Spin relaxation of two-dimensional electrons with an odd Landau-level filling factor in a strong magnetic field

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Spin relaxation is investigated in a system of interacting 2D electrons in a strong magnetic field perpendicular to the layer. The process corresponds to the annihilation of two-dimensional spin excitons with the emission (or absorption) of a phonon when the spin–orbit terms in the Hamiltonian are simultaneously taken into account. At low temperatures the deviation from equilibrium is uniquely specified by the number of "zero-point" excitons, which form a two-dimensional condensate. Annihilation in the condensate is accompanied by the acquisition of finite momentum by one of the zero-point excitons. When the z component of the total spin approaches the equilibrium value, relaxation is driven by the annihilation of "thermal" excitons, whose distribution is characterized by a Bose function. © *1996 American Institute of Physics.* [S1063-7761(96)01907-5]

1. INTRODUCTION

Spin polarization and relaxation in a system of two-dimensional electrons under the conditions of the quantum Hall effect (QHE) has already been the subject of numerous experimental (see, for example, Refs. 1–4) and theoretical[5–9] studies. There is interest in this particular case, because such investigations provide a way to determine the fundamental properties of a 2D system in a strong magnetic field. One common feature of all the experiments cited was the fact that the two-dimensional layer was formed in them by a GaAs/AlGaAs semiconductor heterostructure, although the main results of the theoretical studies are valid, in principle, in the case of diamond-like semiconductors (a Ge/Si junction). The other experimental conditions, which of course also determine the specific spin relaxation mechanism, were considerably more diverse. In formulating the problem we adhere mainly to the experimental conditions in Ref. 3, i.e., we restrict ourselves to a spin–orbit relaxation mechanism in a homogeneous (without consideration of the edge effects), "pure" (with no paramagnetic impurities) system, and we neglect the exchange interaction of the electrons with band holes (see Ref. 10) in view of the small concentration of the latter. In addition, in accordance with Ref. 3, we assume that initially the deviation of the polarization from its equilibrium value is generally considerable, and that the total number of electrons in the system $\mathcal{N}_0$ is fixed during the entire relaxation period, the filling factor ν being odd:

$$\nu=2n+1, \qquad \text{where } \nu=2\pi\lambda^2\mathcal{N}_0/L^2. \tag{1.1}$$

Here $L\times L$ is the size of the 2D system, $\lambda=(c\hbar/eB)^{1/2}$ is the magnetic length, $\mathbf{B}\|z$ (the magnetic field is perpendicular to the plane of the layer), n is the number of the half-filled Landau level (in Ref. 3, $n=0$, 1).

The Zeeman energy $g\mu_bBS_z$ removes the degeneracy with respect to the z component S_z of the total spin. This is why the ground state of the system is a state with $|S_z|$ equal to the highest possible value $S=\mathcal{N}/2$, where $\mathcal{N}=L^2/2\pi\lambda^2$ is the number of orbital states or the number of electrons in the nth Landau level, which amounts to the same thing, even when the Coulomb interaction is taken into account (in the absence of any off-diagonal terms with respect to S_z in the Hamiltonian). We stress that such a simple "ferromagnetic" type of ground state is a consequence of the condition (1.1). The recent experiment in Ref. 4, in which the magnetization of a 2D system was determined (from the NMR Knight shift) as a function of ν and the temperature, underlines the validity of this assertion for $\nu=1$, on the one hand, but demonstrates the radical alteration of the ground-state structure when ν deviates from unity, on the other. As was asserted in some theoretical studies,[7–9] a complex spin and charge texture appears in this case in the form of a lattice of so-called Skyrmions with a characteristic period proportional to $|1-\nu|^{-1/2}$. We assume that (1.1) holds at least to an accuracy of 1–2%; therefore, the ground state remains "ferromagnetic," and the lower part of the spectrum, which is pertinent to the low-temperature range, is determined by spin-wave excitations (in accordance with the measurements in Ref. 4), which are the spin analog of two-dimensional Mott excitons.[11–17]

A special role is played in the spectrum of excited states by zero-point spin excitons, i.e., the lowest states in the spectrum, which are separated from the nominal vacuum by the Zeeman gap $\varepsilon=|g|\mu_bB$ and have a two-dimensional momentum $\hbar\mathbf{q}$ equal to zero. The generation of zero-point excitons signifies only a change in the component S_z of the system without a change in its spatial wave function. The corresponding transition is described by the action of the creation operator Q_0^+ for zero-point excitons on an arbitrary

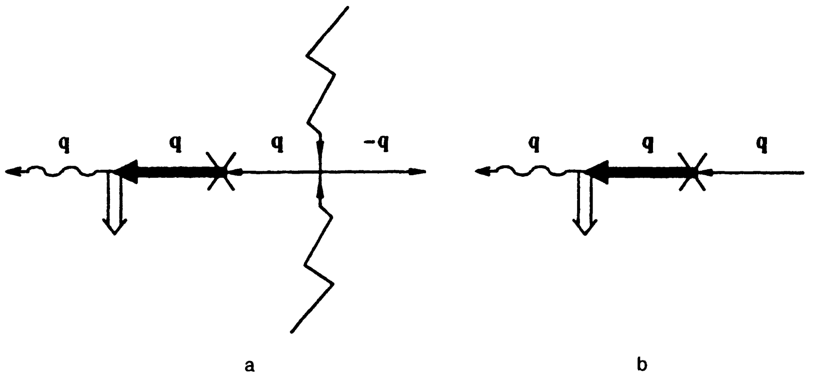

FIG. 1. a) Diagram of the annihilation of a zero-point exciton in a condensate. b) Diagram of the annihilation of a supracondensate non-zero-point exciton.

state. Up to a normalization factor, this operator raises or lowers the spin index (depending on the sign of g):

$$Q_0^+=\begin{cases}\mathcal{N}^{1/2}\sum\limits_i \sigma_+^{(i)}, & g>0\\ \mathcal{N}^{1/2}\sum\limits_i \sigma_-^{(i)}, & g<0,\end{cases} \tag{1.2}$$

where the index i labels the electrons, the σ are Pauli matrices, and $\sigma_\pm=(\sigma_x\pm i\sigma_y)/2$. If the Hamiltonian of the system $\mathcal{H}$ is diagonal with respect to S_z, we have the commutation relations $[\mathcal{H},Q_0^\pm]=\pm\varepsilon Q_0^\pm$ (here $Q_0^-=Q_0=(Q_0^+)^+$), from which it follows that the state obtained by applying Q_0^+ to the original eigenstate of the system will also be an eigenstate with the same value of the total spin S, but with an energy exceeding the initial value by ε. In such a system the zero-point excitons are noninteracting, and no exchange energy appears when they are generated.

Clearly, a significant deviation of $|S_z|$ from its equilibrium value (which is close to $\mathcal{N}/2$) in terms of exciton states can be interpreted as the appearance of a macroscopically large number of zero-point excitons in the system, which form a two-dimensional condensate (see Ref. 12). The return of S_z to the equilibrium value under these conditions corresponds to the "spin–orbit destruction" of the condensate or annihilation of the zero-point excitons, which amounts to the same thing.

At the same time, it is clear that spin relaxation is simultaneously also energy relaxation (when one exciton is annihilated, a quantity of energy close to ε is lost); therefore, it is possible, for example, when the nonstationary field due to the interaction of the electronic system with lattice vibrations is taken into account. To visualize this, we illustrate a relaxation process in Fig. 1a. The broken lines in the diagram depict two zero-point spin excitons, which transform virtually into two spin excitons with finite momenta$^{1)}$ (depicted by the thin lines). Then, an electron in one of these "non-zero-point" excitons passes to another Landau level with simultaneous spin flip by means of the spin–orbit interaction (the corresponding vertex is denoted by a cross), so that $|S_z|$ increases by unity, and the spin exciton transforms virtually into a purely orbital or cyclotron exciton, conserving the same **q** (the thick line in the diagram). Finally, the interaction with the lattice returns the electron that has already been flipped to the original level, the orbital exciton vanishes (outlined arrow), and the momentum is imparted to a phonon (wavy line). The process as a whole involves annihilation of one of the two original zero-point excitons, with simultaneous conversion of the other exciton into a non-zero-point exciton and the emission of a phonon. The probability of annihilation is given by the square of the absolute value of the corresponding vertex part. The rate of such relaxation in the condensate is proportional to the square of the number of zero-point excitons, and the relaxation process is nonexponential.

The direct annihilation of non-zero-point excitons governs the relaxation process at a later stage, when the condensate is depleted and the system is close to equilibrium. The corresponding process is illustrated in Fig. 1b. The relaxation rate is clearly proportional to the number of supracondensate non-zero-point excitons, which, in turn, is governed by the equilibrium Bose momentum distribution function. The functional dependences of the rate on the number of zero-point and non-zero-point excitons are thus fundamentally different. As equilibrium is approached, the relaxation process naturally becomes exponential.

In the method used here we redefine the exciton states. For this purpose, in the one-particle Hamiltonian we go over to a basis which diagonalizes it in first order with respect to the spin–orbit interaction (Sec. 2), and in this basis we define the exciton creation operator, which we use to construct the states of the many-particle problem (Secs. 3 and 4). Thus, each exciton ceases to be a pure spin exciton, and it contains a small "admixture" (as indicated by the ratio of the spin–orbit energy to the cyclotron energy) of an orbital excitation. As a result, the electron–phonon interaction operator in the excitonic representation includes terms that do not conserve the number of excitons in the system, and the calculation of the vertex parts in the diagrams in Fig. 1 reduces directly to a calculation of the matrix element of the transition between two states differing with respect to the total number of excitons by unity.

It must be borne in mind that apart from the processes illustrated in Fig. 1, at finite temperature there are analogous processes involving the absorption of a phonon. In addition, when S_z of the system approaches the equilibrium value, the reverse processes of exciton generation must be taken into account (at equilibrium the annihilation and generation fluxes become equal). The calculation of the transition matrix elements corresponding to all of these processes is elaborated in Sec. 5.

Section 6 contains a direct calculation of the dependence of the spin polarization on time, and the summation is performed over the statistical distribution of the possible exciton states. We also assume there that the electronic system is a quasiequilibrium system characterized by the same temperature as the phonons. The time for establishing thermal equilibrium is determined by the spin-diagonalized electron–phonon scattering matrix elements, and it is consequently considerably shorter than the spin relaxation time.

Section 7 is devoted to a discussion of the calculation results and a comparison with presently available measurements.

2. ONE-PARTICLE HAMILTONIAN

As in the theoretical studies in Refs. 11–17, we shall assume that the usual conditions of the QHE, a two-dimensional system, and the presence of a strong magnetic field hold:[2)]

$$d<\lambda<r_B. \tag{2.1}$$

Here d is the characteristic thickness of the layer, and $r_B=\hbar^2\varepsilon_0/m_ce^2$ is the Bohr radius of the electron in the material. In energy units these inequalities take the form $\mathcal{U}=e^2/\varepsilon_0\lambda<\hbar\omega_c<\hbar^2/m_cd^2$, where ω_c is the cyclotron frequency and $\mathcal{U}$ is the interaction parameter, which is the characteristic value of the Coulomb energy per electron. As a result, for two-dimensional electrons the one-particle Hamiltonian can be written in the effective-mass approximation with consideration of the spin–orbit terms in the form

$$H_1=\hbar\omega_c\lambda^2k_+k_- - \frac{\epsilon\sigma_z}{2} - \alpha\lambda(k_+\sigma_- + k_-\sigma_+)$$

$$-i\beta\lambda(k_+\sigma_+ - k_-\sigma_-),\quad \text{where } \hbar\omega_c\lambda^2=\frac{\hbar^2}{m_c}. \tag{2.2}$$

We used the usual notation:

$$k_\pm=\mp\frac{i}{\sqrt{2}}(k_x\pm ik_y),\qquad \mathbf{k}=-i\nabla+\frac{e}{c\hbar}\mathbf{A}.$$

The energy is measured relative to the zeroth Landau level. The sign of the second term was chosen on the basis of the assumption that $g<0$. The form of the third and fourth spin–orbit terms is determined by the specific details of the two-dimensional problem, although they have different origins.

The term with the coefficient α was first introduced into the effective-mass Hamiltonian in Ref. 18 to describe the band structure of semiconductors with a wurtzitic crystal lattice. In cubic semiconductors it appears when the symmetry is broken, for example, as a result of uniform deformation in a certain direction not coinciding with one of the principal axes of the crystal;[20] however, in our case this term in (2.2) is associated with the anisotropy of the quantum well in the z direction. Usually α is assumed to be proportional to the mean electric field $\bar{E}=-\langle dV_{\text{eff}}/dz\rangle$ (this quantity can be nonzero only because of a difference between the effective masses on different sides of the heterojunction, see Ref. 19), although there is a similar spin–orbit correction to the Hamiltonian in the case of deformation that is nonuniform in the z direction.[20] The value of α was calculated specifically for the Kane model in Ref. 21. Estimates based on experimental results were presented in Ref. 22. If it is assumed that $\lambda\sim10^{-6}$ cm, we obtain $\alpha\sim1$ K. Here and in the following it is assumed in all the evaluations that we are dealing with magnetic fields $B>10$ T and that the material constants correspond to GaAs; therefore, in particular, $\mathcal{U}>130$ K, and $\hbar\omega_c>200$ K.

The last term in (2.2) appears because of the "three-dimensional" removal of the degeneracy with respect to the spin for the S band in crystals with no inversion center (for example, in GaAs), and is proportional in the three-dimensional case to the cube of the components of $\mathbf{k}$ (Ref. 23). The first power of the components of $\mathbf{k}$ appears in the two-dimensional case,[24] and the first of the conditions (2.1) must hold for a system in a strong magnetic field, in which the scale of the variation of the one-particle wave functions in the (x,y) plane is characterized by λ. In addition, we assume that the principal axis of the crystal is parallel to the z axis. It is known that

$$\beta=\gamma_{\text{so}}\frac{\hbar^2}{2m_cd^2}\left(\frac{\hbar\omega_c}{G}\right)^{1/2} \tag{2.3}$$

(see Refs. 5, 6, and 24), where G is the band-gap width of the semiconductor, and γ_{so} is the spin–orbit constant, which is 0.07 for GaAs.[20]

The parameter d in this formula has a specific meaning. It is determined by averaging the square of the z component of the wave vector of a 2D electron in the layer, i.e.,

$$d^{-2}=-\langle f|d^2/dz^2|f\rangle,$$

where $f(z)$ is the corresponding size-quantized function in the heterojunction. As a result, we obtain $\beta\sim1$ K. Also taking into account that $g=-0.44$, we find the dependence of the Zeeman gap on the magnetic field measured in teslas: $\epsilon\simeq0.3B$ K. Therefore, in any case the inequality

$$\epsilon\ll\mathcal{U} \tag{2.4}$$

holds with a broad margin.

We use the following dimensionless quantities everywhere below: $x,y,z\to x\lambda,y\lambda,z\lambda$; $L,L_z,d\to L\lambda,L_z\lambda,d\lambda$, and $\mathbf{k}\to\mathbf{k}/\lambda$. The expression (2.1) remains the same form here, and we need only set $\lambda=1$ in it. In the Landau gauge $\mathbf{A}=(0,\ Bx,\ 0)$, the first term of the Hamiltonian can be diagonalized in the basis

$$\psi_{np}(x,y)=L^{-1/2}e^{ipy}\varphi_n(p+x),\quad \text{where } \varphi_n(x)$$

$$=(2^nn!\sqrt{\pi})^{-1/2}e^{-x^2/2}H_n(x) \tag{2.5}$$

($H_n(x)$ is a Hermite polynomial). The action of the operators is governed here by the rule $k_\pm\xi_0(x+p)=\xi_{\pm1}(x+p)$. For convenience here we introduced the following notation

$$\xi_0=\varphi_n,\qquad \xi_{\pm1}=\varphi_{n\pm1}\sqrt{n+\frac{1\pm1}{2}}. \tag{2.6}$$

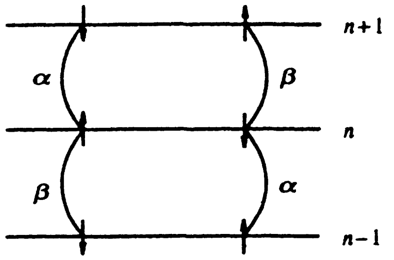

FIG. 2. Mixing of spin states in different Landau levels $(n,n\pm1)$. The type of spin–orbit interaction is indicated by the corresponding Greek letter.

When the spin-dependent terms in (2.2) are taken into account, the state of the electron can be represented in the spinor basis

$$\psi_{np\uparrow}=\begin{pmatrix}\psi_{np}\\ 0\end{pmatrix},\qquad \psi_{np\downarrow}=\begin{pmatrix}0\\ \psi_{np}\end{pmatrix},$$

and the action of the operators $\sigma_\pm$ reduces, as usual, to raising or lowering of the spin index.

If $\alpha=0$, the Hamiltonian is easily diagonalized for any values of ω_c, ε, and β (Ref. 6), since the diagonalization can be performed independently for each pair of states $(\psi_{np\uparrow},\psi_{n-1p\downarrow})$. In exactly the same manner, when $\beta=0$, only the states $(\psi_{np\uparrow},\psi_{n-1p\downarrow})$ undergo pairwise mixing, and the Hamiltonian can also be diagonalized exactly. A diagram of the mixing of different states is shown in Fig. 2. If $\alpha\beta\neq0$, exact diagonalization is impossible, but in our case it is sufficient to use perturbation theory, since

$$\alpha,\beta\ll\hbar\omega_c\pm\epsilon, \tag{2.7}$$

after which in first order we find the eigenstates

$$\psi_{npa}=\frac{e^{ipy}}{\sqrt{L}}\begin{pmatrix}\xi_0\\ v\xi_1+iu\xi_{-1}\end{pmatrix},$$

$$\psi_{npb}=\frac{e^{ipy}}{\sqrt{L}}\begin{pmatrix}-v\xi_{-1}+iu\xi_1\\ \xi_0\end{pmatrix}, \tag{2.8}$$

which diagonalize the Hamiltonian (2.2) to second order in

$$v=\frac{\alpha}{\hbar\omega_c+\epsilon}\approx\frac{\alpha}{\hbar\omega_c}\quad\text{and}\quad u=\frac{\beta}{\hbar\omega_c-\epsilon}\approx\frac{\beta}{\hbar\omega_c}.$$

The corrections to the energy reduce to renormalization of the gap and weak inequivalence between the spin sublevels,

$$\epsilon_n=E_{nb}-E_{na}=\epsilon+(2n+1)\left(\frac{\alpha^2}{\hbar\omega_c+\epsilon}-\frac{\beta^2}{\hbar\omega_c-\epsilon}\right). \tag{2.9}$$

Any state in the conduction band can be expanded in the spinor basis (2.6), (2.8),

$$\psi=\sum_{np}(a_{np}\psi_{npa}+b_{np}\psi_{npb}). \tag{2.10}$$

Treating the coefficients in (2.10) as particle annihilation operators, we obtain the Hamiltonian in the second-quantization representation

$$\mathcal{H}_1=\langle\psi|H_1|\psi\rangle=\sum_n\left(E_{na}\sum_p a^+_{np}a_{np}+E_{nb}\sum_p b^+_{np}b_{np}\right). \tag{2.11}$$

Here $E_{na}=\hbar\omega_c n-\epsilon/2-v^2(n+1)(\hbar\omega_c+\epsilon)+u^2n(\hbar\omega_c-\epsilon)$. The basis (2.6), (2.8) is normalized up to linear terms, and its normalization to terms quadratic in u and v would provide only superfluous accuracy. In precisely the same manner, when the spin is determined in states a and b, we have $s_z^{(a,b)}\approx\pm1/2$ to second order in u and v.

3. CLASSIFICATION OF SPIN-EXCITON STATES

Let us first consider the states of a many-electron system in the zeroth approximation with respect to $\mathcal{H}_{\rm so}$ (for the sake of brevity, we refer to spin–orbit terms in the Hamiltonian as such). The states obtained as a result of the N-fold action of the operator Q_0^+ on the ground state are of primary interest. When only zero-point excitons are present in the system, we call such states zero-point exciton states. The total spin S for them is identical and is always equal to the maximum value $\mathcal{N}/2$. Due to the presence of the Zeeman gap, the zero-point exciton states are not degenerate with respect to S_z, and since (1.2) is a pure spin operator, their spatial wave function is the same as in the ground state, and the energy measured from the ground state is simply equal to $N\epsilon$. For its part, the energy of the Coulomb interaction is determined completely by the spatial wave function. Hence it follows that for a prescribed value of S_z, the minimum energy of the system, which is equal to $N\epsilon$, is achieved in a zero-point exciton state, and that

$$N=\frac{\mathcal{N}}{2}-S_z(g<0). \tag{3.1}$$

When there are other excitations in addition to the zero-point exciton states, the energy is no longer a multiple of ϵ. The state with a nonzero two-dimensional momentum $\hbar\mathbf{q}$ that is closest to the zero-point exciton state with an assigned value of S_z is a quasiparticle state of the spectrum with $N-1$ zero-point excitons (N being determined from (3.1)) and one excited (non-zero-point) exciton. In principle, the spectrum can contain excitations of the molecular-exciton type; however, we assume that the concentration of non-zero-point excitons is always small; therefore, their interaction and the probability of formation of excitonic molecules can be neglected. Of course, the creation operator $Q_{\mathbf{q}}^+$ of non-zero-point spin excitons is defined (see Refs. 16, 17, and 25) such that its action on the eigenstates of the system would also diminish S_z by unity. At the same time, it is not difficult to see that generation of a non-zero-point exciton in the system should also reduce S by unity (see Ref. 15), and, as we shall see, the operator $Q_{\mathbf{q}}^+$ does not commute with $\mathbf{S}^2$.

Indeed, let us first consider a state for which $N=1$ and $\mathbf{q}\neq0$, and which can clearly be expanded in a basis of states with different values of S, but with the same momentum and the same $S_z=\mathcal{N}/2-1$. The expansion will clearly contain only one term, i.e., a state with a total spin equal to $\mathcal{N}/2-1$, since the possible values of S in the terms of the expansion must not be smaller than S_z, and any state with

S equal to $\mathscr{N}/2$ vanishes (it is a zero-point exciton state, and its momentum must unequivocally be $\mathbf{q}=0$). Because Q_0^+ and $\mathbf{S}^2$ commute, this result can easily be extended to any state with one non-zero-point exciton for an arbitrary value of S_z. As a result, considering N-exciton states in which no more than one exciton is non-zero-point, we find for them

$$S=\begin{cases}\mathscr{N}/2, & \mathbf{q}=0\\ \mathscr{N}/2-1, & \mathbf{q}\neq 0.\end{cases} \tag{3.2}$$

We turn our attention to the fact that for a real macroscopic system, $\Delta S=S-S_z$ must be proportional to the order parameter characterizing the presence of a spin condensate in the system when the exciton–exciton interaction is neglected (i.e., under the conditions of a "rarefied" gas of nonzero-point excitons). In fact, because of (3.2), the total spin S of the system differs from $\mathscr{N}/2$ by the number of non-zero-point excitons. Comparing with (3.1), we see that above the critical point (where all the excitons are non-zero-point $\Delta S=0$, and below it ΔS is equal to the macroscopically large number of zero-point excitons in the condensate. This quantity is determined by the deviation of the total "classical" spin of the system as a whole from the z axis. The angle of deviation θ can be chosen as the order parameter, and then $\Delta S=S(1-\cos\theta)$.

Before analyzing how (3.1) and (3.2) change when the corrections due to $\mathscr{H}_{\mathrm{so}}$ are taken into account, we note that (3.1) can be regarded as an operator equality. This means that the operator S_z is specified in the excitonic representation (i.e., it is expressed in terms of the exciton number operator N). In precisely the same manner it can be found in the excitonic representation that

$$\mathbf{S}^2=\mathscr{N}Q_0^+Q_0+(\mathscr{N}/2-N)^2+\mathscr{N}/2-N. \tag{3.3}$$

Basis of exciton states with consideration of $\mathscr{H}_{\mathrm{so}}$

Since the basis of one-particle states of sublevels a and b introduced in the preceding section diagonalizes the Hamiltonian $\mathscr{H}_1$ to second order in $\mathscr{H}_{\mathrm{so}}$, it is natural now to redefine the creation operator for zero-point excitons: $Q_0^+=\mathscr{N}^{-1/2}\Sigma_p b_p^+a_p$. Here and in the following we omit the subscript n, since within the basis (2.8) we restrict ourselves to the single-band approximation, i.e., all the Fermi operators always have the same orbital index n. When $\mathscr{H}_{\mathrm{so}}=0$, this definition of Q_0^+ reverts to Eq. (1.2).

We now present the expression for the creation operator for non-zero-point excitons,

$$Q_{\mathbf{q}}^+=\mathscr{N}^{-1/2}\sum_p \exp(-iq_xp)b^+_{p+q_y/2}a_{p-q_y/2} \tag{3.4}$$

(compare with Refs. 16, 17, and 25). In the next section we see that just this form of $Q_{\mathbf{q}}^+$ gives a solution of the two-particle problem with a pairwise interaction within the single-band (with respect to n) approximation. The ground state $|0\rangle$ must obviously be defined as a state with a completely filled a subband and a completely empty b subband, so that we have $a_p^+|0\rangle=b_p|0\rangle\equiv 0$ for any p. Of course, the zero-point exciton state and the state with one non-zero-point exciton, which we considered above, are now defined in the following manner:

$$\begin{aligned}&|N,0\rangle=(Q_0^+)^N|0\rangle,\\ &|N,\mathbf{q}\rangle=(Q_0^+)^{N-1}Q_{\mathbf{q}}^+|0\rangle\\ &\qquad\times(\langle\mathbf{q}_1,N_1|N_2,\mathbf{q}_2\rangle\propto\delta_{N_1,N_2}\delta_{\mathbf{q}_1,\mathbf{q}_2})\end{aligned} \tag{3.5}$$

(compare Ref. 16). It is easy to see that the exciton number operator is diagonal in the basis of these states, i.e.,

$$\hat{N}=\sum_p b_p^+b_p,\qquad [\hat{N},Q_{\mathbf{q}}^+]=Q_{\mathbf{q}}^+, \tag{3.6}$$

and that its eigenvalues are equal to N. The orthogonality of the states (3.5) can be verified directly using the commutation rules of the exciton operators presented in Appendix A.

We can treat (3.1) and (3.3) as operator equalities after replacing N by $\hat{N}$ in them. It is understood that the operators S_z and $\mathbf{S}^2$ thus defined are no longer pure spin operators, but their properties with respect to the basis (3.5) remain absolutely unchanged. The "internal structure" of the one-particle states a and b in this case is of no significance, since the form of the excitonic distribution for $\mathbf{S}$ and S_z remain the same as when $\mathscr{H}_{\mathrm{so}}$ is equal to zero. Thus, the states (3.5) are eigenstates for the "new" operators S_z and $\mathbf{S}^2$, and they correspond to the same eigenvalues of S_z and different eigenvalues of $\mathbf{S}^2$ in accordance with the rule (3.2). This can be seen directly, for which it is convenient to use, for example, Eqs. (A12) and (A13) from Appendix A.

4. SPECTRUM OF THE SYMMETRIC MODEL. SPIN–ORBIT CORRECTIONS TO THE EXCITON STATES

The Hamiltonian of the electron–electron interaction $\mathscr{H}_{\mathrm{int}}$ is defined in the usual manner (see, for example, Ref. 14) in terms of the Fermi ψ operators of 2D electrons and the Coulomb potential of the pairwise interaction averaged in the layer over z (using the size-quantized wave function $f(z)$), $\bar{V}(\mathbf{r})$ ($\mathbf{r}$ is a two-dimensional vector). If the expansion (2.10) is substituted for the ψ operators, in which case we are confined to the single orbital state n (the summation over n is eliminated) and the operator terms proportional to the small parameters u and v are totally neglected, we obtain the Hamiltonian of the symmetric model

$$\begin{aligned}\mathscr{H}_{\mathrm{int}}^{(0)}=\frac{1}{2}\sum_{p,p_1,\,p_1',p'=2\pi/L}^{L}&W_0(p,p_1,p_1',p')(a_p^+a_{p_1}^+a_{p_1'}a_{p'}\\ &+b_p^+b_{p_1}^+b_{p_1'}b_{p'}+2a_p^+b_{p_1}^+b_{p_1'}a_{p'})\end{aligned} \tag{4.1}$$

(the model is called symmetric because the same vertex W_0 appears in front of all 4-operator terms in (4.1)). The vertex in (4.1) has the form $W_0=L^{-1}\delta_{p+p_1,\,p_1'+p'}J_{0000}\times(\Delta p,\delta p)$, where $\Delta p=p_1-p'=p_1'-p$, $\delta p=p-p'=p_1'-p_1$, and J_{0000} is one of the functions that appear when the matrix elements are calculated (see Refs. 14 and 15), and which are defined in the following manner in terms of the Fourier component $V(\mathbf{q})$ of the Coulomb potential:

$$J_{jklm}(\Delta p,\delta p)=\int\frac{dq_x}{2\pi}V(\mathbf{q})h_{jm}(\mathbf{q})h_{kl}(-\mathbf{q})$$

$$\times\exp(iq_x\Delta p),\quad\text{where}\quad q_y=\delta p,$$

$$j,k,l,m=0,\pm 1 \tag{4.2}$$

and

$$h_{jm}(q_x,\delta p)=\int dx\,\xi_j\left(x+\frac{\delta p}{2}\right)e^{iq_xx}\xi_m\left(x-\frac{\delta p}{2}\right)$$

$$=\left[\frac{iq_x+\delta p\ \mathrm{sgn}(j-m)}{\sqrt{2}}\right]^{|j-m|}L_{\tilde{n}}^{|j-m|}\left(\frac{q^2}{2}\right)\bigg|_{q_y=\delta p},$$

$$\tilde{n}=n+\min(j,m),\qquad jm=0. \tag{4.3}$$

This expression contains one-particle functions that have been redefined according to the rule (2.6), and it is assumed that at least one of the indices j or m equals zero. L_n^j is a Laguerre polynomial ($L_n^0=L_n$).

Before proceeding to an analysis of the next higher order corrections in u (and v) to the interaction Hamiltonian, let us briefly dwell on the known results that have been obtained for the symmetric model.[12,13,15,17] The states (3.5) defined in the preceding section are eigenstates of the Hamiltonian $\mathscr{H}_1+\mathscr{H}_{\mathrm{int}}^{(0)}$. This can easily be seen, if we take into account the commutation relations $[\mathscr{H}_1,Q_{\mathbf{q}}^+]=\varepsilon_nQ_{\mathbf{q}}^+$ and $[\mathscr{H}_{\mathrm{int}}^{(0)},Q_0^+]=0$, as well as the fact that the state $|1,\mathbf{q}\rangle$ is an eigenstate of the two-particle problem:

$$\mathscr{H}_{\mathrm{int}}^{(0)}|1,\mathbf{q}\rangle=[\mathscr{E}(\mathbf{q})+E_0]|1,\mathbf{q}\rangle.$$

These relations can be proved comparatively easily using the commutation rules (A1) and (A2) (see also Refs. 12–16), from which E_0 and $\mathscr{E}(\mathbf{q})$ can also be determined. Obviously, the constant E_0 simply renormalizes the ground-state energy, and $\mathscr{E}(\mathbf{q})$ is the Coulomb part of the energy of a non-zero-point exciton, which has an exchange origin (we recall that generation of a non-zero-point exciton changes the total spin). Thus, the energy of the system in the states (3.5) is

$$E(N,\mathbf{q})=N\epsilon_n+\mathscr{E}(\mathbf{q}). \tag{4.4}$$

The characteristic values q are specified by the temperature of the system,

$$\mathscr{U}q^2\lesssim T\lesssim\epsilon. \tag{4.5}$$

Hence, in any case it follows that $q\ll 1$; therefore, it is sufficient to restrict ourselves to the quadratic approximation

$$\mathscr{E}(\mathbf{q})=\frac{q^2}{2M_n},\quad\text{where}\quad M_n^{-1}=\int_0^\infty\frac{p^3dp}{4\pi}V(p)$$

$$\times\exp\left(-\frac{p^2}{2}\right)\left[L_n\left(\frac{p^2}{2}\right)\right]^2. \tag{4.6}$$

The first of the conditions (2.1) allows us, in addition, to assume that $V(p)\approx 2\pi\mathscr{U}/p$. In particular, for the zeroth and first Landau levels we then obtain

$$M_0\approx 2\mathscr{U}^{-1}\sqrt{2/\pi},\qquad M_1\approx 4M_0/7 \tag{4.6'}$$

(see Refs. 11–13,15 and 16).

Besides the states (3.4), in the following we shall also encounter the state

$$|N,\mathbf{q}_1,\mathbf{q}_2\rangle=Q_{\mathbf{q}_1}^+Q_{\mathbf{q}_2}^+|N-2,0\rangle, \tag{4.7}$$

which is not an eigenstate of the Hamiltonian (4.1) and is not orthogonal to the basis state (3.5) with $\mathbf{q}$ equal to $\mathbf{q}_1+\mathbf{q}_2$. The degree of this nonorthogonality is, however, small. Using (A5)–(A13), (A18), and (A19), we can show that the properly normalized convolution of this basis state with the state (4.7) is at most of order $\mathscr{N}^{-1/2}$, if $\mathscr{N}-N\sim\mathscr{N}$ and $q_1(\mathbf{q}_1+\mathbf{q}_2)\neq 0$. The mean energy in the state (4.7) can also be calculated using (A1), (A2), and (A19):

$$\frac{\langle\mathbf{q}_2,\mathbf{q}_1,N|\mathscr{H}_{\mathrm{int}}^{(0)}|N,\mathbf{q}_1,\mathbf{q}_2\rangle}{\langle\mathbf{q}_2,\mathbf{q}_1,N|N,\mathbf{q}_1,\mathbf{q}_2\rangle}=E_0+N>\epsilon_n+\mathscr{E}(\mathbf{q}_1)+\mathscr{E}(\mathbf{q}_2)$$

$$+O\left(\frac{\mathscr{U}}{\mathscr{N}}\right)\quad\text{for }\mathscr{N}-N\sim\mathscr{N}. \tag{4.8}$$

If a and b are particle states that correspond to the basis (2.8), Eqs. (2.8), (2.9), (3.5), (4.4), and (4.6) give a solution of the spectral problem to first order in u for the states and, accordingly, to second order in β for the energy. At the same time, these response functions comprise the zeroth approximation in $\mathscr{U}/\hbar\omega_c$, since the interaction in them is taken into account (exactly!) only within the single-band model with respect to n. Clearly, the same result can be obtained if the problem is solved in the reverse order, i.e., if the "pure spin" single-band states (3.5) of the interacting 2D electrons are first found in the zeroth approximation in $\mathscr{H}_{\mathrm{so}}$ and then, by applying the operator $\mathscr{H}_{\mathrm{so}}$ to these states, corrections are found in first-order perturbation theory, which reduce to redetermination of the spin sublevels and their conversion into sublevels a and b. We now ascertain the consequences of the main corrections that have yet to be taken into account.

The next most important correction is the first order spin–orbit correction with consideration of the interaction, i.e., a correction to the state that is first order in $\mathscr{H}_{\mathrm{so}}$ and simultaneously of the same order in $\mathscr{U}/\hbar\omega_c$. This correction can be calculated exactly within our approximation. For this purpose we must first calculate the interaction Hamiltonian in the next approximation after (4.1): $\mathscr{H}_{\mathrm{int}}=\mathscr{H}_{\mathrm{int}}^{(0)}+\mathscr{H}_{\mathrm{int}}^{(1)}$. The operator $\mathscr{H}_{\mathrm{int}}^{(1)}$ "ruins" the symmetric model. It is proportional to the first powers of u and v and contains 4-operator terms with the combinations a^+a^+ab, a^+b^+bb, b^+a^+aa, and b^+b^+ba, which "toss" an electron from one sublevel to another. The coefficients of these terms differ, although as before, they are functions of Δp and δp due to the homogeneity of the system. As a result, we can redefine the corrections to the eigenstates (3.5)

$$\overline{||N,\mathbf{q}\rangle\rangle}=||N,\mathbf{q}\rangle\rangle+C_N^-(\mathbf{q})||N-1,\mathbf{q}\rangle\rangle+C_N^+(\mathbf{q})||N+1,\mathbf{q}\rangle\rangle. \tag{4.9}$$

Here the overline shows that we have passed from the state defined by Eqs. (3.4) and (3.5) to an improved state, in which the number of excitons is no longer a quantum number. The doubled notation $||\rangle\rangle$ means that the exciton states (3.5) were properly normalized (see Eqs. (A17) and (A18) in Appendix A). To avoid misunderstanding, we note that an asymmetric model, but only one in which the number of particles in the sublevels was conserved, was also considered in Ref. 14.

The coefficients $C^{\pm}$ are small. It is not difficult to show on the basis of general arguments that they vanish if $\mathbf{q}=0$, and that they are of order $qu\mathcal{U}/\varepsilon$ when $N\sim\mathcal{N}$ and $q\lesssim 1$. As a result, the corresponding contribution to the spin relaxation process in the range of parameters of interest to us (which is determined by the experimental conditions and partially by the need to obtain final response functions in analytic form) is negligible, as we shall see below (see Sec. 6). Omitting the details of a rather cumbersome calculation, which reduces to a determination of the matrix element

$$\mu(N,\mathbf{q})=\langle\langle\mathbf{q},N||\mathcal{H}_{\text{int}}^{(1)}||N+1,\mathbf{q}\rangle\rangle$$

(since $C_N^{-}=\epsilon^{-1}\mu(N-1,\mathbf{q})$ and $C_N^{+}=-\epsilon^{-1}\mu^{+}(N,\mathbf{q})$), we find

$$\mu(N,\mathbf{q})=\frac{\sqrt{N(\mathcal{N}-N)}}{\mathcal{N}}\frac{iuq_{+}+vq_{-}}{\pi\epsilon q}\int_0^{\infty}dpp^2V(p)J_1(pq)\times\left[L_n\left(\frac{p^2}{2}\right)\right]^2e^{-p^2/2},\tag{4.10}$$

where

$$q_{\pm}=\mp\frac{i}{\sqrt{2}}(q_x\pm iq_y),$$

and $J_1(x)$ is a Bessel function. In addition, we took advantage of the fact that the interaction is isotropic, and we assumed that $\mathcal{N},\mathcal{N}-N\gg 1$. If we also take into account the first of the conditions (2.1) and the condition (4.5), in the linear approximation we have

$$C_N^{-}(\mathbf{q})=2\frac{\sqrt{N(\mathcal{N}-N)}}{\mathcal{N}}\frac{iuq_{+}+vq_{-}}{\epsilon M_n},\quad q\ll 1.\tag{4.11}$$

In physical terms, the coefficients $C_N^{\mp}$ specify the probability of quantum fluctuations from the state $|N,\mathbf{q}\rangle$ to the states $|N\pm 1,\mathbf{q}\rangle$. The fact that this probability of the creation or annihilation of a zero-point exciton vanishes if $\mathbf{q}=0$ allows us to interpret such events as results of interactions with a non-zero-point exciton (we are considering deviations from the symmetric model, so that the zero-point excitons become interacting). Naturally, the probability must be proportional to the concentration $N/\mathcal{N}$; therefore, in a weakly excited system, in which $N\ll\mathcal{N}$, the $C_N^{\mp}$ have an additional small factor that is proportional to $(N/\mathcal{N})^{1/2}$.

We note that the response functions (4.9)–(4.11) found here can also be obtained, in principle, in an alternative manner via systematic perturbation theory in $\mathcal{H}_{\text{int}}^{(0)}$ and $\mathcal{H}_{\text{so}}$, and that they reduce to second-order corrections specified by matrix elements of the form $\langle\langle\mathbf{q},N\pm 1||\mathcal{H}_{\text{so}}\times\mathcal{H}_{\text{int}}^{(0)}||N,\mathbf{q}\rangle\rangle$. Under such an approach it must obviously be assumed that the basis states (3.5) are defined in the zeroth approximation in $\mathcal{H}_{\text{so}}$.

To conclude this section we dwell on several features of the spectrum that appear when second-order corrections in $\mathcal{H}_{\text{so}}$ are taken into account. As we see, qualitatively the structure of the eigenstates generally undergoes significant changes in this case, which nevertheless are not reflected to any appreciable extent in the spin relaxation process.

According to perturbation theory, the quadratic spin–orbit corrections in the first approximation in $\mathcal{H}_{\text{int}}$ are described by matrix elements of the operators

$$\mathcal{H}_{\text{so}}\times\mathcal{H}_{\text{int}}^{(0)}\times\mathcal{H}_{\text{so}},\qquad\mathcal{H}_{\text{so}}\times\mathcal{H}_{\text{so}}\times\mathcal{H}_{\text{int}}^{(0)},\qquad\mathcal{H}_{\text{int}}^{(0)}\times\mathcal{H}_{\text{so}}\times\mathcal{H}_{\text{so}}.\tag{4.12}$$

There are obviously corrections of two kinds. The first corresponds to matrix elements that are diagonal with respect to N, have a relative magnitude of order u^2 (as well as of order v^2 and uv), and can be interpreted as being due to a weak interaction between excitons (including zero-point excitons) that does not alter their numbers. These matrix elements give corrections of order $u^2\mathcal{U}$ to the exciton energies, which are small compared to the corrections already taken into account. The inequality

$$(u^2+v^2)\mathcal{U}\ll\epsilon\tag{4.13}$$

along with (1.1), (2.1), (2.7), and (4.5) is assumed to be a necessary condition for solving the current problem. We note that together with (4.5), this condition ensures that the coefficients $C_N^{\mp}$ in (4.9) are small. As can easily be shown, it holds over the parameter range of interest with a wide margin.

Corrections of the second type appear as a consequence of the existence of off-diagonal matrix elements with respect to N for the operators (4.12) and describe quantum fluctuations between $|N,\mathbf{q}\rangle$ and $|N\pm 2,\mathbf{q}\rangle$. In the present case these matrix elements, unlike (4.10), do not vanish when $\mathbf{q}=0$ in the general case. Therefore, their magnitude is determined not by the interaction between zero-point and non-zero-point excitons, but by the interaction of all N excitons with one another, which results in the pairwise creation or annihilation of zero-point excitons. If N is macroscopically large, then they, like the diagonal matrix elements, are proportional to N, i.e., they are also macroscopically large. Then two states with "microscopically" similar energies differing by 2ϵ in the zeroth approximation in $\mathcal{H}_{\text{so}}$ mix. The situation is essentially similar to what occurs when a perturbation theory is devised for a degenerate or almost degenerate spectrum. This problem is known to reduce to the investigation of a matrix secular equation. In our case it is an $\mathcal{N}\times\mathcal{N}$ matrix, in which the Nth diagonal element in the leading approximation is equal to $N\epsilon$ and the off-diagonal elements are determined by the operators (4.12) between the bras and kets $\langle\langle 0,N||\ldots||N\pm 2,0\rangle\rangle$ and $\langle\langle 0,N\pm 2||\ldots||N,0\rangle\rangle$ (when N is large, the presence or absence of a non-zero-point exciton in the system does not have any effect on the analysis of the effects under consideration) and by the denominator $(\hbar\omega_c)^2$. As an analysis shows, when $\alpha\beta=0$, the off-diagonal elements vanish identically. Along with the possibility of exact diagonalization of the one-particle Hamiltonian (see Sec. 2), this is another manifestation of the latent symmetry maintained under this condition. If $\alpha\beta\neq 0$, the off-diagonal elements are equal in order of magnitude to $Nuv\mathcal{U}$.

Investigating the matrix equation, we can easily understand how the eigenstate of the system with a mean number of excitons N is organized. The state $|N,0\rangle$ clearly "spreads out" into a large number of $|N\pm 2s,0\rangle$ states. The absolute values of the coefficients in this expansion decrease with increasing $|s|$, differing appreciably from the value for $s=0$ only when $|s|\gtrsim(Nuv\mathscr{U}/\varepsilon)^{1/2}$. Thus, the "spread" of the number of excitons ΔN is small compared with N, although it is a large number: $\Delta N\sim(Nuv\mathscr{U}/\varepsilon)^{1/2}$. The requirement that the eigenstates corresponding to different mean values of N be orthonormal imposes certain conditions on the coefficients of the expansion in the basis (3.5). In particular, they must be rapidly alternating functions of the index. Utilizing this and the other properties of the coefficients, we can evaluate the matrix elements of the electron–phonon Hamiltonian for transitions with a change in the mean value of N. As expected, in the final analysis the role of the "spreading" reduces to small corrections proportional to $(uv)^2$ for the parameters characterizing the spin relaxation process.

5. ELECTRON–LATTICE INTERACTION. SELECTION RULES. MATRIX ELEMENTS OF THE ONE-PHONON PROCESS

Let us, first of all, dwell on several features of the electron–lattice interaction in a 2D system in a strong magnetic field, and, in particular, let us ascertain the specific details of the action of the corresponding operator on the spin-exciton states under consideration. The three-dimensional Hamiltonian of the interaction with the lattice for 2D electrons in the one-phonon approximation is known to have the form (see, for example, Ref. 26)

$$\mathscr{H}_{\mathrm{e,\,ph}}=\frac{1}{L}\left(\frac{\hbar}{L_z}\right)^{1/2}\sum_{\mathbf{q},k_z,s}\tilde{U}_s^*(\mathbf{k})P^*_{\mathbf{k},s}H_{\mathrm{e,ph}}(\mathbf{q})+\mathrm{H.c.},\qquad(5.1)$$

where

$$H_{\mathrm{e,ph}}(\mathbf{q})=\int\exp(-i\mathbf{qr})\Psi^+(\mathbf{r})\Psi(\mathbf{r})d^2\mathbf{r},$$

$$\mathbf{r}=(x,y),\quad \mathbf{k}=(\mathbf{q},k_z).\qquad(5.2)$$

$\Psi^+(\mathbf{r})$ is the electron creation operator, $P^+_{\mathbf{k},s}$ is the phonon creation operator (s is the polarization), L_z is the dimension of the sample along z, and $\tilde{U}_s(\mathbf{k})$ is the renormalized vertex, which includes the micro- and macroscopic fields created by the phonon. The integration with respect to z has already been performed, and reduces to the renormalization

$$\tilde{U}_s(\mathbf{k})=U_s(\mathbf{k})\int f^*(z)e^{ik_zz}f(z)dz,$$

the wave function of the corresponding size-quantized level $f(z)$ being identical for all $\mathscr{N}$ electrons by virtue of (2.1).

If we now set $\mathbf{q}=0$, (5.2) will simply be equal to the operator of the number of 2D electrons in the layer, i.e., the parameters are conserved. Any off-diagonal matrix elements of the Hamiltonian (5.1), (5.2), including the ones corresponding to transitions with a change in the number of excitons, vanish in this case. Spin relaxation (the annihilation of an exciton) is thus impossible, if the participating phonon has a vanishing momentum component in the plane of the layer. We stress that this selection rule is a result of the two-dimensional character of the problem (the first of the inequalities (2.1)), and it holds to all orders of approximation in $\mathscr{H}_{\mathrm{so}}$.

The two-dimensional character of the layer and the presence of a strong magnetic field are responsible for another specific selection rule, which we first formulate for $\mathscr{H}_{\mathrm{so}}=0$. The transitions defined by the matrix elements of the Hamiltonian (5.1), (5.2) are possible only with conservation of the spin state. In particular, under the condition (1.1) this means that the action of the operator (5.2) on the ground state of the system can specify transitions only to states of other magnetic bands when $\mathbf{q}$ is nonzero, since the exciton states closest to the ground state in the same band are spin excitations. However, a transition to a band with a Landau level having a different number is in fact energy-forbidden in our case, since it is assumed that the cyclotron frequency is greater than the permissible frequency of a phonon; therefore, the action of (5.2) on the ground state is identically equal to zero or reduces to trivial multiplication by a constant, if $\mathbf{q}=0$. We obviously arrive at precisely the same result in the case of an arbitrary zero-point exciton state. After the action of the spin-zero operator (5.2) on it, a state with nonzero $\mathbf{q}$, but with the same value of S equal to $\mathscr{N}/2$ as in the original state would be obtained, in clear contradiction to the condition (3.2).

Thus, if the presence of other magnetic bands is ignored and $\mathscr{H}_{\mathrm{so}}$ is neglected, both transitions with a change in N (i.e., to other spin states) and transitions with a change in $\mathbf{q}$ from zero to a nonzero value and vice versa (even with conservation of the same value of N) are forbidden for the Hamiltonian (5.1). Only transitions between states with nonzero but unequal initial and final momenta and with conservation of the value of N are allowed.

We now ascertain how these rules should be reformulated, even to first order in to $\mathscr{H}_{\mathrm{so}}$. In this case the operator $\mathscr{H}_{\mathrm{e,ph}}$ clearly includes terms that do not conserve the number of excitons in the system; therefore, the result of its action on a zero-point exciton state is now not zero, but a state that can be expanded in the basis (3.5). At the same time $\mathscr{H}_{\mathrm{e,ph}}$ contains terms that are diagonal with respect to N. As can easily be shown, the latter can be specified only by corrections of even order in $\mathscr{H}_{\mathrm{so}}$; therefore, within our approximation they should remain unchanged (the same as for $\mathscr{H}_{\mathrm{so}}=0$). Precisely these corrections will determine the matrix elements of the transitions between the states (3.5) with conservation of the number of excitons. As a result, such transitions are impossible if either of the states (the "initial" or "final" state) is a zero-point exciton state, this being a direct consequence of the maintenance of the selection rules for the eigenvalues of the operator (3.3). In other words, the rule selecting against zero-point exciton states remains in force, but applies only to transitions without a change in N.

One-phonon Hamiltonian in the excitonic representation

We would like, above all, to find the one-phonon Hamiltonian to first order in $\mathscr{H}_{\mathrm{so}}$. We substitute the expansion (2.10) into Eq. (5.2), and restricting ourselves, as before, to only interband (with respect to n) transitions and to the first order in u and v, we see that the convolution of the spinors (2.8) reduces to the integrals (4.3), after which we can use the familiar difference equation for Laguerre polynomials, $L_n^j(x)-L_{n-1}^j(x)=L_n^{j-1}(x)$. As a result, bearing in mind the definitions for $Q_\mathbf{q}^+$ (3.4), as well as for $A_\mathbf{q}^+$ and $B_\mathbf{q}^+$ (see Eq. (A4) in Appendix A), we find the operator that acts on the electron states in the excitonic representation

$$H_{\mathrm{e,ph}}(\mathbf{q})=L_n(q^2/2)e^{-q^2/4}\{\mathcal{N}(A_\mathbf{q}+B_\mathbf{q})+\mathcal{N}^{1/2}[(vq_+ + iuq_-)Q^+_{-\mathbf{q}}+(iuq_+-vq_-)Q_\mathbf{q}]\}. \tag{5.3}$$

The operators $A_{\pm\mathbf{q}}$ and $B_{\pm\mathbf{q}}$ are diagonal with respect to N in the space of the basis states (3.5), i.e., they do not alter the total number of excitons, but only change the exciton momentum by $\mp\mathbf{q}$ because of the commutation relations (A9). As it should be when there is no spin–orbit interaction, the electron–phonon Hamiltonian is diagonal with respect to the spin. It is also not difficult to see that the matrix elements calculated using (5.3) satisfy all the selection rules discussed above.

Matrix elements of the operator (5.3)

The representation (5.3) must now be substituted into (5.1). We are interested in processes that alter the z component S_z of the total spin of the system. They should be specified by the matrix elements of the Hamiltonian (5.1) between states with different mean numbers of excitons. But in the present approximation, the mean number of excitons in the states (4.9) is simply equal to the quantum number N; therefore, remaining within the first approximation with respect to the spin–orbit interaction, we must calculate

$$\overline{\langle\langle\mathbf{q}_1-\mathbf{q},N\pm 1||H_{\mathrm{e,ph}}(\mathbf{q})||N,\mathbf{q}_1\rangle\rangle}=L_n(q^2/2)e^{-q^2/4}\mathcal{F}^{\pm}(\mathbf{q},\mathbf{q}_1,N). \tag{5.4}$$

The Hermitian conjugate of (5.3) is obtained by simply replacing $\mathbf{q}$ by $-\mathbf{q}$, whence

$$\mathcal{F}^+(\mathbf{q},\mathbf{q}_1,N-1)=[\mathcal{F}^-(-\mathbf{q},\mathbf{q}_1-\mathbf{q},N)]^*. \tag{5.5}$$

Substituting the expression in square brackets in (5.3) and the states given by (3.5) and (3.9)–(3.11) into (5.4), we see that to first order in $\mathcal{H}_{\mathrm{so}}$, the calculation reduces to the sum of the expressions

$$\mathcal{N}C_N^-(\mathbf{q}_1)\langle\langle\mathbf{q}_1-\mathbf{q},N-1||A_\mathbf{q}+B_\mathbf{q}||N-1,\mathbf{q}_1\rangle\rangle-\mathcal{N}C_N^-(\mathbf{q}_1-\mathbf{q})\langle\langle\mathbf{q}_1-\mathbf{q},N||A_\mathbf{q}+B_\mathbf{q}||N,\mathbf{q}_1\rangle\rangle \tag{5.6}$$

and

$$\mathcal{N}^{1/2}(iuq_+-vq_-)\langle\langle\mathbf{q}_1-\mathbf{q},N-1||Q_\mathbf{q}||N,\mathbf{q}_1\rangle\rangle. \tag{5.7}$$

The normalization of the basis states (3.5) and the calculation of the "4-exciton" convolutions (A10) and (A11) are performed differently, depending on whether one of the states (the initial or the final state) is a zero-point exciton state or not. In addition, if one of the bras or kets in (5.6) is a zero-point exciton state, then according to the properties of the operators $A_\mathbf{q}$ and $B_\mathbf{q}$ (A8) and (A9′), by calculating (5.6) we obtain a value identically equal to zero, in complete agreement with the general rule for the action of the electron–phonon Hamiltonian on zero-point exciton states. Stated differently, the vertex parts of the diagrams in Figs. 1a and b are described in the excitonic representation only by the matrix element (5.7) with $\mathbf{q}_1=0$ (annihilation of a zero-point exciton, Fig. 1a) and $\mathbf{q}_1=\mathbf{q}$ (annihilation of a non-zero-point exciton, Fig. 1b), respectively. Let us examine these two cases.

1) $\mathbf{q}_1=0$. Substituting the states normalized according to (A17) and (A18) into (5.7) and using (A12) and then (A18) for a second time, we easily find

$$\mathcal{F}_1^-(\mathbf{q},N)=\mathcal{F}^-(\mathbf{q},0,N)=-\mathcal{N}^{-1/2}N(iuq_+-vq_-),\quad N\gg 1. \tag{5.8}$$

2) $\mathbf{q}=\mathbf{q}_1\neq 0$. The calculation in (5.7) reduces simply to a calculation of the norm (A18) with nonzero $\mathbf{q}$, so that for states normalized according to (A17) we obtain

$$\mathcal{F}_2^-(\mathbf{q},N)=\mathcal{F}^-(\mathbf{q},\mathbf{q},N)=\mathcal{N}^{1/2}(1-N/\mathcal{N})(iuq_+-vq_-),\quad N\gg 1. \tag{5.9}$$

Now, generally speaking, we must consider three more cases, in which $\mathbf{q}_1\neq 0$ and $\mathbf{q}\neq\mathbf{q}_1$. Contributions to the annihilation are made here by both matrix elements (5.6) and (5.7). The corresponding calculation can be performed, but here, in addition to (A17) and (A18), Eqs. (A9′), (A14), (A8), and (A9) and Eqs. (A3), (A14), and (A11) must be employed successively in (5.6) and (5.7), respectively. Nevertheless, we shall not present the response function, since in the final analysis the corresponding contribution to the annihilation flux in the present approximation will be small compared to the total contribution of the fluxes determined by cases 1 and 2. In fact, the third case corresponds to a two-exciton process, which results in annihilation of the zero-point exciton, while the momentum of the non-zero-point exciton changes, but it still remains a non-zero-point exciton. The annihilation rate is proportional to the product of the concentration of zero-point and non-zero-point excitons, which in any case under the conditions of a developed condensate is smaller than the square of the concentration of the zero-point excitons (the "gas" of non-zero-point excitons is tenuous), i.e., when $N\sim\mathcal{N}$ the annihilation flux will still be determined by the process corresponding to case 1 ($\mathcal{F}_1\propto N^2/\mathcal{N}$). If the condensate has already been depleted to a considerable degree ($N\ll\mathcal{N}$) and the system is close to the critical point, the process that is linear with respect to the concentration of the non-zero-point excitons becomes the dominant one (case 2, Fig. 1b). Thus, it is clear from general arguments that the annihilation flux in the third case will always be less than the larger of the fluxes specified by $\mathcal{F}_1^-$ and $\mathcal{F}_2^-$.[3)]

The matrix elements (5.8) and (5.9) describe processes accompanied by phonon emission. In case 2 phonon absorption is impossible, in principle, and in case 1 the phonon absorption process is described by a similar matrix element except that it must be replaced by $\mathcal{F}_1^-(-\mathbf{q},N)$.

The matrix elements describing transitions that produce an exciton likewise do not require a separate calculation in the general case, since they are specified by (5.5) (now $\mathcal{F}_2^+$ is defined in terms of $\mathcal{F}_1^{-*}(-\mathbf{q})$, and $\mathcal{F}_1^+$ is defined in terms of $\mathcal{F}_2^{-*}(-\mathbf{q})$). Here, however, one important comment should be made. By restricting the treatment to the basis (3.5) alone, which would seem to be legitimate when the effects of the direct interaction of excitons are neglected, we have, nevertheless, overlooked the possibility of induced exciton generation. Let the system be in the state $|N,\mathbf{q}_1\rangle$, but let another nonzero-point exciton with momentum $\mathbf{q}$ be generated as a result of the absorption of a phonon with momentum $\mathbf{q}$. In our treatment the matrix element $\mathcal{F}_1^{+}(-\mathbf{q},N)$ corresponds to such a process, i.e., the presence of the first exciton is ignored. In the present case this leads to an error. We correct it by taking into account that exciton generation processes are actually significant only when $N \ll \mathcal{N}$. We utilize Eq. (A19), which enables us to calculate the matrix element

$$\langle \mathbf{q}_1,\mathbf{q},N+1 \| H_{\mathrm{e,ph}}(\mathbf{q}) \| N,\mathbf{q}_1 \rangle\rangle, \tag{5.10}$$

as a replacement for (5.4). Here the bra is the two-exciton state (4.7), which was normalized using (A19). Substituting (5.3) into this expression, we find that the matrix element $\mathcal{F}_1^{+}(-\mathbf{q},N)$ should be redefined or, more specifically, replaced by

$$\mathcal{F}_1'^{+}(-\mathbf{q},\mathbf{q}_1,N)=\mathcal{F}_1^{+}(-\mathbf{q},N)\sqrt{1+\delta_{\mathbf{q},\mathbf{q}_1}} \tag{5.11}$$

(which is valid when $N \ll \mathcal{N}$, and for this reason we naturally do not write an overline over the states in (5.10)).

In the next section we already need the actual form of the vertex $U_s(\mathbf{k})$, but here it is sufficient to restrict ourselves to acoustic phonons. The corresponding expression is determined by the deformation and piezoelectric interactions and was presented for cubic crystals, for example, in Refs. 26 and 27. The square of the absolute value of the matrix element always appears in the response function; therefore, restricting ourselves to the case of the isotropic model of a crystal, we at once write the expression for the quantity

$$\left|\sum_s U_s(\mathbf{q},k_z)\right|^2=\frac{\pi\varepsilon_{\mathrm{ph}}(\mathbf{k})}{p_0^3\tau_A}, \quad \text{where} \quad \frac{1}{\tau_A(\mathbf{k})}$$
$$=\frac{1}{\tau_D}+\frac{5(\lambda p_0)^2}{k^6\tau_P}(q^2k_z^2+q_x^2q_y^2). \tag{5.12}$$

This equation, which results from summing over s, is derived in Appendix B. Here $\varepsilon_{\mathrm{ph}}=\hbar c_s k/\lambda =(\hbar c_s\sqrt{q^2+k_z^2})/\lambda$ is the phonon energy. The literal meanings and numerical values of the parameters τ_D, τ_P, and p_0 of the electron–phonon interaction are also defined in Appendix B.

When (4.5) is taken into account, the characteristic energy of a phonon participating in the annihilation of an exciton should be at most of order ε. Hence $|k_z| \lesssim \lambda\varepsilon/\hbar c_s$, which leads to the condition $k_z d < 1$ because of the first inequality (2.1) for a material such as GaAs. This condition, in turn, allows us to assume that

$$\tilde{U}_s(\mathbf{k}) \approx U_s(\mathbf{k}). \tag{5.13}$$

6. EFFECTIVE ANNIHILATION TIME. TIME DEPENDENCE OF THE *z* COMPONENT OF THE TOTAL SPIN OF THE SYSTEM

The effective annihilation time clearly equals

$$\langle 1/\tau \rangle_s=(\mathcal{R}^- - \mathcal{R}^+)/(N-N^{(0)}). \tag{6.1}$$

Here $\mathcal{R}^-(N)$ is the annihilation flux or the mean number of excitons annihilating per unit time; $\mathcal{R}^+(N)$ is the generation flux, i.e., the mean number of excitons generated per unit time; and $N^{(0)}$ is the equilibrium number of excitons in the system. To find the fluxes $\mathcal{R}^{\pm}$, the quantum-mechanical probability of the respective transition must be multiplied by the statistical probability of finding the system in the particular quantum state. Then the summation must be performed over all possible states corresponding to the specified value of N and the temperature of the system.

A certain difficulty arises in carrying out this procedure, because in a literal sense the states (3.5) are not states of the system in which the number of non-zero-point excitons is greater than unity. At the same time, it is clear that neglect of the interaction of the non-zero-point excitons with one another should, in fact, reduce the calculation of the quantum-mechanical transition probability to the calculation of the matrix elements (5.4). In this approximation we can describe a system with N_1 non-zero-point excitons by constructing an N_1-exciton state by the same technique as (4.7), i.e., by applying the operators $Q_{\mathbf{q}_i}^{+}$ to the zero-point exciton state N_1 times. Equations (4.8), (A19), (5.10), and (5.11), which were obtained for a two-exciton state, are generalized in a corresponding manner.

We stress that the "interference" terms appearing because the operators $Q_{\mathbf{q}_i}$ and $Q_{\mathbf{q}_j}^{+}$ with nonzero $\mathbf{q}_i$ and $\mathbf{q}_j$ do not commute (see the commutator (A5)) must be neglected when the matrix elements are calculated. One exception is the case in which $\mathbf{q}_i=\mathbf{q}_j$. Here it turns out that the "interference" term contains a "coherent" component, which is not small compared to the leading components of the matrix element. In (A19) this leads to the presence of $\delta_{\mathbf{q}_1,\mathbf{q}_2}$ on the right-hand side, which is the only difference between the "correct" matrix elements and (5.4).

The appearance of $\delta_{\mathbf{q}_i,\mathbf{q}_j}$ is associated with induced exciton generation. Let $N=N_0+N_1$, where N_0 and N_1 are the numbers of zero-point and non-zero-point excitons, respectively. We recall that a necessary condition is that

$$N_1=\sum_{\mathbf{q}_1\neq 0} n(\mathbf{q}_1) \ll \mathcal{N} \tag{6.2}$$

($n(\mathbf{q}_1)$ is the momentum distribution function of the non-zero-point excitons); therefore, induced generation is much weaker than the spontaneous process. Actually, it must be taken into account when the difference between the fluxes in (6.1) becomes small, i.e., when equilibrium is approached. Below, unless specially stipulated otherwise, we neglect the terms $\delta_{\mathbf{q}_i,\mathbf{q}_j}$ and use the matrix elements (5.8) and (5.9) to calculate the fluxes.

Neglecting the interaction of the excitons means that the calculation of the Gibbs probability of finding the system in a particular quantum state reduces to the Bose distribution[4)] for $n(\mathbf{q}_1)$. The probability of the transitions described by the matrix elements $\mathscr{F}_2^{\pm}$ should be multiplied by $n(\mathbf{q}_1)$ and then summed over the permissible values of $\mathbf{q}_1$ (here $\mathbf{q}=\mp\mathbf{q}_1$) and k_z. The fluxes $\mathscr{R}^{\pm}$ from the zero-point exciton states (the matrix elements $\mathscr{F}_1^{\mp}$) must be calculated separately. Annihilation is significant in these states, only if $N_0 \gg N_1$. Therefore, the presence of non-zero-point excitons can simply be neglected, and it can be assumed that in this case the entire system is in a zero-point exciton state, i.e., the statistical probability is equal to unity.

Kinetics of exciton annihilation and generation processes

The next step needed to calculate the fluxes $\mathscr{R}^{\pm}$ is establishment of the ranges of allowed values of $\mathbf{q}_1$, $\mathbf{q}$, and k_z, over which the summation should be performed. According to time-dependent perturbation theory, these values are determined by the energy conservation law, but here, despite the identity (5.5), there is no symmetry between annihilation and creation. The corresponding equations, from which the phonon component k_z should be determined, have the form

$$\Delta_e^{\mp}(\mathbf{q},\mathbf{q}_1,k_z)=\pm\epsilon_n+\mathscr{E}(\mathbf{q}_1)-\mathscr{E}(\mathbf{q}_1-\mathbf{q})-\varepsilon_{\mathrm{ph}}(\mathbf{q},k_z)=0, \tag{6.3}$$

$$\Delta_a^{\mp}(\mathbf{q},\mathbf{q}_1,k_z)=\pm\epsilon_n+\mathscr{E}(\mathbf{q}_1)-\mathscr{E}(\mathbf{q}_1-\mathbf{q})-\varepsilon_{\mathrm{ph}}(\mathbf{q},k_z)=0, \tag{6.4}$$

where the superscripts $\mp$ refer to the exciton annihilation and creation processes, and the subscripts e and a refer to phonon emission or absorption. One of the $\mathscr{E}$ (see (4.6)) in these equations must vanish in the cases considered in this work. Thus, in case 1 ($\mathbf{q}_1=0$), creation with emission of a phonon is impossible, and in case 2 ($\mathbf{q}=\mathbf{q}_1$) annihilation with absorption is impossible.

It follows from (6.3) and (6.4) that when annihilation with emission of a phonon occurs in case 1, the absolute value of the transverse momentum of the phonon has an upper bound, which is found by solving the inequality

$$\epsilon_n-\mathscr{E}(q)>\hbar c_s\lambda^{-1}q. \tag{6.5}$$

We denote this value by q_e, i.e.,

$$q<q_e=[2M_n\epsilon_n+(\hbar c_sM_n/\lambda)^2]^{1/2}-\hbar c_sM_n/\lambda. \tag{6.5'}$$

The analogous process in case 2 can occur with any value of $\mathbf{q}_1$, since ε_n is known to be smaller than the maximum energy of an acoustic phonon.

On the other hand, when annihilation occurs with absorption, the absolute value of the transverse momentum has a lower bound:

$$\mathscr{E}(q)-\epsilon_n>\hbar c_s\lambda^{-1}q. \tag{6.6}$$

The solution of this inequality is written in the form

$$q>q_a=[2M_n\epsilon_n+(\hbar c_sM_n/\lambda)^2]^{1/2}-\hbar c_sM_n/\lambda. \tag{6.6'}$$

Figures 3a and b graphically illustrate the solution of the inequalities (6.5) and (6.6) for case 1, Fig. 3a corresponding to the annihilation of a zero-point exciton with emission of a phonon, and Fig. 3b describing the process with absorption.

Let us now examine Eqs. (6.3) and (6.4) with the superscript $+$. The emission of a phonon in case 2 is governed by (6.6) and (6.6′) with the replacement of $\mathbf{q}$ by $\mathbf{q}_1$, so that the range of permissible values of q_1 in this process has a lower bound: $q_1>q_a$ ($\mathbf{q}=\mathbf{q}_1$). The absorption of a phonon is always possible in case 1, and in case 2 it is specified by (6.5) and (6.5′), again with the replacement of $\mathbf{q}$ by $\mathbf{q}_1$, i.e., in this case $q_1<q_e$ and $\mathbf{q}=-\mathbf{q}_1$.

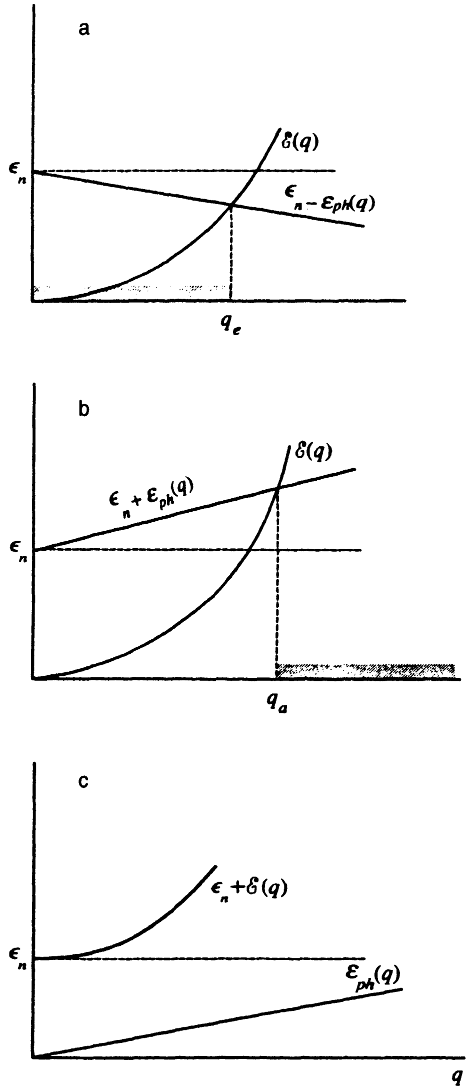

FIG. 3. a, b) Graphical determination of the range of possible values of the transverse momentum of the phonon (hatched area) in processes which alter the number of zero-point excitons in the system: a–annihilation (case 1) or creation (case 2) accompanied, respectively, by the emission or absorption of a photon (inequality (5.5)), b–annihilation in case 1 with absorption of a phonon or creation in case 2 with emission (inequality (5.6)). c. Relationship between the energies in the annihilation (with phonon emission) and creation (with absorption) of a non-zero-point exciton in the case in which the number of zero-point excitons in the system remains unchanged.

Thus, the creation of an exciton in case 2 is illustrated by the same Figs. 3a (absorption of a phonon) and 3b (emission). Figure 3c shows the relationship between the energies in cases in which the momenta of the phonon and the exciton are formally unrestricted (annihilation with emission of a phonon in case 2 and generation with absorption in case 1).

As we know, when the annihilation and creation fluxes are calculated, the terms in the sum over the momenta $\mathbf{q}_1$, $\mathbf{q}$, and k_z contain δ functions of $\Delta^{\mp}_{e,a}$ ((6.3) and (6.4)). As a result, the summation over the phonon component k_z is eliminated, so that it can be expressed in a definite manner in terms of the momentum transfer $\mathbf{q}$ and $\mathbf{q}_1$ for each of the processes considered. Solving Eqs. (6.3) and (6.4) for $|k_z|$, in the cases under consideration here we obtain only two significantly different roots: $|k_z|=k_z^{\pm}$. We write them using the quadratic approximation (4.6) for the exciton energy:

$$(k_z^{\pm})^2=-q^2+\left(\frac{\lambda}{\hbar c_s}\right)^2\left(\mp\epsilon_n+\frac{q^2}{2M_n}\right)^2. \qquad (6.7)$$

Expressions for the fluxes $\mathscr{R}^{\pm}$ in a general form

We henceforth utilize the fact that the condition $q_1\ll 1$ leads to $q\ll 1$ for the characteristic phonons that determine the leading contribution to the fluxes; therefore, the factor in front of $\mathscr{F}^{(\pm)}$ in (5.4) can be assumed to be equal to unity. As a result, the processes that are specified by the matrix elements $\mathscr{F}_2^{\pm}$ and accompanied by the emission of a phonon make the following contributions to the creation and annihilation fluxes

$$\mathscr{R}^{\pm}_{2e}=\frac{2\pi}{(p_0\lambda)^3L^2}\sum_{q>q^{\pm}}n(q)\sum_{\varphi}\begin{pmatrix}|\mathscr{F}_1^-(\mathbf{q})|^2\\|\mathscr{F}_2^-(\mathbf{q})|^2\end{pmatrix}\times\frac{k^2(q)}{k_z^{\pm}(q)\tau_A(k_z^{\pm},q,\varphi)}[1+n_{\text{ph}}(k)] \qquad (6.8)$$

(the upper row corresponds to the upper sign +). In this expression $n_{\text{ph}}(k)$ is the Bose distribution function of the phonons. This equation was obtained by summing the square of the absolute value of the matrix element of the Hamiltonian (5.1) (with multiplication by $2\pi\delta(\Delta_e^{\mp})/\hbar)$) over all possible statistical realizations of the exciton and phonon states. In this process we made L and L_z dimensionless and used (5.12) and (5.13). The angle φ specifies the orientation of $\mathbf{q}$ in the plane, and summing over φ reduces to averaging over the directions of $|\mathscr{F}^-_{1,2}|^2/\tau_A$ and can easily be performed, since the remaining quantities do not depend on φ: $k^2=q^2+[k_z^{\mp}(q)]^2$ (see (6.7)). There is a constraint on the sum over q only when an exciton is created; therefore, $q^+=q_a$ (see (5.6′)), and $q^-=0$. To find $\mathscr{R}^+_{2a}$ the following changes must be made in (6.8): only the upper row is left, but k_z^+ is replaced by k_z^- the summation over q is performed from zero to q_e (see (6.5′)), and the 1 added to n_{ph} is removed.

Similarly, in case 1 we have

$$\mathscr{R}^{\pm}_{1a}=\frac{2\pi}{(p_0\lambda)^3L^2}\sum_{q>q^{\mp}}\sum_{\varphi}\begin{pmatrix}|\mathscr{F}_2^-(\mathbf{q})|^2\\|\mathscr{F}_1^-(\mathbf{q})|^2\end{pmatrix}\times\frac{k^2(q)}{k_z^{\pm}(q)\tau_A(k_z^{\pm},q,\varphi)}n_{\text{ph}}(k). \qquad (6.9)$$

Here $q^{\pm}$ were defined above. Similarly, $[k^{\pm}(q)]^2$ in the sum with q^2 gives $k^2(q)$. Averaging over the angle is performed just as in (6.8), since the matrix elements (5.8) and (5.9) give an identical dependence on $\mathbf{q}$. The equation for $\mathscr{R}^-_{1e}$ has a form similar to the lower row in (6.9), except that the sum over q should not have a lower bound, but should be bounded from above by q_e, $k_z^-(q)$ must be replaced by $k_z^+(q)$ (see (6.7)), and, in addition, $1+n_{\text{ph}}$ must be substituted for n_{ph}. Finally, we recall that, generally speaking, the matrix element (5.10) must be calculated instead of $\mathscr{F}_1^+(\mathbf{q})$.

Calculation of the exciton annihilation time in the strongly nonequilibrium case within a condensate

The calculation based on Eqs. (6.10)–(6.12) must be performed in reference to a specific situation. The characteristic values of the momentum transfer of a phonon and an exciton are determined to a considerable degree by the temperature, which is restricted, first of all, by the requirement (6.2). If a system of 2D electrons is in equilibrium, the number of excitons in it is prescribed by the requirement that the chemical potential μ_{ex}, which appears as a parameter in the distribution $n(\mathbf{q}_1)$, be equal to zero. In this case, going over from the summation in (6.2) to integration, we easily obtain (see, for example, Ref. 12)

$$N=N^{(0)}=-\mathscr{N}M_nT\ln(1-e^{-\epsilon_n/T}). \qquad (6.10)$$

Hence it follows that in or near equilibrium the condition (6.2) holds even when $T\ll\mathscr{U}$. A deviation of the z component of the total spin from its equilibrium value causes the exciton chemical potential to begin to increase. Nevertheless, it cannot exceed the width of the Zeeman gap, but, on the other hand, even when it is equal to ϵ_n, the number of non-zero-point excitons remains restricted. This is because the smallest absolute value of the non-zero-point momentum in the sum (6.2) is "cut off" in any case by the quantity $2\pi/L$ or by the quantity specified by consideration of the exciton–exciton interaction. The latter is found by requiring that the energy of the dipole–dipole interaction of a non-zero-point exciton with the mean field created by the other excitons is $\mathscr{U}_{q_1q_{1T}}(\sqrt{N_1}/L)^3$, where $q_{1T}\sim(M_nT)^{3/2}\mathscr{N}/N_1$. The interaction energy must be lower than the kinetic energy $q_1^2/2M_n$, whence we obtain the lower cut-off parameter q_1. Moving over to the integral expression in (6.2), we find that for $\mu_{\text{ex}}=\epsilon_n$ the critical value of the number of non-zero-point excitons is

$$N_{1\max}=N_c=\mathscr{N}\int_{\tilde{\mathbf{q}}_1}^{\infty}\frac{q_1dq_1}{\exp(q_1^2/2M_nT)-1}\approx\mathscr{N}M_nT\mathscr{L}, \qquad (6.11)$$

$$\mathscr{L}=\ln(2M_nT/\tilde{q}_1^2),\quad \tilde{q}_1=\max(2\pi/L,M_n^2T^2\mathscr{L}^{1/2}). \qquad (6.12)$$

Thus, when $N>N_c$, a condensate is present in the system,[12] and the condition (6.2) leads to the requirement

$$M_nT\mathscr{L}\ll 1,\quad \text{where } \mathscr{L}\simeq 14\text{–}16. \qquad (6.13)$$

It is easy to show that in the present, highly nonequilibrium case our treatment is valid if the temperature does not exceed 1 K. In a real situation, this means, in particular, that

$$T\ll\epsilon,\hbar c_s/\lambda,\quad \text{where } \epsilon\gtrsim\hbar c_s/\lambda. \qquad (6.14)$$

In the components $\mathscr{R}^-_{(1,2)(r,a)}$ the same function can be summed (integrated) over the angle. As a result, with consideration of (5.12) we find

$$\int_0^{2\pi} d\varphi \frac{|iuq_+ - vq_-|^2}{\tau_A} = \pi(u^2+v^2) \times \left[\frac{q^2}{\tau_D} + \frac{5(\lambda p_0)^2 q^2}{\tau_P (k_z^2+q^2)^3} \left(k_z^2 q^2 + \frac{q^4}{8} \right) \right]. \tag{6.15}$$

Now we can substitute (6.15) into (6.8) and (6.9) and integrate over q. Here the different dependences of the PA and DA interactions on q and k_z cause the contribution to the integrals that are proportional to τ_D^{-1} and τ_P^{-1} for $\mathscr{R}^-_{(e,a)}$ to be determined by different characteristic values of q. For example, for $\mathscr{R}^-_{1e}$ we have

$$\begin{aligned} DA:&\quad q^2 \sim M_n\epsilon, \qquad k_z \approx k \sim \lambda\epsilon/\hbar c_s, \quad \text{but} \\ &\quad q^2 < q_e^2 \approx 2M_n\epsilon; \\ PA:&\quad 0 < q_e^2 - q^2 \lesssim \hbar c_s (2M_n\epsilon)^{3/2}/\lambda\epsilon \ll q_e^2, \\ &\quad k_z^2 \sim k^2 \sim 2M_n\epsilon. \end{aligned} \tag{6.16}$$

The flux $\mathscr{R}^-_{1e}$ also includes induced phonon emission, which is proportional to $n_{\text{ph}}(\varepsilon_{\text{ph}})$, but in the case of the DA interaction this part of the flux can be neglected by virtue of (6.16) and the condition (6.14). At the same time, the flux associated with the PA interaction is determined mainly by low-energy phonons with $\varepsilon_{\text{ph}} \sim (\hbar c_s/\lambda)\sqrt{M_n\epsilon} \ll \epsilon$, and, generally speaking, the induced emission must be taken into account. In precisely the same manner, $\mathscr{R}^-_{1a}$, which is proportional to n_{ph}, is determined in our approximation only by the PA interaction, and the characteristic values of q and k_z for it correspond to the second row in (6.16) (after the replacement of q_e by q_a). As a calculation shows, in this case $\mathscr{R}^-_{1a}$ is simply equal to the component of $\mathscr{R}^-_{1e}$ specified by the induced emission. Finally, calculating the sum $\mathscr{R}^-_1 = \mathscr{R}^-_{1e} + \mathscr{R}^-_{1a}$ (see (6.9) and the remarks pertaining to this equation), with consideration of (6.14)–(6.16) we find after integration with respect to q

$$\mathscr{R}^-_1 = \frac{N^2}{\tau_{0s}\mathscr{N}}, \quad \text{where} \quad \tau_{0s}^{-1} = \frac{M_n^2 \epsilon^3 (u^2+v^2)}{6\hbar c_s p_0^3 \lambda^2} \left[\frac{1}{\tau_D} + \frac{15}{2\tau_P} \left(\frac{\hbar c_s p_0}{\epsilon} \right)^2 \Phi(\xi) \right], \tag{6.17}$$

$$\Phi(\xi) = \int_1^\infty \frac{4x^2 - 7/2}{(x^2-1)^{1/2} x^4} \left(\frac{e^{x/\xi}+1}{e^{x/\xi}-1} \right) dx,$$

$$\xi = \frac{T\lambda}{\hbar c_s} (2M_n\epsilon)^{-1/2}.$$

The asymptotic values of $\Phi(\xi)$ are

$$\Phi(\xi) = \begin{cases} 5/3, & \xi \ll 1 \\ 11\pi\xi/16, & \xi \gg 1. \end{cases}$$

We note that Eq. (6.17) (like the other response functions presented below for the fluxes and relaxation times) was obtained for both the DA and PA interactions to leading order in $(\hbar c_s/\lambda)^2 M_n/\varepsilon$, which is clearly small because of (2.4) and (6.14). As we have already stated, the flux thus obtained describes the annihilation associated with the spin–orbit interaction of zero-point excitons with one another, the presence of non-zero-point excitons being ignored. Therefore, Eq. (6.17) can be used only if $N \gg N_c$.

Let us now determine $\mathscr{R}^-_2$. The characteristic values of q and k_z for this flux with consideration of the condition (6.14) are identical in both cases of DA and PA electron–phonon interactions,

$$\mathscr{R}^-_{2e}: \quad q^2 = q_1^2 \sim M_n T, \qquad k_z \approx k \approx \frac{\lambda\epsilon}{\hbar c_s}. \tag{6.18}$$

In addition, we always have $\mathscr{R}^-_{2a} \ll \mathscr{R}^-_{2e}$. Substituting (6.15) into (6.8), for $\mu_{\text{ex}} = \varepsilon_n$ in the leading approximation we obtain

$$\mathscr{R}^-_2 = \mathscr{N} \left(1 - \frac{N}{\mathscr{N}} \right)^2 \frac{(\pi M_n T) 2\epsilon (u^2+v^2)}{6\hbar c_s p_0^3 \lambda^2} \left[\frac{1}{\tau_D} + \frac{M_n T (2\pi p_0/\lambda)^2}{\tau_P} \left(\frac{\hbar c_s}{\epsilon} \right)^4 \right], \qquad N > N_c. \tag{6.19}$$

Here we have formally retained the factor $(1 - N/\mathscr{N})^2$ to show that the flux vanishes when $N = \mathscr{N}$. In fact, comparing (6.17) and (6.19), we find

$$\frac{\mathscr{R}^-_2}{\mathscr{R}^-_1} \sim \left[\frac{T}{\epsilon} \left(\frac{\mathscr{N}}{N} - 1 \right) \right]^2, \tag{6.20}$$

therefore, when $N \sim \mathscr{N}$, $\mathscr{R}^-_1$ is the leading contribution to the flux.

Let us trace how the response function varies as N decreases. It follows from (6.20) that when

$$N_c \ll N \sim \mathscr{N} T/\epsilon, \tag{6.21}$$

both (6.17) and (6.19) are valid. However, when (6.12) and (6.13) are taken into account, it is not difficult to see that this relation can actually hold only in comparatively weak magnetic fields, for which, in turn, the second of the conditions (2.1) is violated. When N decreases to values of order N_c, (6.17) has, as we have seen, superfluous accuracy, and (6.19) gives a correct response function only if the right-hand side of (6.20) is much greater than unity. Hence we obtain the condition $M_n \varepsilon \mathscr{L} \ll 1$. If this condition is not satisfied, both (6.17) and (6.19) give an estimate that is correct only in order of magnitude when $N \sim N_c$.

We note that in the case considered here of a strong deviation of the z component of the total spin from equilibrium, the contribution of the reverse creation fluxes $\mathscr{R}^+_j$ to (6.1) is naturally small compared with the contribution of the annihilation fluxes $\mathscr{R}^-_j$ This, however, can be proved directly using (6.8) and (6.9) and taking into account the remarks pertaining to them. Thus, when the conditions (6.13) and (6.14) hold, we have

$$\langle 1/\tau \rangle_s = \begin{cases} (\mathscr{R}^-_1 + \mathscr{R}^-_2)/N \quad \text{[Eqs. (6.17)} \\ \quad \text{and (6.19)],} \qquad \text{if} \quad N \gg N_c; \\ \mathscr{R}^-_2/N \quad \text{[Eq. (6.19)],} \\ \quad \text{if } N \sim N_c \quad \text{and} \quad M_n \epsilon \mathscr{L} \ll 1. \end{cases} \tag{6.22}$$

The time dependence of the number of excitons in the system is given by

$$-\frac{dN}{dt}=\mathcal{R}^- - \mathcal{R}^+. \tag{6.23}$$

Therefore, when $N \gg N_c$, substituting (6.17) into this expression, we can easily determine the explicit time dependence of S_z (we recall that S_z is related to N by (3.1) with an accuracy to the neglect of small corrections, whose relative magnitude is of order u^2),

$$S_z(t)=\frac{S_z(0)+t[\mathcal{N}/2-S_z(0)]/2\tau_{0s}}{1+t[1/2-S_z(0)/\mathcal{N}]/\tau_{0s}}. \tag{6.24}$$

Exciton annihilation time below the critical point ($N<N_c$) and near the equilibrium value $N^{(0)}$

The flux $\mathcal{R}_1^-$ is determined by the interaction of zero-point excitons with one another. Formally, the number of zero-point excitons in the system N_0 should appear in Eqs. (5.8) and (6.17) instead of N; therefore, below the critical point $\mathcal{R}_1^-$ is equal to zero in principle, so that the annihilation flux is $\mathcal{R}_2^-$. It must be taken into account in the calculation that the chemical potential μ_{ex} is now smaller then ϵ_n. It is a function of the total number of excitons $N=N_1$, and in the present case in the model of an ideal two-dimensional Bose gas it is specified by the simple formula

$$\mu_{\text{ex}}=\epsilon_n-T\zeta\left(\frac{N}{\mathcal{N}}\right), \quad \text{where } \zeta=-\ln\left[1-\exp\left(-\frac{N}{\mathcal{N}M_nT}\right)\right], \tag{6.25}$$

which is valid when $N_c-N>\mathcal{N}M_nT$. All that remains is to substitute the Bose functions of the excitons and phonons

$$n(q)=\frac{1}{\exp[\zeta+\mathcal{E}(q)/T]-1},$$

$$n_{\text{ph}}(q)=\frac{1}{\exp[\varepsilon_{\text{ph}}(q)/T]-1} \tag{6.26}$$

(here $\varepsilon_{\text{ph}}(q)=\epsilon_n+\mathcal{E}(q)$) into (6.8) and to integrate the total momentum over them. When we go to equilibrium, where the number of zero-point excitons is considerably less than N_c, we no longer need to require the conditions (6.13) and (6.14). Here we assume that the temperature is restricted only by $T \ll \mathcal{U}$, which allows us to use the quadratic approximation for the exciton energy (4.6).

One significant point is the need to now take into account the reverse creation flux, which is of order $\mathcal{R}_2^-$ near equilibrium. In the present case the reverse flux, which is not associated with a direct exciton–exciton interaction, is $\mathcal{R}_{1a}^+$. The absolute value of the corresponding matrix element $\mathcal{F}_1^+(-\mathbf{q})$ is equal to that of the annihilation matrix element $\mathcal{F}_2^-(\mathbf{q})$ (see (6.9)) and governs the spontaneous creation of an exciton from the vacuum. The correction for the possibility of induced creation with consideration of the remark made in the preceding section simply reduces, according to Eq. (5.1), to the multiplication of (6.9) by $1+\delta_{\mathbf{q},\mathbf{q}_1}$. The presence of the δ term signifies doubling of the quantum-mechanical probability of exciton creation, as soon as an exciton with the same wave vector as the exciton being created is present in the system. The induced creation flux is proportional to the number of occurrences of such doubling, i.e., to $n(\mathbf{q}_1)$.

Thus, the integrand for the reverse flux contains the factor $n_{\text{ph}}(q)[1+n(q)]$ and the same matrix element as the forward annihilation flux. Subtracting the creation flux from the sum $\mathcal{R}_{2e}^-+\mathcal{R}_{2a}^+$ we obtain the response function

$$\mathcal{R}^- - \mathcal{R}^+ = \mathcal{N}\frac{(M_nT)^2\epsilon(u^2+v^2)}{\hbar c_s p_0^3\lambda^2}\left[\frac{1}{\tau_D}\phi_1(N)+\frac{10M_nT}{\tau_P}\phi_2(N)\left(\frac{p_0}{\lambda}\right)^2\left(\frac{\hbar c_s}{\epsilon}\right)^4\right]. \tag{6.27}$$

Here the functions $\phi_{1,2}$ can be expressed in the following manner in terms of integral Bose functions

$$\phi_\kappa(N)=F_\kappa\left[\zeta\left(\frac{N}{\mathcal{N}}\right)\right]-F_\kappa\left(\frac{\epsilon_n}{T}\right), \quad \text{where } F_\kappa(\zeta)=\int_0^\infty\frac{x^\kappa dx}{e^{\zeta+x}-1}. \tag{6.28}$$

Their asymptotes are well known. When $\zeta \ll 1$, they take the values $F_1=\pi^2/6$ and $F_2=\pi^4/15$, and at large values of ζ they are exponentially small and equal respectively to $\kappa e^{1-\zeta}$. Therefore, we once again obtain Eq. (6.19) by the time $N \gg \mathcal{N}M_nT$ (provided, of course, in addition, $T \ll \epsilon$).

As equilibrium is approached, the response function is obtained by expanding $F_\kappa(\zeta)$ near ϵ_n/T:

$$\phi_\kappa=\frac{N-N^{(0)}}{\mathcal{N}M_nT}\gamma_\kappa(\epsilon_n/T), \tag{6.29}$$

where

$$\gamma_\kappa\left(\frac{\epsilon_n}{T}\right)=(e^{2\epsilon_n/T}-e^{\epsilon_n/T})\int_0^\infty\frac{e^x x^\kappa dx}{(e^{\epsilon_n/T+x}-1)^2}=\begin{cases}\kappa!, & \epsilon \gg T\\ (\epsilon/T)\chi(\kappa) & \epsilon \ll T\end{cases},$$

$\chi(1)=\ln(T/\epsilon)$, and $\chi(2)=\pi^2/3$. The equilibrium number of excitons is specified by Eq. (6.10). Substituting (6.29) into (6.27) and (6.1), we find the annihilation time near equilibrium

$$1/\tau_{qs}=\frac{M_nT\epsilon(u^2+v^2)}{\hbar c_s p_0^3\lambda^2}\left[\frac{1}{\tau_D}\gamma_1\left(\frac{\epsilon_n}{T}\right)+\frac{10M_nT}{\tau_P}\gamma_2\left(\frac{\epsilon_n}{T}\right)\times\left(\frac{p_0}{\lambda}\right)^2\left(\frac{\hbar c_s}{\epsilon}\right)^4\right]. \tag{6.30}$$

Equations (6.29) and (6.30) hold if $N-N^{(0)} \ll \mathcal{N}M_n\epsilon(e^{\epsilon_n/T}-1)$.

The general expression (6.27) under the appropriate conditions (see (6.13) and (6.14)) is suitable for application up to values of N of order N_c, so that, in principle, (6.27) together with (6.22) defines, according to Eq. (6.23), the time dependence of S_z over the entire range of values of N. Near equilibrium the spin relaxation process becomes exponential with the characteristic time (6.30).

7. DISCUSSION AND CONCLUSIONS

Let us obtain concrete numerical values of the inverse relaxation times for GaAs as a function of the magnetic field and the temperature. We neglect the spin–orbit constant α, and in the expression for $\beta(B)$ (2.3) we assume that $d=33$ Å. Substituting the corresponding values of the material constants, as well as the values of p_0, τ_D, and τ_P given in Appendix B, into (6.17) and (6.30), we find

$$1/\tau_{0s}=\mathscr{A}_n[\Phi(\xi)+\mathscr{K}B^2],\quad \xi=r_nTB^{-3/4};$$
$$1/\tau_{qs}=\mathscr{C}_nT^2B^{-3}\gamma_2(w)+\mathscr{D}_nTB^{1/2}\gamma_1(w),\quad w=\eta B/T. \tag{7.1}$$

The first of these quantities specifies relaxation in a condensate (see (6.24)), and the second describes exponential relaxation near the equilibrium value $S_z=\mathscr{N}/2-N^{(0)}$. The functions $\Phi(\xi)$ and $\gamma_\kappa(w)$ are defined in the next section. If the field and the temperature are measured in teslas and Kelvins, respectively, and the times (7.1) are measured in microseconds, the constants appearing in (7.1) have the following values for the zeroth and first Landau level:5) $\eta\simeq0.3$; $\mathscr{K}\simeq3.5\times10^{-3}$;

$$\mathscr{A}_0\simeq0.037;\quad r_0\simeq6.8;\quad \mathscr{C}_0\simeq44;\quad \mathscr{D}_0\simeq0.29;$$
$$\mathscr{A}_1\simeq0.012;\quad r_1\simeq9.1;\quad \mathscr{C}_1\simeq15;\quad \mathscr{D}_1\simeq0.16.$$

Thus, the relaxation of a spin condensate under the characteristic conditions of QHE should take several microseconds, and relaxation near the equilibrium value of S_z is measured in tenths of a microsecond.

The peculiar nonmonotonic dependence of the parameters (7.1) on the magnetic field is due to the competition between two opposite tendencies. On the one hand, as B increases, the spin–orbit interaction effectively weakens ($u^2\propto1/B$), and the exciton effective mass decreases, but, on the other hand, the increase in the Zeeman gap causes exciton annihilation to occur in the large phase volume of possible wave vectors of the phonon being emitted. The dependences of the PA and DA interactions on $\mathbf{k}$ are different, and thus there are significantly different dependences of the first (PA) and second (DA) terms in the final response functions (7.1) on B. As can easily be seen, deformation scattering has an appreciable influence on relaxation in a condensate only at fields greater than 15 T and temperatures below 0.1 K.

Let us compare our response functions with the results of the measurements in Ref. 3. If we formally substitute the values $T=1.5$ K and $B=4.9$ T for $n=0$ and $T=1.5$ K and $B=2.6$ T for $n=1$ into (7.1), we obtain $\tau_{0s}\simeq5.6$ and 10 μs and $\tau_{qs}=0.48$ and 0.16 μs, respectively, for the zeroth and first levels. Such fundamental deviations from the values of the characteristic relaxation times found experimentally in Ref. 3, viz., 10 ns ($n=0$) and 4 ns ($n=1$), must be explained.

The first reason for this disparity lies in the fact that there was obviously no spin condensate in Ref. 3, since the condition (6.13) was not satisfied. In terms of spin-exciton excitations, the spin system in Ref. 3 was initially a cluster of strongly interacting nonzero-point excitons, so that relaxation far from equilibrium took place, of course, according to a scenario differing from the one described here. When equilibrium is approached, agreement between the experimental data and the values of τ_{qs} might be expected, but here, too, as we see, the difference amounts to more than an order of magnitude. Nevertheless, this fact is not surprising. First of all, the parameter d in the definition of the spin–orbit energy (2.3) is quite vague. The effective thickness of the layer in this equation must correspond to the characteristic thickness of the heterojunction only in order of magnitude. On the other hand, a twofold decrease in d results in a 16-fold increase in the rate of the relaxation process, and then our values of τ_{qs} turn out to be close to the experimental values. It must also be taken into account that the first of the inequalities in (2.1) was satisfied in Ref. 3 only by a small margin, and the second inequality actually even had the opposite sign. In particular, this allows the effective mass, for example, to differ significantly (in the upward direction) from the values defined in (4.6′), which we used to obtain (7.1). As an example, we calculate M_0 directly from Eq. (4.6) using the model size-quantized function $f(z)=2d^{-3/2}ze^{-z/d}$. For d equal to 33 Å and a field equal to 4.9 T we find that the exciton mass increases in comparison to (4.6′) by a factor of approximately 1.5. The violation of the second inequality in (2.1), which leads to alteration of the matrix element (5.4), can have an even more significant effect on the result.

To verify our theoretical results experiments must be set up at temperatures below 0.1 K in fields no weaker than 10 T. In addition, to investigate relaxation under the conditions of a spin condensate it would be desirable to use a procedure under which a strong deviation of S_z from its equilibrium value would not have an effect, as much as possible, on the thermal equilibrium of the electronic system with the lattice.

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APPENDIX A

We define the exciton annihilation operator as the Hermitian conjugate of the operator (3.4), and then taking into account that a^+, a, b^+, and b are Fermi operators, we can easily find

$$[Q_\mathbf{q},a_p^+]=[Q_\mathbf{q},b_p]=[Q_\mathbf{q}^+,a_p]=[Q_\mathbf{q}^+,b_p^+]=0; \tag{A1}$$

$$[Q_\mathbf{q}^+,a_p^+]=\mathscr{N}^{-1/2}\exp\{-iq_x(p+q_y/2)\}b_{p+q_y}^+,$$
$$[Q_\mathbf{q},a_p]=-\mathscr{N}^{-1/2}\exp\{iq_x(p+q_y/2)\}b_{p+q_y}, \tag{A2}$$
$$[Q_\mathbf{q},b_p^+]=\mathscr{N}^{-1/2}\exp\{iq_x(p-q_y/2)\}a_{p-q_y}^+,$$
$$[Q_\mathbf{q}^+,b_p]=\mathscr{N}^{-1/2}\exp\{-iq_x(p-q_y/2)\}a_{p-q_y}.$$

In addition,

$$[Q_{\mathbf{q}_1}^+,Q_{\mathbf{q}_2}^+]=[Q_{\mathbf{q}_1},Q_{\mathbf{q}_2}]=0. \tag{A3}$$

It is useful to introduce the operators

$$A_{\mathbf{q}}^{+}=A_{-\mathbf{q}}=\mathcal{N}^{-1}\sum_{p}\exp(-iq_xp)a_{p+q_y/2}^{+}a_{p-q_y/2},$$

$$B_{\mathbf{q}}^{+}=B_{-\mathbf{q}}=\mathcal{N}^{-1}\sum_{p}\exp(-iq_xp)b_{p+q_y/2}^{+}b_{p-q_y/2}, \qquad (A4)$$

in terms of which we can express the commutator

$$[Q_{\mathbf{q}_1},Q_{\mathbf{q}_2}^{+}]=\exp(i\Theta_{12})A_{\mathbf{q}_1-\mathbf{q}_2}-\exp(-i\Theta_{12})B_{\mathbf{q}_1-\mathbf{q}_2}. \qquad (A5)$$

We used the notation

$$\Theta_{12}=\Theta(\mathbf{q}_1,\mathbf{q}_2)=(q_{1x}q_{2y}-q_{2x}q_{1y})/2. \qquad (A6)$$

In a special case

$$[Q_{\mathbf{q}},Q_{\mathbf{q}}^{+}]=[Q_0,Q_0^{+}]=(A_0^{+}-B_0^{+})=1-2\frac{\hat{N}}{\mathcal{N}}. \qquad (A7)$$

In (A7) it was taken into account that the operator identity $A_0^{+}+B_0^{+}=1$ holds in our system under the condition (1.1).

The following rules governing the action of the operators on the ground state are obvious:

$$Q_{\mathbf{q}}|0\rangle=0, \qquad A_{\mathbf{q}}^{+}|0\rangle=\delta_{\mathbf{q},0}|0\rangle,$$

$$B_{\mathbf{q}}^{+}|0\rangle=0, \qquad [Q_{\mathbf{q}_1},Q_{\mathbf{q}_2}^{+}]|0\rangle=\delta_{\mathbf{q}_1,\mathbf{q}_1}|0\rangle. \qquad (A8)$$

The commutation rules for the operators (A4) with the exciton operators $Q_{\mathbf{q}}^{+}$ and $Q_{\mathbf{q}}$ have the following forms:

$$e^{i\Theta_{12}}[A_{\mathbf{q}_1},Q_{\mathbf{q}_2}]=-e^{-i\Theta_{12}}[B_{\mathbf{q}_1},Q_{\mathbf{q}_2}]=\mathcal{N}^{-1}Q_{\mathbf{q}_2+\mathbf{q}_1},$$

$$e^{i\Theta_{12}}[A_{\mathbf{q}_1},Q_{\mathbf{q}_2}^{+}]=-e^{-i\Theta_{12}}[B_{\mathbf{q}_1},Q_{\mathbf{q}_2}^{+}]=-\mathcal{N}^{-1}Q_{\mathbf{q}_2-\mathbf{q}_1}^{+}. \qquad (A9)$$

In a special case

$$[A_{\mathbf{q}}+B_{\mathbf{q}},Q_0^{+}]=[A_{\mathbf{q}}+B_{\mathbf{q}},Q_0]=0. \qquad (A9')$$

As a result, we can use (A3)–(A9) to calculate the convolutions

$$\langle 0|Q_{\mathbf{q}}Q_{-\mathbf{q}}(Q_0^{+})^2|0\rangle=2\delta_{\mathbf{q},0}-2/\mathcal{N}, \qquad (A10)$$

$$\langle 0|Q_{\mathbf{q}_2}Q_{\mathbf{q}}Q_0^{+}Q_{\mathbf{q}_1}^{+}|0\rangle=\delta_{\mathbf{q},0}\delta_{\mathbf{q}_1,\mathbf{q}_2}+\delta_{\mathbf{q}_2,0}\delta_{\mathbf{q},\mathbf{q}_1}-\frac{2\cos\Theta_{12}}{\mathcal{N}}\delta_{\mathbf{q}_2+\mathbf{q},\mathbf{q}_1}, \qquad (A11)$$

as well as to establish the following rules for the action of the excitonic operators on the states (3.5)

$$Q_{\mathbf{q}}|N,0\rangle=N\left(\delta_{\mathbf{q},0}-\frac{N-1}{\mathcal{N}}\right)|N-1,-\mathbf{q}\rangle, \qquad (A12)$$

$$Q_{\mathbf{q}_1}|N,\mathbf{q}_2\rangle=\left(\delta_{\mathbf{q}_1,\mathbf{q}_2}-2\frac{N-1}{\mathcal{N}}\cos\Theta_{12}\right)|N-1,\mathbf{q}_2-\mathbf{q}_1\rangle+(N-1)\left(\delta_{\mathbf{q}_1,0}-\frac{N-2}{\mathcal{N}}\right)Q_{-\mathbf{q}_1}^{+}|N-2,\mathbf{q}_2\rangle. \qquad (A13)$$

Next, using (A12) and (A13) with $\mathbf{q}_1=0$, we obtain the rules for the action of powers of Q_0 on the states (3.5):

$$(Q_0)^M|N,\mathbf{q}\rangle=\frac{(N-1)!(\mathcal{N}-N+M-1)!}{\mathcal{N}^M(N-1-M)!(\mathcal{N}-N-1)!}|N-M,\mathbf{q}\rangle, \qquad (A14)$$

if $\mathbf{q}\neq 0$ and $M\leqslant N-1$,

$$(Q_0)^M|N,0\rangle=\frac{N!(\mathcal{N}-N+M)!}{\mathcal{N}^M(N-M)!(\mathcal{N}-N)!}|N-M,0\rangle, \quad \text{for } M\leqslant N \qquad (A15)$$

and, at the same time,

$$(Q_0)^{M-1}|N,\mathbf{q}\rangle|_{\mathbf{q}\neq 0}=(Q_0)^M|N,0\rangle=0 \quad \text{for } M\geqslant N+1. \qquad (A16)$$

Hence the normalization rule of the states (3.5) is established in the special case in which $M=N$. Calculating $\langle\mathbf{q},N|N,\mathbf{q}\rangle=R(N,\mathbf{q})$, we find

$$||N,\mathbf{q}\rangle\rangle=R^{-1/2}(N,\mathbf{q})|N,\mathbf{q}\rangle, \qquad (A17)$$

where

$$R(N,\mathbf{q})=\begin{cases} N!\mathcal{N}![\mathcal{N}^N(\mathcal{N}-N)!]^{-1}, & \text{if } \mathbf{q}=0 \text{ and } N\leqslant\mathcal{N}; \\ (N-1)!(\mathcal{N}-2)![\mathcal{N}^{N-1}(\mathcal{N}-N-1)!]^{-1}, & \\ \text{if } \mathbf{q}\neq 0 \text{ and } 1\leqslant N\leqslant\mathcal{N}-1. & \end{cases} \qquad (A18)$$

Using (A5)–(A13) and (A14)–(A18), we can show that the norm of a two-exciton state that is not an eigenstate of the symmetric model is

$$\langle\mathbf{q}_2,N|Q_{\mathbf{q}_1}Q_{\mathbf{q}_1}^{+}|N,\mathbf{q}_2\rangle=[(1-N/\mathcal{N})^2+\delta_{\mathbf{q}_1,\mathbf{q}_2}+O(\mathcal{N}^{-1})]R(N,\mathbf{q}_1)$$

$$q_1q_2\neq 0. \qquad (A19)$$

This equation was obtained for the case in which $N\gg 1$, $\mathcal{N}\gg 1$, and $\mathcal{N}-N\sim\mathcal{N}$.

APPENDIX B

The isotropic model[27] enables us to take into account the deformation and piezoelectric fields independently. In addition, we are dealing with a cubic crystal, in which the band extremum is located at the point $\mathbf{k}=0$. Then contributions from the *DA* interaction are made only by longitudinal phonons; therefore the "deformation" part of the desired quantity (the left-hand side of (5.12)) contains only one term. As a result, this quantity can easily be found (see Refs. 26 and 27):

$$|U^D(\mathbf{q},k_z)|^2=(\pi/p_0^3\tau_D)\varepsilon_{\text{ph}}(\mathbf{k}), \quad \text{where } \tau_D^{-1}=\frac{\Xi^2p_0^3}{2\pi\hbar\rho c_s^2}. \qquad (B1)$$

The nominal time τ_D was introduced in accordance with Ref. 27. It can be expressed in terms of the deformation potential Ξ, the velocity of longitudinal sound c_s, the density ρ, and the nominal momentum p_0, whose symbolic and numerical values were presented in Ref. 27; in particular, $p_0=2.52\times 10^6$ cm^{-1} for GaAs.

It is somewhat more difficult to obtain the corresponding "piezoelectric" contribution, since the longitudinal mode and both transverse modes must be taken into account:

$$|U^P(\mathbf{q},k_z)|^2=\left|U_l^P(\mathbf{q},k_z)+\sum_{t=t_1,t_2}U_t^P(\mathbf{q},k_z)\right|^2. \tag{B2}$$

Here

$$U_s^P=\frac{4\pi e e_{14}}{\epsilon_0}\sqrt{\frac{\hbar}{2\rho\epsilon_{\mathrm{ph}}}}\left[\frac{q_xq_yd_z^{(s)}+k_z(q_xd_y^{(s)}+q_yd_x^{(s)})}{q^2+k_z^2}\right] \tag{B3}$$

(e_{14} is the piezoelectric constant of the cubic crystal, ϵ_0 is the dielectric constant, and e is the electron charge). The polarization unit vectors $\mathbf{d}^{(s)}$ in the long-wavelength approximation depend only on the direction of propagation of the phonon. Therefore, $\mathbf{d}^{(l)}=(\mathbf{q}/k,k_z/k)$, and the transverse vectors $\mathbf{d}^{(t)}$ can be chosen arbitrarily in a plane perpendicular to $\mathbf{k}$. After raising (B3) to the second power, the appropriate averaging must be performed in (B2).

We introduce the coordinate system (x',y',z'), in which the z' axis is directed along $\mathbf{k}$, and the x' axis follows the straight line along which the (x,y) and (x',y') planes intersect. Then the polarization vector for each transverse mode can be expressed in terms of the unit vector $\mathbf{t}$ in the (x',y') plane, for example

$$d_x^{(t_1)}=t_{1x}q_y/q+t_{1y}q_xk_z/qk,\qquad d_y^{(t_1)}=-t_{1x}q_x/q$$
$$+t_{1y}q_yk_z/qk,\qquad d_z^{(t_1)}=-t_{1y}q/k.$$

Substituting this and the analogous expression for $\mathbf{d}^{(t_2)}$ into (B3), and then everything into (B2), and averaging over the $\mathbf{t}_1$ and $\mathbf{t}_2$ directions ($\overline{t_x^2}=\overline{t_y^2}=1/2$, $\overline{t_x}=\overline{t_y}=\overline{t_xt_y}=0$; therefore, the interference terms vanish), we obtain an expression, which, when added to (B1), gives Eq. (5.12). The nominal PA scattering time is defined in the following manner:

$$\tau_P^{-1}=\left(\frac{ee_{14}}{\epsilon_0}\right)^2\frac{8\pi p_0}{5\hbar\rho c_s^2}. \tag{B4}$$

The coefficient 5 in (5.12) and, accordingly, the factor 1/5 in (B4) were introduced so that our expression for τ_P would correspond to the "three-dimensional" quantity. This coefficient appears because the formal averaging of the quantity in parentheses in (5.12) over all the directions of $\mathbf{k}$ in three-dimensional space gives $k^4/5$, which is consistent with the result of the averaging performed, for example, in Ref. 28.

The values of the material constants appearing in the definitions (B1) and (B4) can be found in reference texts (for example, Ref. 29). In particular, the data presented on the deformation potential in GaAs (in the case of A_3B_5 semiconductors, it is the quantity denoted by $a(\Gamma_{1c})$ or $a(\Gamma_{6c})$ in the generally accepted terminology) suggest that $|\Xi|\simeq 17.5\pm 1.0$ eV (with consideration of the small spread in the experimental values obtained by different methods). Therefore, the corresponding time τ_D equals 0.8 ps, which is five times smaller than the numerical value of this time in Table A5 in Ref. 27. To determine the numerical value of τ_P, the piezoelectric constant of the crystal must be known. The data presented in Refs. 27 and 29 were obtained from the experiment in Ref. 30 ($e_{14}=0.16$ C/m^2). As a result, we find $\tau_P\simeq 35$ ns.

[1)] The possibility of such a transformation stems from the fact that the exciton creation and annihilation operators are neither Bose nor Fermi operators and that, in particular, the "kinematic" identity $(Q_0^+)^2=\mathcal{N}^{-1}\Sigma_{\mathbf{q}}Q_{-\mathbf{q}}^+Q_{\mathbf{q}}^+$ holds (the expression for $Q_{\mathbf{q}}^+$ is presented in Sec. 3).

[2)] Satisfaction of the weaker inequality $d^2\ll\lambda r_B$ is sufficient for "uncoupling" the coordinates (x,y) and z in the three-dimensional Schrödinger equation and neglecting the interaction when the size-quantization problem in a layer is solved, but for reasons that will become clear below we assume that d satisfies (2.1). The second inequality in (2.1) means that to determine the state of the interacting two-dimensional electrons we can consider the combinations of one-particle wave functions of only a single Landau level (the one which corresponds to the lowest unfilled magnetic band), taking into account, if necessary, the states of other levels only to leading order.

[3)] These arguments are of course quite superficial. A detailed analysis, however, confirms the validity of the statement made in regard to an equilibrium system (with respect to the temperature, but not with respect to S_z). At the same time, we can imagine a situation in which a highly nonequilibrium fluctuation of nonzero-point excitons with a large characteristic value of the momentum $|\mathbf{q}_1|\gtrsim 1$ might appear in a system with a developed condensate. In this case, even if the number of such excitons is small compared with $\mathcal{N}$ (i.e., if the rarefaction condition holds), their contribution to annihilation can be greater than the "background" contribution of zero-point excitons in the condensate. Then the flux specified by the third case of annihilation can be dominant.

[4)] We recall that the Bose–Einstein distribution is valid in all cases in which an unlimited number of identical noninteracting particles can be found in the same state and that, generally speaking, the particles themselves need not be described by quantum-mechanical Bose creation and annihilation operators. Strictly speaking, in our case the Bose distribution is only approximately valid, since the possible number of non-zero-point excitons in each assigned state is restricted to $\mathcal{N}-N_0-1$. Naturally, we assume that $\mathcal{N}-N_0\gg 1$.

[5)] The first terms in Eqs. (7.1) are determined by the PA interaction, for which the phonons of all three polarizations are significant. Since we are using an isotropic model, a certain mean value of the velocities of longitudinal and transverse sound should be employed as c_s in the corresponding places (including for τ_P) in (6.17), (6.19), (6.27), and (6.30). In our calculations we assume that it is equal to 0.7 times the velocity of longitudinal sound (see Ref. 27). Contributions to the DA interaction are made only by longitudinal phonons; therefore, $\mathcal{K}$, $\mathcal{A}_n$, and $\mathcal{D}_n$ are specified by the corresponding formulas, in which c_s is equal to the velocity of longitudinal sound.

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[1] S. Bar-Ad and I. Bar-Joseph, Phys. Rev. Lett. **68**, 349 (1992).

[2] G. Muller, L. Weiss, A. V. Khaetskii, K. von Klitzing *et al.*, Phys. Rev. B **45**, 3932 (1992).

[3] V. E. Zhitomirskiĭ, V. E. Kirpichev, A. I. Filin, V. B. Timofeev, B. N. Shepel', and K. von Klitzing, JETP Lett. **58**, 439 (1993).

[4] S. E. Barret, G. Dabbagh, L. N. Pfeiffer *et al.*, Phys. Rev. Lett. **74**, 5112 (1995).

[5] A. V. Khaetskii, Phys. Rev. B **45**, 13 777 (1992).

[6] G. Bastard, Phys. Rev. B **46**, 4253 (1992).

[7] S. L. Sondhi, A. Karlhede, S. A. Kivelson, and E. H. Rezayi, Phys. Rev. B **47**, 16 419 (1993).

[8] H. A. Fertig, L. Brey, R. Côté, and A. H. MacDonald, Phys. Rev. B **50**, 11 018 (1994).

[9] L. Brey, H. A. Fertig, R. Côté, and A. H. MacDonald, Phys. Rev. Lett. **75**, 2562 (1995).

[10] G. L. Bir, A. G. Aronov, and G. E. Pikus, Zh. Éksp. Teor. Fiz. **69**, 1382 (1975) [Sov. Phys. JETP **42**, 705 (1976)].

[11] I. V. Lerner and Yu. E. Lozovik, Zh. Éksp. Teor. Fiz. **78**, 1167 (1980) [Sov. Phys. JETP **51**, 588 (1980)].

[12] I. V. Lerner and Yu. E. Lozovik, Zh. Éksp. Teor. Fiz. **80**, 1488 (1981) [Sov. Phys. JETP **53**, 763 (1981)]; Zh. Éksp. Teor. Fiz. **82**, 1188 (1982) [Sov. Phys. JETP **55**, 691 (1982)].

[13] Yu. A. Bychkov, S. V. Iordanskiĭ, and G. M. Éliashberg, JETP Lett. **33**, 143 (1981); Poverkhnost' **5**, 3 (1983).

[14] Yu. A. Bychkov and É. I. Rashba, Zh. Éksp. Teor. Fiz. **85**, 1826 (1983) [Sov. Phys. JETP **58**, 1062 (1983)].

[15] C. Kallin and B. I. Halperin, Phys. Rev. B **30**, 5655 (1984).

[16] A. B. Dzyubenko and Yu. E. Lozovik, Fiz. Tverd. Tela (Leningrad) **25**, 1519 (1983) [Sov. Phys. Solid State **25**, 874 (1983)]; Fiz. Tverd. Tela (Leningrad) **26**, 1540 (1984) [Sov. Phys. Solid State **26**, 938 (1984)].

[17] A. B. Dzyubenko and Yu. E. Lozovik, J. Phys. A **24**, 415 (1991).

[18] É. I. Rashba and V. I. Sheka, in *Solid State Physics* [in Russian], A. F. Ioffe (ed.), Akad. Nauk SSSR (1959), Vol. 2, p. 162.

[19] T. Ando, A. B. Fowler, and F. Stern, "Electronic properties of two-dimensional systems," Rev. Mod. Phys. **54**, 437–672 (1982).

[20] G. E. Pikus and A. N. Titov, *Optical Orientation*, North-Holland, Amsterdam (1984), p. 73.

[21] F. J. Ohkawa and Y. Uemura, J. Phys. Soc. Jpn. **37**, 1325 (1974).

[22] Yu. A. Bychkov and É. I. Rashba, JETP Lett. **39**, 78 (1984).

[23] G. Dresselhaus, Phys. Rev. **100**, 580 (1955).

[24] M. I. D'yakonov and V. Yu. Kachorovskiĭ, Fiz. Tekh. Poluprovodn. **20**, 178 (1986) [Sov. Phys. Semicond. **20**, 110 (1986)].

[25] Yu. Bychkov and S. Iordansky, in *High Magnetic Fields in Semiconductor Physics II*, G. Landwehr (ed.), Springer-Verlag, New York (1989), p. 85.

[26] S. V. Iordanskiĭ and B. A. Muzykantskiî, Zh. Éksp. Teor. Fiz. **96**, 1783 (1989) [Sov. Phys. JETP **69**, 1006 (1989)].

[27] V. F. Gantmakher and I. B. Levinson, *Carrier Scattering in Metals and Semiconductors*, Wiley, New York, (1987).

[28] Sh. M. Kogan, Fiz. Tverd. Tela (Leningrad) **4**, 2474 (1962) [Sov. Phys. Solid State **4**, 1813 (1962)].

[29] *Landolt-Börnstein, Vol. 17a*, Springer-Verlag, Berlin (1982); *Landolt-Börnstein, Vol. 22a*, Springer-Verlag, Berlin (1987).

[30] G. Arlt and P. Quadflieg, Phys. Status Solidi **25**, 323 (1968).

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