside layers larger than those of inside layers. In such shells we have $N>-1$. The square of the instability growth rate κ divided by the classical limiting growth rate $\sqrt{gk}$ is plotted against N. The curves *1*, *2*, and *3* correspond to the indices $\gamma=20$, 5/3, and 1.2, respectively. The extreme values of γ are close to those in the incompressible and isothermal cases.

the large scale $(-y_a)$ are more important because a large-scale mode takes a long time to develop.

An important point is that the modes *1* related to the ablation front and modes *2* due to the wide spread of the profile are separated in space. The modes *1* are concentrated near the ablation front, and the modes *2* are near the boundary $(-y)=0$. The development of the dangerous modes *2*, which can disrupt the high-density shell, is independent of the situation on the ablation front, which is far from the region of the modes *2*.

The growth rate $\Gamma^2=-\Omega^2$ of the modes *2* are determined by Eq. (69) with the minus sign. Our interest is in the fastest growing mode of the countable set of the modes *2*. Its squared growth rate is calculated by taking $m=0$:

$$\Gamma^2=\sqrt{\left[\gamma\frac{N+2}{2(N+1)}\right]^2+\frac{1-N(\gamma-1)}{N+1}}-\gamma\frac{N+2}{2(N+1)}. \tag{72}$$

The analysis of Eq. (72) indicates that the squared growth rate Γ^2 can be scaled down by taking a broader profile (Fig. 5). The instability exists in the region

$$-1<N<1/(\gamma-1), \quad \gamma>\theta>0.$$

In the case of incompressible media $(\gamma=\infty)$ the instability condition is $N<0$. The value $N=-1$ defines a natural physical limit for the class of possible power-function distributions. In the limit $N\rightarrow-1$ or $\theta\rightarrow\gamma$ the temperature gradient becomes infinite. This case corresponds to a profile in which the masses of the consecutive layers are equal to each other. It follows from Eq. (72) that in this case the squared growth rate satisfies $\Gamma^2\rightarrow 1$ (see also Fig. 5). Thus, in order to reduce the squared growth rate, one must create a distribution of mass among layers such that the integral $\int\rho dy$ diverges at the low-density boundary. This statement is illustrated by Fig. 5.

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1)Lebo *et al.*[39] simulated the 2D dynamics of a target coated with a low-density ablator. Gus'kov *et al.*[40] and Lebo *et al.*[41] studied 1D and 2D models, respectively, of accelerated targets coated with foam and placed inside a closed chamber (Hohlraum).

2)Note that, like in the case of a polytrope, the isothermal condition is usually related to the hydrostatic distribution of temperature. In the general case, when the adiabatic exponent is arbitrary and the energy flow is described by Eq. (3), the temperature perturbations are nonzero. They tend to zero when the exponenet tends to unity. This is also valid in case (c), when the density perturbation is nonzero for an arbitrary exponent, although it tends to zero when the exponent tend to infinity (incompressible liquid).

3)These solutions are closely related to classical trochoidal waves (see Refs. 27, 29, 30, and references therein).

4)See for example Ref. 30 and references therein.

5)The eigenfunctions of the direct and inverse problems are different.

6)The squared frequency $\Omega^2_{\min}$ is negative (see above), corresponding to instability. The largest negative value means the smallest growth rate. Hence the optimal profile is such that the rate at which the instability develops should be a minimum.

7)The spectrum of an arbitrary profile consists of p-modes with an accumulation point at infinity and g-modes which accumulate at zero frequency and are bounded by two f-modes. These f-modes are extreme g-modes. In fact, p- and g-modes have different asymptotic forms in the limit of incompressible liquid and in the long-wave limit, for example, in the case of a layer bounded by two free surfaces. The analysis of the asymptotic limits of these f-modes indicates that they are g-modes.

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