Motion in the vicinity of the separatrix of a nonlinear resonance in the presence of high-frequency excitations

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Motion in the chaotic layer of a nonlinear resonance in the presence of weak high-frequency forces is studied both analytically and numerically. It is found that the secondary harmonics (among which there can generally be low-frequency harmonics) emerging even in second order in the small parameter at combinations of the primary frequencies may provide the leading contribution to the amplitude of the separatrix map of the system and to formation of the chaotic layer. © *1996 American Institute of Physics.* © *1996 American Institute of Physics.* [S1063-7761(96)02406-7]

1. INTRODUCTION

The interaction of nonlinear resonances and the related chaotic modes of dynamical Hamiltonian systems belong to the most important and complex problems of the modern theory of nonlinear oscillations.[1,2] Usually the initial states are chosen near one of the resonances, which is assumed to be the leading resonance, while the others are interpreted as perturbations. In some cases the problem is reduced to studying a dynamical model interpreted as a pendulum (the leading resonance) subjected to periodic forces:[3–5]

$$H(x,p,t)=p^2/2+\cos x+V(x,t), \tag{1}$$

$$V(x,t)=\varepsilon_n \cos(x-\Omega_n t), \quad n=1,2,\dots,N, \tag{2}$$

where N is the total number of the perturbing resonances, and there is summation over the index n. We think of $V(x,t)$ as a weak ($\varepsilon_n \ll 1$) and high-frequency ($\Omega_n \gg 1$) perturbation.

As is known, the phase space of a Hamiltonian system in which the number of degrees of freedom is greater than one is generally separated into regular and chaotic components.[1,2] The formation of the chaotic component, or dynamic chaos, was found to be related to the splitting of the separatrices, an effect qualitatively described in the 19th century by Henri Poincaré.[3,4] As shown by Mel'nikov's quantitative study,[6] in order of magnitude the separatrix splitting (and everything that depends on it) decreases faster than any power of the small perturbation parameter, with the result that the splitting cannot be discovered by applying the ordinary technique of a series expansion in this parameter. In recent years rigorous analytical estimates of this exponentially small effect have appeared, both for canonical maps[7,8] and for continuous Hamiltonian systems.[9]

Another characteristic of dynamic chaos important for applications is the full area of the unstable region in the vicinity of a disintegrated separatrix, which became known as the chaotic layer.[4] Mel'nikov's approach provides no means for estimating the width of this layer and its relation to the separatrix splitting. Today this problem has found its complete solution only for the case of a single-frequency perturbation in (1) on the basis of building what became known as a separatrix map. This map, first introduced by Zaslavskiĭ and Filonenko,[4] provides an approximate description of the dynamical state of the leading resonance of the system moving near a separatrix at the times when the system passes the points of stable equilibrium. If the perturbation of the system (1) contains only one harmonic of the form

$$V(x,t)=\varepsilon \cos(mx/2-\Omega t), \tag{3}$$

where m is a parameter, the amplitude W_{MA} of the separatrix map is related by the formula

$$W_{MA}(\Omega)=\varepsilon\Omega A_m(\Omega) \tag{4}$$

to the Mel'nikov–Arnol'd integrals[1]

$$A_m(\Omega>0)=\frac{2\pi}{(m-1)!}\frac{e^{\pi\Omega/2}}{\sinh(\pi\Omega)}(2\Omega)^{m-1}\times[1+f_m(\Omega)],$$
$$f_1=f_2=0, \quad f_{m+1}=f_m-(1+f_{m-1})m(m-1)/4\Omega^2. \tag{5}$$

By employing the resonance overlap criterion and the properties of a standard map it was found that here the chaotic instability occurs in a layer $|w|\lesssim w_s$ whose width is[1]

$$w_s\approx\Omega|W_{MA}(\Omega)|; \tag{6}$$

here and in what follows $w=\frac{1}{2}p^2+\cos x-1$ is the dimensionless deviation (in energy) from the unperturbed separatrix. Below it is shown that if the perturbation (2) of the system (1) contains several distinct frequencies ($N>1$), the mechanism for formation of the chaotic layer differs drastically from the respective mechanism in the single-frequency case.

The separatrix map of a multifrequency ($N>1$) system (1) with (2) can be written as

$$\bar{w}=w+W_l\sin(\Omega_l t_\pi), \quad \bar{t}_\pi=t_\pi+\ln\frac{32}{|\bar{w}|}, \tag{7}$$
$$l=1,2,\dots,L,$$

where t_π are the times when the system passes the points of stable equilibrium $x=\pi$, L is the total number of harmonics,

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and summation takes place over the index l. If the frequencies are incommensurable, the moments t_π are measured on the continuous time scale. If the frequencies are commensurable and are integral multiples of a reference frequency Ω_0, the second relationship in (7) can be represented in the following form:[1]

$$\bar{\tau}_\pi=\tau_\pi+\Omega_0 \ln\frac{32}{|\bar{w}|}, \quad \tau_\pi=\Omega_0 t_\pi \pmod{2\pi}.$$

The very first numerical experiments in measuring the amplitudes of the harmonics of map (7) showed that not only are all the N primary frequencies (those that the Hamiltonian $H(x,p,t)$ contains) present in the spectrum of the map, but so are some combinations of these frequencies, i.e., L is always larger than N.

Section 2 describes the mechanism by which the harmonics of the combination frequencies enter into the separatrix map and gives estimates of the amplitudes of these harmonics. In Sec. 3 the results are applied to the standard map. Section 4 is devoted to brief remarks on the calculation technique and gives some numerical results.

2. AMPLITUDES OF THE HARMONICS OF COMBINATION FREQUENCIES OF A SEPARATRIX MAP

In analyzing the motion in the chaotic layer of the leading resonance of the system (1), (2) it has proved convenient to introduce a new position variable and a new momentum variable describing the deviation of the phase variables $x(t)$ and $p(t)$ from their values on the unperturbed separatrix $x_s(t)$ and $p_s(t)$:

$$y(t)=x(t)-x_s(t), \quad u(t)=\dot{y}(t)=p(t)-p_s(t),$$

$$x_s(t)=r \arctan(e^t), \quad p_s(t)=2\sin(x_s(t)/2).$$

Introducing $F_2(u,x,t)=[p_s(t)+u][x-x_s(t)]$ as the generating function, we obtain the following expression for the Hamiltonian of the y-motion (summation over n is implied):

$$H(y,u,t)= u^2/2+\cos y[\cos x_s+\varepsilon_n \cos(x_s-\Omega_n t)] - \sin y[\sin x_s+\varepsilon_n \sin(x_s-\Omega_n t)]+y \sin x_s. \tag{8}$$

Assuming $|y(t)|\ll 1$ because of the weakness of the perturbation and performing the approximate substitutions $\sin y\to y$ and $\cos y\to 1-\frac{1}{2}y^2$ in the Hamiltonian (8), we arrive at the following equation of motion:

$$d^2y/dt^2 =y[\cos x_s+\varepsilon_n \cos(x_s-\Omega_n t)] +\varepsilon_n \sin(x_s-\Omega_n t). \tag{9}$$

Here we are interested only in the forced solution y_ε (vanishing as $\varepsilon\to 0$), which can be built by successive approximations. We introduce the following notation for the difference of the left- and right-hand sides of Eq. (9):

$$\Delta y_\varepsilon=d^2y/dt^2 -y[\cos x_s+\varepsilon_n \cos(x_s-\Omega_n t)] -\varepsilon_n \sin(x_s-\Omega_n t). \tag{10}$$

The approximation process must reduce this difference to zero.

For the first approximation we take the expression

$$y_\varepsilon^{(1)}=-\frac{\varepsilon_n}{(p_s-\Omega_n)^2}\sin(x_s-\Omega_n t),$$

and then calculate the difference (10) for it:

$$\Delta y_\varepsilon^{(1)}=-\frac{\varepsilon_n}{(p_s-\Omega_n)^2}\sin(\Omega_n t)-\frac{\varepsilon_n\varepsilon_m}{2}\times\left[\frac{1}{(p_s-\Omega_n)^2}-\frac{1}{(p_s-\Omega_m)^2}\right]\times\sin(\Omega_m-\Omega_n)t-\cdots.$$

We see that the terms of the type $\varepsilon_n \sin(x_s-\Omega_n t)$ have vanished from Δy_ε, but instead new terms have appeared. To compensate for these new terms we need the second approximation

$$y_\varepsilon^{(2)}=-\frac{\varepsilon_n}{(p_s-\Omega_n)^2}\sin(x_s-\Omega_n t)-\frac{\varepsilon_n}{\Omega_n^2(p_s-\Omega_n)^2}\times\sin(\Omega_n t)-\frac{\varepsilon_n\varepsilon_m}{2(\Omega_m-\Omega_n)^2}\left[\frac{1}{(p_s-\Omega_n)^2}-\frac{1}{(p_s-\Omega_m)^2}\right]\times\sin(\Omega_m-\Omega_n)t. \tag{11}$$

Knowing $y_\varepsilon^{(2)}$ makes it possible to calculate $\Delta y_\varepsilon^{(2)}$ and continue the approximation process.

Note that even if the initial system (1), (2) contains a fairly small number of terms, the number of terms in the expressions for $\Delta y_\varepsilon^{(1)}$, $y_\varepsilon^{(2)}$, $\Delta y_\varepsilon^{(2)}, \ldots$ proves to be large and snowballs as the approximation order increases. The solution $y_\varepsilon(t)$ can be constructed accurately, as a rule, only by applying the techniques of computer algebra which proves extremely cumbersome. The aim of this study, however, is not to find exact dependences but to obtain order-of-magnitude estimates. Hence we restrict our study to the second approximation (11) for $y_\varepsilon^{(2)}$.

Returning to the system (1), (2), we assume that the perturbation consists of only two high-frequency harmonics ($\Omega_1,\Omega_2\gg 1$) and consider the first harmonic, $\varepsilon_1 \cos(x-\Omega_1 t)$. In this expression, for the motion near a slightly deformed separatrix the phase variable x can be represented by the sum of the unperturbed motion and the deviation from such motion: $x\approx x_s+y_\varepsilon^{(2)}$. When combined with the approximate substitutions $\sin y\approx y_\varepsilon^{(2)}$ and $\cos y\approx 1$ and with the condition that both Ω_1 and Ω_2 are much larger than $p_{s,\max}\approx 2$, this representation yields

$$\varepsilon_1 \cos(x-\Omega_1 t) \approx\varepsilon_1 \cos(x_s-\Omega_1 t)+\varepsilon_1\varepsilon_2 \times\left[\frac{1}{\Omega_2^2}\sin(x_s-\Omega_2 t)+\frac{1}{\Omega_2^4}\sin(\Omega_2 t)+\cdots\right] \times\sin(x_s-\Omega_1 t).$$

Clearly, in addition to the primary harmonics with the frequencies Ω_1 and Ω_2, secondary harmonics with frequencies equal to the sum and difference of Ω_1 and Ω_2 appear in the perturbed system, with the latter contributing to the separatrix map spectrum and to the formation of the chaotic layer. Note that one of the secondary harmonics may prove to be low-frequency, with the result that it plays the principal role in forming the chaotic layer (see Sec. 4).

The reader will recall that in constructing a separatrix map one must calculate the variations of the system Hamiltonian over one period of rotation or half-period of oscillation, and the amplitude of each explicitly time-dependent harmonic is proportional (see Eq. (4)) to the corresponding value of the Mel'nikov–Arnol'd integral.[1] Employing this technique, we can find the amplitudes W of the separatrix map harmonics generated by the secondary harmonics of the perturbation for the following combinations $\Delta\Omega$ of the primary frequencies:

(1) Two primary frequencies Ω_1 and Ω_2 are much higher than unity, and the frequency of the secondary harmonic is $\Delta\Omega=\Omega_1+\Omega_2>0$:

$$W(\Delta\Omega)=\frac{4\pi}{3}\varepsilon_1\varepsilon_2\left[\frac{1}{\Omega_1^2}+\frac{1}{\Omega_2^2}\right]\frac{e^{\pi\Delta\Omega/2}}{\sinh(\pi\Delta\Omega)} \times\Delta\Omega^4\left(1-\frac{2}{\Delta\Omega^2}\right); \tag{12}$$

(2) Two primary frequencies Ω_1 and Ω_2 are much higher than unity, and the frequency of the secondary harmonic is $\Delta\Omega=\Omega_1-\Omega_2>0$:

$$W(\Delta\Omega)=\pi\varepsilon_1\varepsilon_2\left[\frac{1}{\Omega_1^2}-\frac{1}{\Omega_2^2}\right]\frac{e^{\pi\Delta\Omega/2}}{\sinh(\pi\Delta\Omega)}; \tag{13}$$

(3) Three primary frequencies Ω_1, Ω_2, and Ω_3 are much higher than unity, and the frequency of the secondary harmonic is $\Delta\Omega=\Omega_1+\Omega_2-\Omega_3>0$:

$$W(\Delta\Omega)=2\pi\varepsilon_1\varepsilon_2\varepsilon_3\frac{e^{\pi\Delta\Omega/2}}{\sinh(\pi\Delta\Omega)}\Delta\Omega^2\left[\frac{1}{\Omega_1\Omega_2(\Omega_1+\Omega_2)^2} +\frac{1}{\Omega_1\Omega_3^2(\Omega_3-\Omega_1)}+\frac{1}{\Omega_2\Omega_3^2(\Omega_3-\Omega_2)}\right]. \tag{14}$$

Note that in case (3) it would be more natural to talk about a tertiary harmonic rather than a secondary because W depends on the third power of the perturbation parameter.

3. THE SEPARATRIX MAP AMPLITUDE FOR A STANDARD MAP

We wish to link the first example of applying the above results with the standard Chirikov map, which in the notation of Ref. 1 assumes the form

$$\bar{I}=I+K\sin\theta,\quad \bar{\theta}=\theta+\bar{I}. \tag{15}$$

The standard map is used as a model in solving many problems of modern nonlinear dynamics. Hence the numerous studies done with this model and devoted to it.

It has long been known that the values of the separatrix map amplitudes for the system (15) calculated via the Mel'nikov–Arnol'd integral (W_{MA}) and those measured in numerical experiments (W_E) differ considerably. According to Chirikov's results,[1] the ratio of these values is independent of the single parameter K in the system and is equal to

$$R_E=W_E/W_{MA}\approx 2.15. \tag{16}$$

This ratio can also be obtained from the data of Ref. 7, where the splitting angle of the standard map separatrices was determined both analytically and numerically:

$$R_S=|\theta_1|/16\pi^3=2.2552\ldots, \tag{17}$$

with $|\theta_1|=1118.82\ldots$, a constant found in Ref. 7. Note that the same constant enters into the rigorous asymptotic estimates of the upper and lower bounds on the chaotic layer width of the standard map.

In Ref. 1 already it was assumed that the discrepancy between theory and calculations is due to the anomalously strong effect of the higher harmonics in the perturbation. This assumption has been verified, and it was found that secondary harmonics of the standard map play the leading role.

Introducing other variables and using the properties of a periodic delta function, we can show that a continuous system equivalent to the standard map (15) has the form[1]

$$H=\frac{p^2}{2}+\sum_{n=-\infty}^{n=\infty}\cos\left(x-\frac{2\pi nt}{\sqrt{K}}\right), \tag{18}$$

i.e., is an expanded version of the system (1), (2) with an equidistant spectrum with the reference frequency $\Omega_0=\Omega=2\pi/\sqrt{K}$ and the parameters $\varepsilon_n=1$ for all values of n. To be able to assume that we are dealing with a high-frequency perturbation we set $K\ll 1$.

Let us now use Eqs. (4) and (5) to find the contribution to the separatrix map of each harmonic in (18) on the assumption that the specified harmonic is the only perturbation. For $\Omega\gg 1$ we can easily see that in any type of motion only the harmonic with the lowest frequency ($n=1$ for phase rotation with $p>0$ and $n=-1$ for the phase rotation with $p<0$) is important. This harmonic generates a single-frequency separatrix map with an amplitude (Eqs. (4) and (5) with $m=2$)

$$W_{MA}(\Omega)=8\pi\Omega^2e^{-\pi\Omega/2}. \tag{19}$$

This reasoning was considered sufficient to drop the infinite number of terms with $|n|>1$ from (18).

Bearing in mind the above mechanism of secondary harmonic formation, we note that each pair of terms of the form

$$\cos[x-(n+1)\Omega t]+\cos(x+n\Omega t),\quad n\neq 0$$

in the Hamiltonian (18) generates secondary harmonic of the principal frequency Ω at the sum of the frequencies, thus enhancing the action of the leading term in the perturbation and adding a certain quantity to the amplitude (19). Let us find the total value $\Delta W_1(\Omega)$ of all these corrections by employing (12) for the amplitude of the secondary harmonic of the separatrix map at the sum of the frequencies, where we must put $\Omega_1=(n+1)\Omega$, $\Omega_2=-n\Omega$, $\Delta\Omega=\Omega\gg 1$, and $\varepsilon_1=\varepsilon_2=1$:

$$\Delta W_1(\Omega)=\frac{8\pi}{3}\Omega^2 e^{-\pi\Omega/2}\sum_{n\geq 1}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$$

$$=\frac{W_{MA}(\Omega)}{3}\left(\frac{\pi^2}{3}-1\right)\approx 0.76W_{MA}(\Omega). \quad (20)$$

The numerical factor in this relationship is independent of Ω and, hence, of the standard map parameter K. By employing the formulas of Sec. 2 we can easily verify that for $\Omega\gg 1$ the contributions of secondary harmonics with other frequency combinations are infinitesimal.

The amplitude ratio calculated in this way,

$$R_C=1+\Delta W_1(\Omega)/W_{MA}(\Omega)\approx 1.76,$$

is considerably closer to the experimental value (16), but the difference is still large.

Note that all the above relationships were derived on the assumption that the perturbation is weak, $\varepsilon_n\ll 1$, which is definitely not the case for the standard map (18), where $\varepsilon_n=1$ holds for all values of n, and this fact will have an effect on R_C.

Since we have already allowed for the effect of the higher harmonics ($|n|>1$) by calculating their contribution (20), we can now ignore them and instead of (18) consider a reduced system:

$$H(x,p,t)=\frac{p^2}{2}+\cos x+\frac{a}{2}\cos(x-\Omega t)+\frac{a}{2}\cos(x+\Omega t)$$

$$=\frac{p^2}{2}+\cos x+a\cos x\cos(\Omega t), \quad (21)$$

where for our case we must put $a=2$. The system (21) was studied by Gelfreich,[9] who determined the correction factor $f(a)$ to the Mel'nikov–Arnol'd integral as a function of the perturbation intensity a. From Fig. 4 of Ref. 9 it follows that $f(a)\approx 1.38$ at $a=2$, and we can now allow for the perturbation intensity as one more correction $\Delta W_2(\Omega)$ to the amplitude (19):

$$\Delta W_2(\Omega)\approx 0.38W_{MA}(\Omega). \quad (22)$$

If the effects considered are assumed to act independently, the resulting amplitude ratio is

$$R_C=1+\frac{\Delta W_1(\Omega)+\Delta W_2(\Omega)}{W_{MA}(\Omega)}\approx 2.14. \quad (23)$$

This practically coincides with the experimental value (16) but is somewhat smaller than the value (17) found from the separatrix splitting angle.

4. NUMERICAL CALCULATIONS

Studying the effects caused by the formation of secondary harmonics required determining numerically the amplitudes of the harmonics of the separatrix map (7) and the following characteristics of split separatrices: the intersection angle γ_s, the phase volume of the region between neighboring intersection points $\mathscr{A}_s$, and the maximum splitting in momentum, $\Delta\mathscr{P}_{\max}$.

The reliability of the results of numerical integration of the Hamilton equations of motion can be guaranteed only if one employs canonical calculation algorithms (i.e., algorithms that preserve the phase volume). Many popular calculation techniques (say, the Runge–Kutta method) do not satisfy this requirement, and hence suppress weak real dynamical effects and introduce spurious dynamical effects. The explicit canonical second-order algorithm used in the present study is described in the Appendix.

The system (1), (2) possesses a symmetry of the type $x,\ p,\ t\leftrightarrow(2\pi-x,p,-t)$, which makes it possible to calculate only one separatrix instead of two. The symmetry also implies that the central homoclinic point $\mathscr{P}_{fb}(\pi)$ lies on the straight line $x=\pi$ and for a small perturbation is close to the value p_s,max=2. To find the separatrix intersection angle γ_s it is sufficient to study a small neighborhood of this point, while calculating the phase volume $\mathscr{A}_s$ requires reaching the neighboring intersection point. In addition, it is necessary to know $\mathscr{P}_{fb}(\pi)$ to guarantee a "hit" at the chaotic layer in finding the amplitudes of the separatrix map harmonics. For this reason the search for $\mathscr{P}_{fb}(\pi)$ practically always constitutes the first stage in the calculations.

In building the separatrix map (7) the initial conditions for the orbits were selected randomly from a narrow interval within the part studied (rotation or oscillation of the phase) of the chaotic layer. Each trajectory either performed the required number of periods of motion (the period of motion T is the time interval between two successive moments t_π of intersection with the stable phase $x=\pi$) or was terminated due to a change in the type of motion. In any case a new random trajectory was initiated and the process was repeated so as to reach the fixed number of periods N_p. For each period the mean energy w was calculated by the formula[1]

$$w=32\exp(-T). \quad (24)$$

Determining the energy variation $\delta w=\bar{w}-w$ for each pair of adjacent periods and assigning it to the moment t_π common for all these periods, we can build the separatrix map (7) $(\delta w_k,t_{\pi,k}),\ k=1,2,\dots,N_p-1$, on the continuous time scale. In the case of commensurable frequencies it has proved convenient to recalculate the map in terms of the phases $\tau_\pi=\Omega_0 t_\pi\ (\mathrm{mod}\,2\pi)$ with respect to the reference frequency Ω_0 (see Sec. 1). Note that this stage can be skipped because the values of the amplitudes of the harmonics do not depend on it, as experience shows. In calculating the periods T we fixed the shortest period $T_{\min}$, which made it possible to estimate the total width of the part of the chaotic layer under investigation:

$$w_{s,m}=32\exp(-T_{\min}). \quad (25)$$

It is interesting to see how the results of the above measurements agree with each other and with the theory. To this end the system (1), (2) with a single perturbation harmonic

$$V(x,t)=\varepsilon\cos(x-\Omega t),\quad \varepsilon=0.075,\quad \Omega=10.0, \quad (26)$$

was studied. In this and other examples of the present section the separatrix map was calculated for phase rotation with $p>0$.

The following characteristics of split separatrices were found: the angle $\gamma_s\approx 7.09\times 10^{-5}$, the phase volume $\mathscr{A}_s\approx 5.66\times 10^{-6}$, and the maximum splitting in momentum

$\Delta\mathcal{P}_{\max}\approx 1.44\times 10^{-5}$. All three values yield practically the same expected value W_{sp} for the amplitude of the separatrix map harmonic,

$$W_{sp}\approx\frac{4}{\Omega}\gamma_s\approx\frac{\Omega}{2}\mathcal{A}_s\approx p_{\max}\Delta\mathcal{P}_{\max}\approx 2.84\times 10^{-5},\qquad(27)$$

which agrees well with both the measured value $W_E\approx 2.87\times 10^{-5}$ (see Fig. 1) and the value $W_{MA}\approx 2.84\times 10^{-5}$ calculated via the Mel'nikov–Arnol'd integral (Eq. (4) with $m=2$). The width of the rotational part of the layer found from the minimum period and Eq. (25), $w_{s,m}\approx 2.84\times 10^{-4}$, is also close to the theoretical value $\Omega W_{MA}\approx 2.84\times 10^{-4}$ (see Eq. (6). Five periods of the separatrix map (7) for this case are shown in Fig. 1. The reference frequency was taken at $\Omega_0=2.0$ for convenience of comparison with Fig. 2. In all captions to the figures the numbers in parentheses are the frequencies corresponding to the measured values W_E of the amplitudes of the harmonics.

The addition of a symmetric harmonic to the perturbation changes nothing either qualitatively or quantitatively, and the perturbation assumes the form

$$V(x,t)=\varepsilon_1\cos(x-\Omega_1 t)+\varepsilon_2\cos(x-\Omega_2 t),$$
$$\varepsilon_1=\varepsilon_2=0.075,\quad \Omega_1=10.0,\quad \Omega_2=-10.0.\qquad(28)$$

Such a result can be explained by saying that the only secondary harmonic emerging here (on the difference of frequencies) has the high frequency 2Ω and its effect is extremely small.

The picture changes dramatically when the two perturbation harmonics are asymmetric in frequency:

$$V(x,t)=\varepsilon_1\cos(x-\Omega_1 t)+\varepsilon_2\cos(x-\Omega_2 t),$$
$$\varepsilon_1=\varepsilon_2=0.075,\quad \Omega_0=2.0,\quad \Omega_1=6\Omega_0=12.0,\qquad(29)$$
$$\Omega_2=-5\Omega_0=-10.0.$$

The second and third rows in (29) show that the frequencies of the harmonics are commensurate and that the reference frequency is $\Omega_0=2.0$.

The total separatrix map amplitude measure for this case, $W_E\approx 1.33\times 10^{-4}$ is almost five times larger than for the symmetric perturbation (28), even though the frequency Ω_1 was increased. The reason is the emergence of a secondary harmonic with the low frequency $\Delta\Omega=\Omega_1+\Omega_2=\Omega_0=2.0$. Figure 2 depicts one period of the separatrix map and the amplitudes of all three harmonics. The horizontal scales in Figs. 1 and 2 are the same, and comparison of the two reveals the changes caused by the appearance of a secondary harmonic whose amplitude is larger by several orders of magnitude than the amplitudes of the primary frequency harmonics that generated the secondary harmonic. Consequently, we can assume the map to be single-harmonic with a frequency $\Omega_0=2.0$. The layer width $w_s=\Omega_0 W_E(\Omega_0)\approx 2.89\times 10^{-4}$ calculated by the single-frequency formula (6) as the sum of contributions from all harmonics proved to be smaller than the value $w_{s,m}=3.47\times 10^{-4}$ calculated from the minimum rotation period. Applying formula (12) to this case yields an estimate

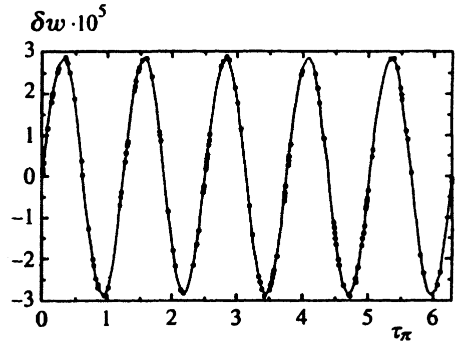

FIG. 1. Numerical integration of the system (1) with the single-frequency perturbation (26). The initial conditions for the orbits were selected within a narrow interval inside the rotational part (with $p>0$) of the chaotic layer. The dots stand for the elements ($\delta w=\bar{w}-w$ and $\tau_\pi=\Omega_0 t_\pi$ (mod 2π)) of the separatrix map (7) with the reference frequency $\Omega_0=2.0$, and the curve represent the result of fitting by the least squares method. The amplitude of the map is $W_E(10.0)\approx 2.87\times 10^{-5}$.

$W(\Omega_0)\approx 2.7\times 10^{-4}$, which agrees, in order of magnitude, with the measured value $W_E(\Omega_0)\approx 1.32\times 10^{-4}$.

The following characteristics of split separatrices with the asymmetric perturbation (29) were found: the angle $\gamma_s\approx 7.15\times 10^{-5}$, the phase volume $\mathcal{A}_s\approx 1.32\times 10^{-4}$, and the maximum splitting in momentum $\Delta\mathcal{P}_{\max}\approx 9.24\times 10^{-5}$. Processing these data by formula (27) shows that only measuring the phase volume $\mathcal{A}_s$ yields a plausible forecast for the expected value of the separatrix map amplitude, $W_{sp}=\frac{1}{2}\Omega_0\mathcal{A}_s\approx 1.32\times 10^{4}$. However, this possibility also vanishes if the separatrix map cannot be thought of as being approximately single-frequency.

A multifrequency separatrix map occurs, for instance, when the perturbation is of the form

$$V(x,t)=\varepsilon_1\cos(x-\Omega_1 t)+\varepsilon_2\cos(x-\Omega_2 t)+\varepsilon_3\cos(x-\Omega_3 t),$$
$$\varepsilon_1=0.1,\quad \varepsilon_2=0.6,\quad \varepsilon_3=1.2,\qquad(30)$$
$$\Omega_1=13.0,\quad \Omega_2=19.0,\quad \Omega_3=31.0.$$

Figure 3 shows that three secondary harmonics are added to the three primary harmonics, with the secondary harmonic at

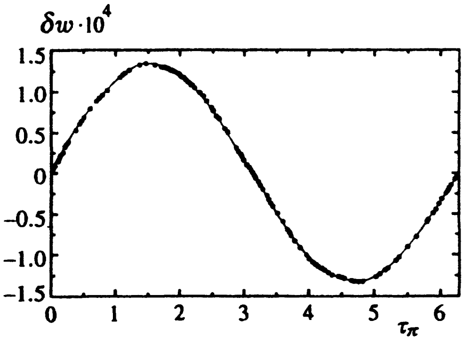

FIG. 2. The same as in Fig. 1 but with the asymmetric double-frequency perturbation (29). The amplitudes of the primary separatrix-map harmonics are $W_E(12.0)\approx 1.91\times 10^{-6}$ and $W_E(-10.0)\approx -2.13\times 10^{-7}$, and the amplitude of the secondary harmonic is $W_E(2.0)\approx 1.32\times 10^{-4}$.

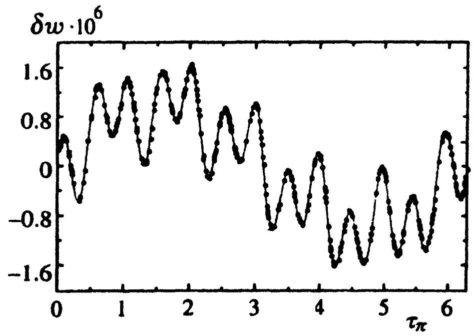

FIG. 3. The same as in Fig. 1 but with the triple-frequency perturbation (30). The amplitudes of the primary harmonics of the separatrix map: $W_E(13.0)\approx 5.88\times 10^{-7}$, $W_E(19.0)\approx 2.48\times 10^{-8}$, and $W_E(31.0)\approx 1.91\times 10^{-8}$; the amplitudes of the secondary harmonics: $W_E(1.0)\approx 9.78\times 10^{-7}$, $W_E(6.0)\approx -3.67\times 10^{-7}$, and $W_E(12.0)\approx -1.36\times 10^{-9}$.

the frequency combination $\Delta\Omega=\Omega_1+\Omega_2-\Omega_3=1.0$ having the largest amplitude. Applying formula (14) yields the estimate $W(1.0)\approx 2.4\times 10^{-6}$, which agrees, in order of magnitude, with the measured value $W_E(1.0)\approx 1.0\times 10^{-6}$. The width of the chaotic layer calculated by the minimum rotation period, $w_{s,m}\approx 1.22\times 10^{-5}$, in this case proved to be close to the sum of the contributions of all the separatrix map harmonics calculated by the single-frequency formula (6).

5. CONCLUSION

We have established that the motion of the Hamiltonian system (1), (2) in the vicinity of the separatrix of the principal resonance with several perturbing harmonics of distinct frequencies differs dramatically from the well-studied case of a single-frequency perturbation. Not only does the interaction of the perturbation and the principal resonance have a strong effect on the spectrum of the separatrix map of the system and the formation of a chaotic layer, but so does the interaction of the perturbing resonances with one another, the result of which is secondary harmonics at combinations of the primary frequencies. The exponential frequency dependence of the separatrix map amplitude leads to a situation in which even very weak but low-frequency perturbations may play the leading role in dynamic chaos formation. Generally the width of the chaotic layer is not determined by the simple formula (6) and can substantially exceed the sum of contributions of all the terms in the separatrix map calculated by the single-frequency theory.

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APPENDIX THE CANONICAL INTEGRATION ALGORITHM

We used the method of building canonical generating functions developed in Ref. 11, which makes it possible to replace continuous Hamiltonians of the equation of motion by their discrete analogs, canonical maps. One iteration of a map is equivalent to integrating equations on a time interval τ with an error amounting to $O(\tau^{n+1})$, where n is the order of the map. The method allows, at least in principle, maps of any order to be constructed, but we restrict our discussion to the second order ($n=2$), which for systems of type (1) implies that the new variables depend explicitly on the old.

After we have gone over to the extended phase space of system, the system (1) becomes

$$K(x,p,h,t)=p^2/2+\cos x+V(x,t)+h=0,$$

$$h=-H(x,p,t).$$

We assume that the generating function depends on the old coordinates x and t and the new momenta $\bar{p}$ and $\bar{h}$ and introduce the following compact notation:

$$\frac{\partial^{m+n}}{\partial t^m \partial x^n}[\cos x+V(x,t)]=f_{mn}.$$

The second-order generating function

$$S=x\bar{p}+t\bar{h}+\tau(\bar{p}^2/2+\bar{h}+f_{00})+\tau^2/2(\bar{p}f_{01}+f_{10})$$

generates an explicit canonical map of the form

$$\bar{p}=\frac{p-\tau f_{01}-(\tau^2/2)f_{11}}{1+(\tau^2/2)f_{02}},\quad \bar{x}=x+\tau\bar{p}+(\tau^2/2)f_{01},$$

$$\bar{h}=h-\tau f_{10}-(\tau^2/2)(\bar{p}f_{11}+f_{20}).$$

Because of phase-space conservation, the calculation error oscillates instead of building up as the number of iteration increases. The amplitude of these oscillations allows for an analytical estimate:[11]

$$\delta K\approx(\tau^3/6)(\bar{p}^3 f_{03}+3\bar{p}^2 f_{12}+3\bar{p}f_{21}+f_{30}).$$

The last relationship helps to select the required time step τ, while the presence of the integral of motion $K(x,p,h,t)=0$ makes its possible to effectively monitor the accuracy in the calculation process.

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