Nonlinear dynamics of short wave trains in dispersive media

E. M. Gromov and V. I. Talanov

Institute of Applied Physics, Russian Academy of Sciences, 603600 Nizhniĭ Novgorod, Russia

We study the nonlinear dynamics of short wave trains (two to three wavelengths long) in dispersive media in the higher-order approximations of dispersion theory. New classes of "soliton" phase-modulated solutions of the equations of the theory are found. For time-dependent wave packets we observe effects not present in the parabolic approximation, such as the dependence of the packet's velocity on its intensity (nonlinear dispersion) and length (linear aberration). We also study the dynamics of short high-intensity wave trains and the modulation instability of plane electromagnetic and Langmuir waves in an isotropic plasma. Finally, we show that when the wave's amplitude exceeds a certain critical value, which depends on the nonlinear dispersion, the instability disappears. © *1996 American Institute of Physics.* [S1063-7761(96)01107-9]

1. INTRODUCTION

The propagation of high-frequency wave packets

$$\varphi = \psi(x,t)\exp(i\omega_0 t - ik_0 x)$$

in nonlinear dispersive media is usually analyzed in the quasi-optical approximation of nonlinear dispersion theory,[1] which corresponds to using the "parabolic" approximation of the dispersion law, $\omega = \omega(k, |\psi|^2)$, near the point $\omega_0 = \omega(k_0, 0)$ and to allowing for the local part of the nonlinearity in an additive manner. The envelope $\psi(x,t)$ in this approximation in the first order in intensity is described by the well-known nonlinear Schrödinger equation

$$2i\frac{\partial \psi}{\partial t} - \omega''_{kk}\frac{\partial^2 \psi}{\partial \xi^2} + 2\alpha|\psi|^2\psi = 0, \tag{1}$$

where $\xi = x - V_g^L t$, with $V_g^L = (\partial\omega/\partial k)|_{k=k_0, |\psi|=0}$ the group velocity of linear waves, $\omega''_{kk} = \partial^2\omega/\partial k^2|_{k=k_0, |\psi|=0}$, and $\alpha = (\partial\omega/\partial|\psi|^2)|_{k=k_0}$ is a nonlinear parameter. This equation provides a correct description of the evolution of extended wave packets whose temporal and spatial spectra are narrow enough, i.e.,

$$\frac{\Delta\omega}{\omega} \sim \frac{\Delta k}{k} \sim |\psi| \sim \nu \ll 1,$$

and has been thoroughly studied.[2–5] For wave packets several wavelengths long ($\Delta \sim (2-3)\lambda$), the spectrum is not narrow, which requires allowing for higher-order terms in Eq. (1) and, as a result, dropping the quasi-optical approximation. Up to now this procedure has basically been limited to allowing for nonlinear dispersion terms that lead to the formation of envelope shock waves. Ostrovskiĭ[6] was the first to discuss the possibility of formation of such waves. In optics this effect has been studied both experimentally[7] and theoretically[8] in nonlinear geometrical optics and has become known as the self-steepening effect. For deep-water gravitational waves, Dysthe[9] derived an equation for the packet's envelope in higher-order approximations of nonlinear dispersion theory, which was used to study the modulation instability of a plane wave. In particular, it was found

that the increments diminish and the ranges of parameters corresponding to the instability are narrower than those in the parabolic approximation.

In the present paper we analyze the general properties of the solutions of the equations for the envelope of a one-dimensional wave packet in higher-order approximations of nonlinear dispersion theory. New classes of steady-state non-linear localized waves (solitons) of the given equations are discovered. We study the dynamics of time-dependent wave packets. We observe effects that are not present in the quasi-optical approximation, such as the dependence of the packet's velocity on its intensity (nonlinear dispersion) and length (linear aberration) and the acceleration of packets in the absence of an inhomogeneous potential. We study the evolution of packets of electromagnetic and Langmuir wave in an isotropic plasma. We also obtain equations for the packet's envelope in the higher-order approximations of nonlinear dispersion theory. Within these approximations we establish the presence of solitons of the electromagnetic and Langmuir waves. Finally, we study the modulation instability of plane waves. The instability is shown to disappear when the wave amplitude exceeds a certain critical value determined by nonlinear dispersion.

2. THE GENERAL PROPERTIES OF HIGHER-ORDER APPROXIMATIONS

Allowing for the possible combinations of higher-order terms, the equation for ψ in a reference frame moving with a velocity equal to the group velocity of the linear waves can be written as

$$2i\frac{\partial \psi}{\partial t} + q\frac{\partial^2 \psi}{\partial \xi^2} + \alpha|\psi|^2\psi$$

$$= -2i\left(\beta|\psi|^2\frac{\partial \psi}{\partial \xi} + \mu\psi\frac{\partial|\psi|^2}{\partial \xi}\right) - i\gamma\frac{\partial^3 \psi}{\partial \xi^3} - \psi U(|\psi|^2). \tag{2}$$

The left-hand side of Eq (2) contains terms and the functional U of second order in the small quantity
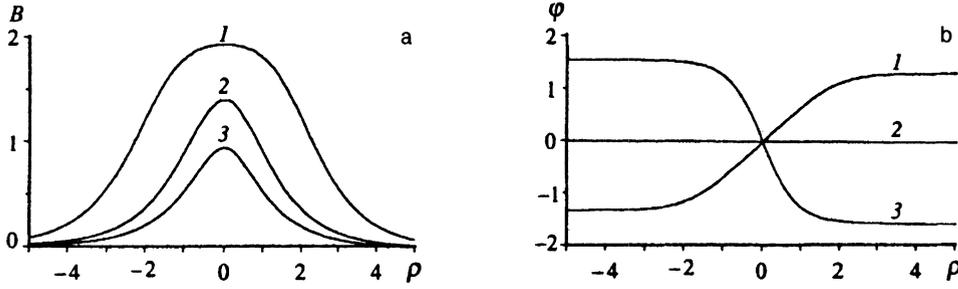
FIG. 1. The dependence of (a) the envelope B of a soliton and (b) its phase φ on the coordinate ρ for different values of r: 1), $r= -2990/16000$; 2), $r=0$; 3), $r=2$.

$$v=\left|\frac{\psi}{\partial\psi/\partial\xi}\right|^{-1}\sim|\psi|.$$

We examine the role of higher-order terms on the right-hand-side of Eq. (2) by using examples of solutions of this equation in the form of steady-state localized nonlinear waves (solitons) and the dynamics of time-dependent wave packets.

2.1. "Soliton" solutions

Allowing for higher-order terms in the equation for the packet's envelope leads to a modification of the well-known "soliton" solutions and to the emergence of new "soliton" solutions. We illustrate this with a modified nonlinear Schrödinger equation containing nonlinear dispersion terms:

$$2i\left(\frac{\partial\psi}{\partial t}+\beta|\psi|^2\frac{\partial\psi}{\partial\xi}+\mu\psi\frac{\partial|\psi|^2}{\partial\xi}\right)+\frac{\partial^2\psi}{\partial\xi^2}+|\psi|^2\psi=0. \quad (3)$$

Equation (3) is a special case of a more general nonlinear equation:

$$2i\left(\frac{\partial\psi}{\partial t}+\beta|\psi|^2\frac{\partial\psi}{\partial\xi}+\mu\psi\frac{\partial|\psi|^2}{\partial\xi}\right)$$
$$+q\frac{\partial^2\psi}{\partial\xi^2}+i\gamma\frac{\partial^3\psi}{\partial\xi^3}+\alpha|\psi|^2\psi=0. \quad (4)$$

For $\beta=\gamma=\mu=0$ Eq. (4) becomes the well-known nonlinear Schrödinger equation;[3-5] for $\alpha=q=0$ and a real function ψ Eq. (4) becomes the modified Korteweg–de Vries equation studied in Refs. 10–13. Equation (4) with $\mu=0$ and $2q\beta=3\gamma\alpha$ was studied by Hirota.[14] All these cases (the nonlinear Schrödinger equation, the modified Korteweg–de Vries equation, and the Hirota equation) have been analyzed by the inverse scattering method,[5] and exact N-soliton solutions have been found. In Eq. (3) we have $2q\beta\neq3\gamma\alpha$ and $\mu\neq0$, i.e., Eq. (3) cannot be reduced to any of the above-mentioned equations. At the same time, Eq. (3) has a solution in the form of steady-state localized waves (solitons) moving with a velocity that is generally not equal to the group velocity of linear waves. To demonstrate this fact, we write the solution of (3) in the form of a steady-state wave:

$$\psi(\xi,t)=A(\xi-Vt)\exp\{i\Omega t+i\varphi(\xi-Vt)\}. \quad (5)$$

As a result we arrive at a system of two equations for the amplitude A and phase φ of the packet:

$$A\frac{d^2\varphi}{d\zeta^2}+2\frac{d\varphi}{d\zeta}\frac{dA}{d\zeta}+2(A^2\Theta-V)\frac{dA}{d\zeta}=0, \quad (6)$$

$$\frac{d^2A}{d\zeta^2}-A\left(\frac{d\varphi}{d\zeta}\right)^2+A^3-2A\Omega+2A\frac{d\varphi}{d\zeta}(V-\beta A^2)=0, \quad (7)$$

where $\zeta=\xi-Vt$, and $\Theta=\beta+2\mu$. Integrating (6), we obtain the following relationship for the packet's phase φ:

$$\frac{d\varphi}{d\zeta}=-\frac{1}{2}A^2\Theta+V. \quad (8)$$

Substituting (8) into (7) yields an equation for the packet's envelope A:

$$\frac{d^2A}{d\zeta^2}=(2\Omega-V^2)A-A^3(1-2\beta V)-\Theta\left(\beta-\frac{\Theta}{4}\right)A^5. \quad (9)$$

By introducing the variables

$$\rho=\zeta\sqrt{2\Omega-V^2}, \quad B=A\sqrt{\frac{1-2\beta V}{2\Omega-V^2}} \quad (10)$$

we can reduce Eq. (9) to

$$\frac{d^2B}{d\rho^2}=B-B^3-rB^5, \quad (11)$$

with the single parameter

$$r=\Theta\left(\beta-\frac{\Theta}{4}\right)\frac{2\Omega-V^2}{(1-2\beta V)^2},$$

which allows for nonlinear dispersion. The "soliton" solution of (11) has the form

$$B^2(\rho)=\frac{4}{1+\sqrt{1+16r/3}\,\cosh(2\rho)}. \quad (12)$$

As (12) implies, solitons exist if $r> -3/16$. Note that in the presence of nonlinear dispersion, which tends to shift the peak to the leading or trailing edge of the pulse, the soliton retains its symmetric shape: the nonlinear shift in the peak is balanced by the corresponding phase modulation (8) of the packet. The dependence of the envelope B of a soliton and the soliton's phase φ on the coordinate ρ for different values of r is depicted in Fig. 1: (a) corresponds to the B vs r dependence, and (b) to the φ vs r dependence. Curves 1, 2, and 3 correspond to $r= -2990/16000$, $r=0$, and $r=2$. The soliton's amplitude $B_0=B(0)$ and length Δ increase with r. In the limiting case $r=0$, which corresponds to the absence of nonlinear dispersion and $V=0$, the relationships for the amplitude and phase correspond to the well-known solitons of the nonlinear Schrödinger equation.

The general nonlinear equation (4) also has a solution in the form of steady-state nonlinear waves whose velocity differs from that of linear waves. To demonstrate this feature, in Eq. (4) we go to a reference frame moving with velocity V, i.e., $\varsigma = \xi - Vt$ and $t' = t$, and write the solution of the new equation in the following form (we drop the prime in what follows):

$$\psi(\varsigma,t) = A(\varsigma)\exp[i\Omega t + i\varphi(\varsigma)]. \tag{13}$$

For the amplitude $A(\varsigma)$ and phase $\varphi(\varsigma)$ we obtain

$$\gamma\frac{d^3A}{d\varsigma^3} + 2(\beta+2\mu)A^2\frac{dA}{d\varsigma} + \left[2q\frac{d\varphi}{d\varsigma} - 2V - 3\gamma\left(\frac{d\varphi}{d\varsigma}\right)^2\right]\frac{dA}{d\varsigma}$$
$$+ \left(q - 3\gamma\frac{d\varphi}{d\varsigma}\right)\frac{d^2\varphi}{d\varsigma^2}A = 0, \tag{14}$$

$$\left(q - 3\gamma\frac{d\varphi}{d\varsigma}\right)\frac{d^2A}{d\varsigma^2} + \left(\alpha - 2\beta\frac{d\varphi}{d\varsigma}\right)A^3 - 3\gamma\frac{d^2\varphi}{d\varsigma^2}\frac{dA}{d\varsigma}$$
$$+ \left[\gamma\left(\frac{d\varphi}{d\varsigma}\right)^3 - q\left(\frac{d\varphi}{d\varsigma}\right)^2 + 2V\frac{d\varphi}{d\varsigma} - 2\Omega - \gamma\frac{d^3\varphi}{d\varsigma^3}\right]A = 0. \tag{15}$$

Here we are interested in steady-state waves with linear phase modulation $d\varphi/d\varsigma = k = \text{const}$. In this case, integrating Eq. (14) with respect to ς with vanishing boundary conditions at infinity, $A(\varsigma\to+\infty)\to 0$ and $A''(\varsigma\to +\infty)\to 0$, we arrive at

$$\frac{d^2A}{d\varsigma^2} + \frac{2(\beta+2\mu)}{3\gamma}A^3 - \frac{2(qk-V)-3\gamma k^2}{\gamma}A = 0, \tag{16}$$

$$\frac{d^2A}{d\varsigma^2} + \frac{\alpha-2\beta k}{q-3\gamma k}A^3 - \frac{2Vk-qk^2+\gamma k^3-2\Omega}{q-3\gamma k}A = 0. \tag{17}$$

Equations (16) and (17) form a consistent system of equations if

$$\frac{2(\beta+2\mu)}{3\gamma} = \frac{\alpha-2\beta k}{q-3\gamma k}, \tag{18}$$

$$\frac{2(qk-V)-3\gamma k^2}{\gamma} = \frac{2Vk-qk^2+\gamma k^3-2\Omega}{q-3\gamma k} \tag{19}$$

and have a solution in the form of a single soliton with linear phase modulation $d\varphi/d\varsigma = k = \text{const}$, whose parameter k can be found from Eq. (18):

$$k = \frac{2q(\beta+2\mu)-3\alpha\gamma}{12\mu\gamma}. \tag{20}$$

This implies that by proper selection of the phase modulation parameter k we can make Eq. (4) have a solution in the form of a soliton for arbitrary values of the parameters in this equation. For fixed phase modulation Eq. (19) determines the solition velocity:

$$V = \frac{\Omega\gamma + k(q-2\gamma k)^2}{2(q-2\gamma k)}. \tag{21}$$

The equation for the amplitude A can then be written as

$$\frac{d^2A}{d\varsigma^2} + \frac{2(\beta+2\mu)}{3\gamma}A^3 - \left(\frac{2\Omega}{q-2\gamma k} - k^2\right)A = 0,$$

which has the soliton solution

$$A^2(\varsigma) = \frac{2(q-3\gamma k)[2\Omega/(q-2\gamma k)-k^2]}{(\alpha-2\beta k)\cosh^2[\varsigma\sqrt{2\Omega/(q-2\gamma k)-k^2}]}.$$

This solution exists if

$$\frac{\Omega}{q-2\gamma k} - k^2 > 0, \quad \gamma(\beta+2\mu) > 0.$$

This "soliton" solution can be reduced in special cases to solitons of the nonlinear Schrödinger equation and Hirota solitons. In particular, in the absence of modulation ($k=0$), the relationships (20) and (21) assume a simpler form:

$$2q(\beta+2\mu) = 3\alpha\gamma, \quad V = \gamma\Omega/q \tag{22}$$

(here the velocity V depends on the linear aberration, $V\propto\gamma$), and at $\mu=0$ they are simply the existence condition for solitons in the Hirota equation (4).

2.2. Time-dependent wave packets

Allowing for higher-order terms modifies the dynamics of time-dependent wave packets in comparison to the parabolic approximation. For instance, the first two terms in Eq. (2) correspond to the dependence of the group velocity of the waves on their intensity $|\psi|^2$ (nonlinear dispersion). To illustrate, we put $q = \alpha = \gamma = U = 0$ and reduce the remaining equation to the form

$$\frac{\partial|\psi|^2}{\partial t} + (\beta+2\mu)|\psi|^2\frac{\partial|\psi|^2}{\partial\xi} = 0.$$

Clearly, the sections of the packet with different intensities move with different group velocities:

$$\frac{d\xi}{dt} = \Delta V_g^{NL} = (\beta+2\mu)|\psi|^2. \tag{23}$$

For $\beta+2\mu > 0$ the sections of the packet with higher intensities have higher velocities. This leads to an increase in the steepness of the leading edge of an initially symmetric packet. In the opposite case, $\beta+2\mu < 0$, the sections of the packet with higher intensities have lower velocities, which leads to an increase in the steepness of the trailing edge of an initially symmetric packet.

For subsequent analysis of the role that higher-order approximations play in Eq. (2), we find the velocity of the "center of gravity" of the packet in a reference frame that moves at the velocity of linear waves V_g^L:

$$\overline{\xi(t)} = \frac{1}{N_0}\int_{-\infty}^{\infty}\xi|\psi|^2 d\xi, \tag{24}$$

where

$$N_0 = \int_{-\infty}^{\infty}|\psi|^2 d\xi$$

is the energy of the wave field in the packet. We multiply both sides of Eq. (2) by $\xi\psi^*$, where ψ^* is the complex

conjugate of the field ψ, and add it to the complex conjugate product. Integrating this sum with respect to ξ from $-\infty$ to ξ, for spatially localized wave packets and a real functional U ($U=U^*$) we find that

$$\dot{\xi}(t)=\frac{(\beta+2\mu)}{2N_0}\int_{-\infty}^{\infty}|\psi|^4d\xi-\frac{3\gamma}{2N_0}\int_{-\infty}^{\infty}\left|\frac{\partial\psi}{\partial\xi}\right|^2d\xi$$

$$+\frac{q}{N_0}\int_{-\infty}^{\infty}\frac{\partial\varphi}{\partial\xi}\left|\psi\right|^2d\xi, \tag{25}$$

where the bullet denotes d/dt, and φ is the packet's phase ($\psi=|\psi|\exp(i\varphi)$. If we compare the variation of the packet's velocity, $\Delta V=\dot{\xi}(t)$, with that given by the standard nonlinear Schrödinger equation (the last term on the right-hand side of Eq. (25)), we see that it is described by two terms: $\overline{\Delta V}=\overline{\Delta V^{NL}}+\overline{\Delta V^L}$. The first, $\overline{\Delta V^{NL}}$, depends on the field intensity, and is due to the nonlinear dispersion mentioned above:

$$\overline{\Delta V^{NL}}=\frac{\beta+2\mu}{2N_0}\int_{-\infty}^{\infty}|\psi|^4d\xi. \tag{26}$$

The second, $\overline{\Delta V^L}$ on the right-hand side of Eq. (25), does not depend on the field intensity and corresponds to the term with the third derivative in Eq. (2) ($\gamma\neq0$):

$$\overline{\Delta V^L}\equiv\frac{3\gamma}{2N_0}\int_{-\infty}^{\infty}\left|\frac{\partial\psi}{\partial\xi}\right|^2d\xi. \tag{27}$$

The aberration correction to the velocity is negative ($\overline{\Delta V^L}<0$) for $\gamma>0$ and positive ($\overline{\Delta V^L}>0$) for $\gamma<0$. The absolute value of this correction, $|\overline{\Delta V^L}|$, increases as the packet gets shorter, i.e., as the packet's spectrum broadens.

Allowing for higher-order approximations not only alters the velocity of packet motion in comparison to the parabolic approximation but also introduces additional acceleration. For instance, the acceleration of the "center of gravity" of the packet of Eq. (2) is

$$\ddot{\xi}(t)=\frac{3\alpha\gamma-2q\beta}{2N_0}\int_{-\infty}^{\infty}\frac{\partial^2\varphi}{\partial\xi^2}|\psi|^4d\xi$$

$$+\frac{1}{2N_0}\int_{-\infty}^{\infty}\frac{\partial U}{\partial\xi}|\psi|^2\left(q-3\gamma\frac{\partial\varphi}{\partial\xi}\right)d\xi. \tag{28}$$

The packet's acceleration $\overline{\Delta a}=\ddot{\xi}(t)$ contains two terms: $\overline{\Delta a}=\overline{\Delta a_h}+\overline{\Delta a_{inh}}$. The first, $\overline{\Delta a_h}$, on the right-hand side of Eq. (28) depends on the field intensity and is not related to the inhomogeneous functional U. It emerges in the event of phase modulation with allowance for nonlinear dispersion and linear aberration. The second, $\overline{\Delta a_{inh}}$, is due to the presence of the nonlinear functional U and contains two terms:

$$\overline{\Delta a_{inh}}=\frac{1}{2N_0}\int_{-\infty}^{\infty}\frac{\partial U}{\partial\xi}|\psi|^2\left(q-3\gamma\frac{\partial\varphi}{\partial\xi}\right)d\xi. \tag{29}$$

The first corresponds to the classical acceleration of packets when the motion is described by the Schrödinger equation in an inhomogeneous medium with the potential U:

$$\overline{\Delta a_{clas}}=\frac{q}{2N_0}\int_{-\infty}^{\infty}\frac{\partial U}{\partial\xi}|\psi|^2d\xi. \tag{30}$$

The second corresponds to the acceleration of phase-modulated packets in the presence of an inhomogeneous functional U and aberration ($\gamma\neq0$). Note that in the event of linear phase modulation, $\partial\varphi/\partial\xi=q/3\gamma$, the acceleration of the wave packets is zero even in the presence of an inhomogeneous potential, i.e., phase-modulated packets can move in an inhomogeneous medium at constant velocity.

To examine the behavior of short trains of high-intensity waves we examine the evolution of one-dimensional packets of a strong high-frequency field of linearly polarized electromagnetic or Langmuir plasma waves in a homogeneous plasma with ponderomotive nonlinearity.

3. HIGHER-ORDER APPROXIMATIONS IN THE DESCRIPTION OF SHORT INTENSE TRAINS OF HIGH-FREQUENCY WAVES IN AN ISOTROPIC PLASMA

3.1. The basic equations

We start with the following system of equations:[4]

$$V^2\frac{\partial^2E}{\partial z^2}-\frac{\partial^2E}{\partial t^2}-\omega_p^2\left(1+\frac{\delta n}{N}\right)E=0, \tag{31}$$

$$c_s^2\frac{\partial^2\delta n}{\partial t^2}-\frac{\partial^2\delta n}{\partial z^2}=\frac{c_s^2}{16\pi T_e}\frac{\partial^2\widetilde{E^2}}{\partial z^2}, \tag{32}$$

where E is the electric field strength, δn is the deviation of the plasma concentration from the equilibrium value N, c_s is the velocity of ion-acoustic waves, $\omega_p^2=4\pi e^2N/m_e$, the parameter V in Eq. (31) corresponds to the speed of light for electromagnetic waves and the thermal electron velocity V_{T_e} for Langmuir waves. The wavy line on the right-hand side of Eq. (32) stands for time-averaging of the high-frequency field. Next we introduce dimensionless variables:

$$\tau=\frac{t}{T}, \quad \eta=\frac{z}{l}, \quad \psi=\frac{E}{\bar{E}}, \tag{33}$$

where

$$\omega_pT=1, \quad l=\frac{V}{\omega_p}, \quad \bar{E}=\sqrt{16\pi NT_e}.$$

In terms of these variables the systems of equations (31) and (32) assumes the form

$$\frac{\partial^2\psi}{\partial\eta^2}-\frac{\partial^2\psi}{\partial\tau^2}-(1+n)\psi=0, \tag{34}$$

$$\alpha^2\frac{\partial^2n}{\partial\tau^2}-\frac{\partial^2n}{\partial\eta^2}=\frac{\partial^2\widetilde{\psi^2}}{\partial\eta^2}, \tag{35}$$

where $\alpha=V/c_s\gg1$. Let us write the solution of Eqs. (34) and (35) in the form of a quasi−monochromatic wave packet:

$$\psi(\eta,\tau)=\varphi(\eta,\tau)\exp(-i\omega_0\tau+ik_0\eta), \tag{36}$$

where ω_0 and k_0 are related through the linear dispersion relation $\omega_0^2=1+k_0^2$. In the above relationships we go to the

reference frame moving with the group velocity of the linear high-frequency waves $v \equiv V_g^L = \partial\omega_0/\partial k_0 = k_0/\omega_0$:

$$\xi = \eta - v\tau, \quad \tau' = \tau.$$

In this case we arrive at the following equations for $\varphi(\xi,\tau)$ and $n(\xi,\tau)$:

$$2i\omega_0\frac{\partial\varphi}{\partial\tau} + \frac{1}{\omega_0^2}\frac{\partial^2\varphi}{\partial\xi^2} - n\varphi = \frac{\partial^2\varphi}{\partial\tau^2} - 2v\frac{\partial^2\varphi}{\partial\tau\partial\xi}, \qquad (37)$$

$$(v^2\alpha^2 - 1)\frac{\partial^2 n}{\partial\xi^2} - \frac{\partial^2|\varphi|^2}{\partial\xi^2} = -\alpha^2\left(\frac{\partial^2 n}{\partial\tau^2} - 2v\frac{\partial^2 n}{\partial\tau\partial\xi}\right). \quad (38)$$

The left-hand sides of Eqs. (37) and (38) correspond to the approximation in dispersion theory that leads to the nonlinear Schrödinger equation for the envelope φ of the high-frequency field packet:

$$2i\omega_0\frac{\partial\varphi}{\partial\tau} + \frac{1}{\omega_0^2}\frac{\partial^2\varphi}{\partial\xi^2} - n\varphi = 0, \qquad (39)$$

where

$$n = -\frac{|\varphi|^2}{1 - v^2\alpha^2} \qquad (40)$$

is the variation of the plasma concentration of the field of a steady-state wave. To obtain the next approximation, we substitute the derivative $\partial\varphi/\partial\tau$ from (39) and n from (40) into the right-hand sides of Eqs. (37) and (38) and identify the leading terms. We then have

$$2i\omega_0\frac{\partial\varphi}{\partial\tau} + \frac{1}{\omega_0^2}\frac{\partial^2\varphi}{\partial\xi^2} - n\varphi$$

$$= -i\frac{v}{\omega_0^3}\frac{\partial^3\varphi}{\partial\xi^3} + \frac{v|\varphi|^2}{2\omega_0^2(1 - v^2\alpha^2)}\frac{\partial\varphi}{\partial\xi}$$

$$+ \frac{v}{2\omega_0}\left(\frac{\varphi}{1 - v^2\alpha^2}\frac{\partial|\varphi|^2}{\partial\xi} + \frac{|\varphi|^2}{1 - v^2\alpha^2}\frac{\partial\varphi}{\partial\xi}\right), \qquad (41)$$

$$n = -\frac{|\varphi|^2}{1 - v^2\alpha^2} - \frac{2\alpha^2 v}{2\omega_0^2(1 - v^2\alpha^2)^2}\frac{1}{2i}\left(\varphi^*\frac{\partial\varphi}{\partial\xi} - \varphi\frac{\partial\varphi^*}{\partial\xi}\right), \qquad (42)$$

where now the variation of concentration n has a correction to the steady-state value. The terms on the right-hand side of Eq. (41) and the correction to the steady-state value of concentration variation in (42) are of the third order in

$$\nu = \left|\frac{\varphi}{\partial\varphi/\partial\xi}\right| \sim |\varphi|.$$

Combining (41) and (42), we arrive at an equation for the third approximation to φ:

$$2i\omega_0\frac{\partial\varphi}{\partial\tau} + \frac{1}{\omega_0^2}\frac{\partial^2\varphi}{\partial\xi^2} + \frac{|\varphi|^2}{1 - v^2\alpha^2}\varphi$$

$$= -2i\omega_0\left(\beta|\varphi|^2\frac{\partial\varphi}{\partial\xi} + \mu\varphi\frac{\partial|\varphi|^2}{\partial\xi}\right) - i\frac{v}{\omega_0^3}\frac{\partial^3\varphi}{\partial\xi^3}, \qquad (43)$$

where on the right-hand side we have written the terms of the third approximation with the parameters

$$\beta = \frac{v}{2\omega_0^2(1 - v^2\alpha^2)}\left[1 - \frac{2\alpha^2}{\omega_0^2(1 - v^2\alpha^2)}\right], \qquad (44)$$

$$\mu = \frac{v}{2\omega_0^2(1 - v^2\alpha^2)}\left[1 + \frac{\alpha^2}{\omega_0^2(1 - v^2\alpha^2)}\right]. \qquad (45)$$

According to (23), the nonlinear correction to the group velocity amounts to

$$\Delta v_g^{NL} = \frac{3v|\varphi|^2}{2\omega_0^2(1 - v^2\alpha^2)}. \qquad (46)$$

For $\alpha v < 1$, which corresponds to subsonic motion of the packet, $\Delta v_g^{NL} > 0$. In this case the regions in the packet with larger amplitudes move faster and the leading edge of the packet becomes steeper. For $\alpha v > 1$ (supersonic motion), $\Delta v_g^{NL} < 0$, and the trailing edge becomes steeper. The characteristic "steepness" time t^*, defined as the time it takes the packet's peak to shift by the packet's halfwidth $\Delta/2$, is given by the following formula:

$$t^* = \frac{\Delta}{2\Delta v_g^{NL}} = \frac{\omega_0^2\Delta|1 - v^2\alpha^2|}{3v|\varphi|^2}. \qquad (47)$$

The distance L^* at which a shock wave is formed from the initially symmetric pulse is

$$L^* = t^* v = \frac{\Delta\omega_0^2|1 - v^2\alpha^2|}{3|\varphi|^2}. \qquad (48)$$

The above relationships show that the correction of the third approximation are important near the group synchronism of high- and low-frequency waves, $\alpha v \sim 1$.

3.2. Solitons of high-frequency waves

To illustrate the manifestation of third-order effects, we take solitons of intense high-frequency plane waves in the subsonic mode, $\alpha v < 1$. If in (43) we go to the dimensionless variables

$$\theta = \frac{\tau}{\tau_0}, \quad s = \frac{\xi}{\delta}, \quad \phi = \frac{\varphi}{\bar{\varphi}}, \qquad (49)$$

where

$$\delta = \frac{v}{2\omega_0}, \quad \tau_0 = \frac{v^2}{4}, \quad \bar{\varphi} = \frac{4\omega_0}{v^2}\sqrt{1 - v^2\alpha^2},$$

we have

$$2i\left(\frac{\partial\phi}{\partial\theta} + \beta_1|\phi|^2\frac{\partial\phi}{\partial s} + \mu_1\phi\frac{\partial|\phi|^2}{\partial s}\right)$$

$$+ \frac{\partial^2\phi}{\partial s^2} + |\phi|^2\phi + 2i\frac{\partial^3\phi}{\partial s^3} = 0, \qquad (50)$$

where

$$\beta_1 = 1 - \frac{2\alpha^2}{\omega_0^2(1 - v^2\alpha^2)}, \quad \mu_1 = 1 + \frac{\alpha^2}{\omega_0^2(1 - v^2\alpha^2)}.$$

We write the solution of Eq. (50) in the form of a nonlinear steady-state wave with linear phase modulation, propagating at velocity V:

$$\phi(\rho,\theta) = A(\rho)\exp(i\Omega\theta + ik\rho),\tag{51}$$

where $\rho = s - V\theta$, and k is the phase modulation parameter. The relationships (20) and (21) determining the conditions for the existence of localized nonlinear waves (solitons) assume the following form for Eq. (50):

$$k \equiv 0, \quad V = \gamma\Omega/2q \equiv \Omega.\tag{52}$$

In this case the corresponding "soliton" solution has the form

$$\phi(\rho,\theta) = \frac{A_0}{\cosh(A_0\rho/\sqrt{2})}\exp\left(i\frac{A_0^2}{4}\theta\right).\tag{53}$$

In contrast to the case of the well-known "parabolic approximation" solitons, the velocity of solitons of higher-order approximations differs from that of linear high-frequency waves and depends on the soliton amplitude:

$$V = A_0^2/4.\tag{54}$$

3.3. Instability of plane waves

For further analysis of the role of the third approximation we examine the instability of an intensive plane wave. Let us write the solution of Eq. (50) in the form of a plane wave with a weak perturbation:

$$\phi = (\phi_0 + \widetilde{\phi})\exp(i\phi_0^2\theta/2),\tag{55}$$

where $|\widetilde{\phi}| \ll \phi_0$. Retaining in this relationship only terms linear in $\widetilde{\phi}$, we have

$$2i\left[\frac{\partial\widetilde{\phi}}{\partial\theta} + (\beta_1 + \mu_1)\phi_0^2\frac{\partial\widetilde{\phi}}{\partial s} + \mu_1\phi_0^2\frac{\partial\widetilde{\phi^*}}{\partial s}\right]$$
$$+ \frac{\partial^2\widetilde{\phi}}{\partial s^2} + \phi_0^2(\widetilde{\phi} + \widetilde{\phi^*}) + 2i\frac{\partial^3\widetilde{\phi}}{\partial s^3} = 0.\tag{56}$$

By introducing the variables $\rho = s - (\beta_1 + \mu_1)\phi_0^2\theta$ and $\theta' = \theta$ we can transform Eq. (56) (we drop the prime on θ in what follows) to

$$2i\left(\frac{\partial\widetilde{\phi}}{\partial\theta} + \mu_1\phi_0^2\frac{\partial\widetilde{\phi^*}}{\partial\rho}\right) + \frac{\partial^2\widetilde{\phi}}{\partial\rho^2}$$
$$+ \phi_0^2(\widetilde{\phi} + \widetilde{\phi^*}) + 2i\frac{\partial^3\widetilde{\phi}}{\partial\rho^3} = 0.\tag{57}$$

Writing the spatial perturbations in the form

$$\widetilde{\phi} = A\exp(\Gamma\theta + iq\rho),\tag{58}$$

and employing standard methods, we obtain

$$\Gamma = -iq^3 \pm \frac{1}{2}q\sqrt{2\phi_0^2(1 - 2\mu_1^2\phi_0^2) - q^2 - 4q^4}.\tag{59}$$

According to (59), the growth factor for modulation instability is

$$\text{Re}\,\Gamma = \frac{1}{2}q\sqrt{2\phi_0^2(1 - 2\mu_1^2\phi_0^2) - q^2 - 4q^4}.\tag{60}$$

Comparing this with the well-known expression for the modulation-instability factor that follows from the nonlinear Schrödinger equation,

$$(\text{Re}\,\Gamma)_{NSE} = \frac{1}{2}q\sqrt{2\phi_0^2 - q^2},\tag{61}$$

we can easily see that in third order there is stabilization of the instability due to nonlinear dispersion (the term with μ_1^2 in (60)) and due to the cubic linear aberration of the dispersion dependence $\omega(k)$ (the last term under the radical sign). At large amplitude,

$$\phi_0 > \frac{1}{\sqrt{2}\mu_1},$$

modulation instability disappears and stabilization ensues. In the initial variables for the amplitude of the electric field E, the relationship assumes the form

$$\frac{E}{\sqrt{16\pi NT_e}} > \frac{E_c}{\sqrt{16\pi NT_e}} = \frac{4\omega_0^3(1 - V_g^2/c_s^2)^{2/3}}{\omega_p^3(k_0l)^2[(1 - V_g^2/c_s^2) + V^2/c_s^2]}.$$

In particular, for Langmuir waves, when $V = V_{T_e}$ and $l = V_{T_e}/\omega_p = D$, for the value of the critical field we have

$$\frac{E_c}{\sqrt{16\pi NT_e}} = \frac{4\omega_0^3(1 - V_g^2/c_s^2)^{2/3}}{(k_0D)^2\omega_p^3[(1 - V_g^2/c_s^2) + V_{T_e}^2/c_s^2]}.$$

For low group velocities of Langmuir waves, $V_g \ll V_{T_e}$, we have $\omega_0 \simeq \omega_p$, and

$$\frac{E_c}{\sqrt{16\pi NT_e}} = \frac{4c_s^2(1 - V_g^2/c_s^2)^{2/3}}{V_{T_e}^2(k_0D)^2}.\tag{62}$$

This implies that modulation instability may be stabilized at low values of the electric field amplitude: $E_c \ll \sqrt{16\pi NT_e}$. A similar stabilization effect is known to exist for gravitational waves on the surface of a homogeneous deep liquid, where it emerges if one takes into account approximations that lie outside the scope of the standard nonlinear Schrödinger equation.[15]

Thus, the propagation of short trains of intense waves in highly dispersive nonlinear media can be described if one takes into account the approximations that follow the parabolic approximation in the theory of linear and nonlinear dispersion of waves.

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