Elastic waves in randomly inhomogeneous ferromagnets in the vicinity of a magnetoelastic resonance

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The elastic waves in a ferromagnet with zero-mean magnetostriction are considered with an arbitrary angle θ between the wave vector **k** and the magnetic field **H**. The properties of the dispersion curves, the damping, and the polarization of the fundamental waves are studied. It is shown that the dispersion curves in the vicinity of the magnetoelastic resonances corresponding to transverse and longitudinal phonons have qualitatively different dependence on θ. The magnetoelastic analogs of the magneto-optical Faraday and Cotton–Mouton effects are studied. It is shown that the parameters characterizing the inhomogeneous magnetoelastic interaction can be determined, in principle, from the results of experimental observations of these effects. © *1996 American Institute of Physics.* [S1063-7761(96)02304-9]

1. INTRODUCTION

A model of disorder-induced resonance crossing in a system of two scalar waves, one of which has a gapless dispersion law, while there is a gap in the spectrum of the other, was considered in a recent paper.[1] In this model the parameter of the linear relationship between the waves was assumed to be a random function of the coordinates with a mean value equal to zero. It was found that the removal of the degeneracy in the vicinity of the point where the original dispersion curves cross in this model has several significant features not associated with the analogous phenomenon in homogeneous systems.[2] One of the possible physical realizations of this model is magnetoelastic resonance in amorphous ferromagnets. It is known[3] that some of these materials have a nearly zero value for the magnetoelastic constant averaged over the sample, which renders them a convenient object for an experimental investigation of the effects predicted in Ref. 1. The interest in the investigation of the magnetoelastic interaction in such materials is due to the need to develop methods for determining the values of the root-mean-square (rms) fluctuation and the correlation radius of the inhomogeneities in the magnetostriction parameter, which are the principal characteristics of the magnetoelastic properties of such materials.

In Ref. 4 the magnetoelastic resonance in a ferromagnet with a random zero-mean magnetoelastic constant was investigated for the simplest case of a wave propagating parallel to the equilibrium magnetization $\mathbf{M}_0$. It was shown that in this case the equations describing the averaged dynamics of the elastic and spin subsystems basically reduce to the equations investigated in Ref. 1. Numerical evaluations demonstrated the fundamental possibility of using unordered ferromagnets to experimentally investigate the phenomenon of stochastic resonance crossing. It was also shown that the experimental investigation of the magnetic-field and frequency dependences of several parameters (the velocity of sound, damping, etc.) can, in principle, make it possible to measure both the rms fluctuation and the correlation radius of the magnetostriction parameter.

A fundamental difference between coupled magnetoelastic waves and the simple model in Ref. 1 is their vector character. For this reason, apart from the investigation of the dispersion curves, the question of the polarization properties of these waves arises in this case. At lengths much less than the mean free path, the contribution of the fluctuation component to the polarization, which is described by the corresponding coherence matrix $R_{ij}=\langle u_i u_j\rangle-\langle u_i\rangle\langle u_j\rangle$, can be neglected, and the polarization properties can be described in this approximation using the polarization vectors of the mean amplitudes $\langle u_i\rangle$.

In the case of a wave propagating parallel to $\mathbf{M}_0$, which was considered in Ref. 4, these properties are trivial (the fundamental polarizations of the coherent component of the elastic waves are left- and right-circularly polarized waves, whose polarization states do not depend on the parameters of the material or the external conditions); therefore, a more general case, in which the wave propagates at an arbitrary angle θ to $\mathbf{M}_0$ must be considered to study them. The main difference between this situation and the case of $\theta=0$ is that now a longitudinal wave also participates in the interaction along with the transverse elastic wave. It is seen that in this case there are two types of resonance: one for the transverse phonons and the other for the longitudinal phonons. Just this circumstance renders the polarization properties of the waves in such a situation nontrivial.

In this paper we shall investigate the special features of the dispersion curves of coherent elastic waves in the vicinity of each of these resonances and show that, in contrast to the case of a purely determined interaction, the behavior of these curves can be qualitatively different from one to another.

In addition, the problem of the propagation of a monochromatic wave with a given frequency Ω will be addressed in this paper. In contrast to the investigation of the dispersion laws of waves, where a wave vector characterizing different states of a coherent wave is assumed to be real, and the frequency obtained from the solution of the dispersion equation has an imaginary part, in the problem of the propagation of a traveling wave the frequency is a given real quantity, and the wave number k is complex. We shall investigate the dependence of the real and imaginary parts of k on Ω and use the results obtained to analyze the evolution of the polarization state of a wave incident on a medium. The analogs of the magneto-optical Faraday and Cotton–Mouton effects will be considered, and the influence of the longitudinal component of the elastic waves on these effects will be analyzed.

2. MODEL AND EQUATIONS OF MOTION

Just as in Ref. 4, we consider an isotropic (with respect to the elastic properties) ferromagnet, all of whose variables, except the magnetoelastic parameter $P(\mathbf{x})$, are assumed to be constant. Neglecting the magnetic-dipole interaction and the ponderomotive force, we write the energy density in the form

$$H=\frac{\alpha}{2}\left(\frac{\partial \mathbf{M}}{\partial \mathbf{x}}\right)^2-\mathbf{M}\cdot\mathbf{H}+\frac{1}{2}d_{ijkl}u_{ij}u_{kl}+\frac{1}{2}P(\mathbf{x})M_iM_ju_{ij}, \quad (1)$$

where α is the exchange parameter, $\mathbf{H}$ is the external magnetic field, $\mathbf{M}$ is the magnetization, $d_{ijkl}=d_1\delta_{ij}\delta_{kl}+d_2(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$ is the elastic modulus tensor, and u_{ij} is the strain tensor. We represent the magnetoelastic parameter in the following form:

$$P(\mathbf{x})=\langle P\rangle+\Delta P\rho(\mathbf{x}), \quad (2)$$

where $\langle P\rangle$ is the mean value of the magnetoelastic constant, which we henceforth set equal to zero, ΔP is the rms fluctuation, and $\rho(\mathbf{x})$ is a centered normalized homogeneous random function, which is characterized by the correlation function $K(\mathbf{r})$:

$$\langle\rho(\mathbf{x})\rho(\mathbf{x}+\mathbf{r})\rangle=K(\mathbf{r}). \quad (3)$$

Just as in Ref. 4, we select the correlation function in the form

$$K(\mathbf{r})=\exp(-rk_c), \quad (4)$$

where k_c is the correlation wave number, which is inversely proportional to the correlation radius of the inhomogeneities of the magnetoelastic parameter $P(\mathbf{x})$.

The equation of motion for the components of the magnetic moment and the elastic displacement vector have the form

$$\frac{\partial \mathbf{M}}{\partial t}=g(\mathbf{H}^{\text{eff}}\times\mathbf{M}), \qquad G\ddot{u}_i=\frac{\partial\sigma_{ij}}{\partial x_j}, \quad (5)$$

where

$$H_j^{\text{eff}}=\alpha\nabla^2M_j+H_j-\Delta P\rho M_iu_{ij},$$

$$\sigma_{ij}=d_{ijkl}u_{kl}+\frac{1}{2}\Delta P\rho M_iM_j,$$

g is the gyromagnetic ratio, and G is the density of the medium.

We choose the coordinate system so that the z axis is parallel to $\mathbf{H}$ and $\mathbf{M}_0$ and the wave vector $\mathbf{k}$ lies in the xz plane (Fig. 1). Linearizing Eqs. (5) and expanding them in plane waves $\propto\exp[i(\Omega t-\mathbf{k}\cdot\mathbf{x})]$, we obtain

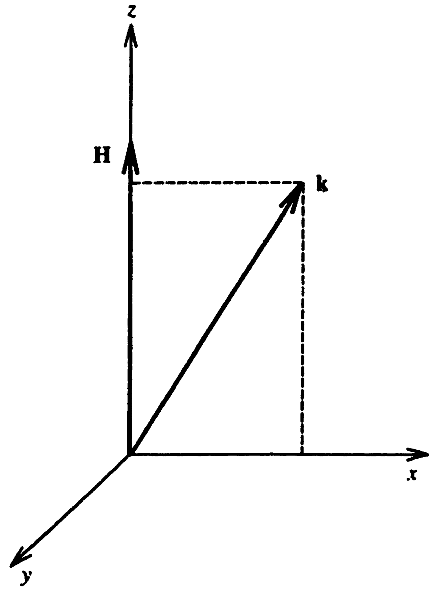

FIG. 1. Coordinate system for Eq. (6).

$$\begin{aligned}
&(\Omega^2-\omega_t^2)u_x-(\omega_l^2-\omega_t^2)\left(u_x\sin^2\theta+\frac{1}{2}u_z\sin2\theta\right)\\
&\quad=-i\zeta_u\cos\theta\hat{\Phi}(m_x),\\
&(\Omega^2-\omega_t^2)u_y=-i\zeta_u\cos\theta\hat{\Phi}(m_y),\\
&(\Omega^2-\omega_t^2)u_z-(\omega_l^2-\omega_t^2)\left(\frac{1}{2}u_x\sin2\theta+u_z\cos^2\theta\right)\\
&\quad=-i\zeta_u\sin\theta\hat{\Phi}(m_x),\\
&(\Omega^2-\omega_s^2)m_x=\zeta_m[i\omega_s\hat{\Phi}(q_xu_z+q_zu_x)-\Omega\hat{\Phi}(q_yu_z\\
&\qquad+q_zu_y)],\\
&(\Omega^2-\omega_s^2)m_y=\zeta_m[i\omega_s\hat{\Phi}(q_yu_z+q_zu_y)+\Omega\hat{\Phi}(q_xu_z\\
&\qquad+q_zu_x)],
\end{aligned} \quad (6)$$

where

$$\zeta_m=\Delta PM_0^2g/2, \qquad \zeta_u=\Delta PM_0k/2G,$$

$$\hat{\Phi}(f)=\int\rho(\mathbf{k}-\mathbf{q})f(\mathbf{q})d\mathbf{q},$$

$$\rho(\mathbf{k})=\frac{1}{(2\pi)^3}\int\rho(\mathbf{r})e^{i\mathbf{kr}}d\mathbf{r}, \qquad \omega_t=v_tk,$$

$$\omega_l=v_lk, \qquad \omega_s=\omega_0+\omega_M\alpha k^2$$

are the original dispersion laws of the transverse and longitudinal phonons and magnons, respectively, $\omega_0=gH_0$, $\omega_M=gM_0$, and θ is the angle between $\mathbf{k}$ and $\mathbf{H}$.

As follows from Refs. 1 and 4, the system of equations (6) describes two types of excitations. One of them is characterized by nonzero values for the mean components of the elastic displacement vector $\langle u_i\rangle\neq0$ and zero-mean values for the transverse components of the magnetization $\langle m_\perp \rangle = 0$. The other type is characterized by the opposite situation, in which $\langle u_i \rangle = 0$ and $\langle m_\perp \rangle \neq 0$. This situation differs significantly from the case of the resonant interaction of waves in a homogeneous medium, where mixed states having nonzero values for the amplitudes of each of the interacting waves appear.

In this paper we restrict ourselves to considering only excitations with nonzero values for the mean elastic displacement amplitudes. The excitations considered arise as a result of the interaction of a coherent elastic wave with fluctuational (scattered) spin waves, as was discussed in detail in Refs. 1 and 4. The equations describing these excitations are obtained from (6) after eliminating the transverse components of the magnetization **m** and averaging, which we perform, just as in Ref. 1, to the lowest nonvanishing order of perturbation theory. The applicability of such an approximation to the description of the situation under consideration was discussed in Ref. 1.

After the averaging we go over to a coordinate system which is more natural for describing elastic waves. For this purpose, we rotate the original coordinate system around the y axis so that the z' axis coincides with the direction of propagation of the wave **k**. In this coordinate system elastic displacement vectors with a zero z' component correspond to transverse waves, and vectors with zero x' and y' components correspond to longitudinal waves. Since this coordinate system will be used everywhere below, we shall omit the prime signs on the coordinate labels.

As a result we obtain a system of equations containing the terms F_- and F_+, which describe the interactions with resonant and nonresonant magnons, respectively:

$$F_\pm = \int \frac{S(\mathbf{k}-\mathbf{q})}{\Omega \pm \omega_s(\mathbf{q})} d\mathbf{q}, \tag{7}$$

where $S(\mathbf{q})$ is the Fourier transform of the correlation function (3). It is seen that in the vicinity of the resonance $F_- \gg F_+$; therefore, we neglect F_+ and omit the subscript in F_-.

As a result, the system of equations for the mean components of the elastic displacement vector $\mathbf{U} = \langle \mathbf{u} \rangle$ takes the form

$$\begin{aligned}
&U_x[\Omega^2 - \omega_t^2 - \zeta\omega_M\omega_t^2 F \cos^2\theta] - U_y i\zeta\omega_M\omega_t^2 F \\
&\quad \times \cos 2\theta \cos\theta - U_z \zeta\omega_M\omega_t^2 F \cos 2\theta \sin 2\theta = 0, \\
&U_x i\zeta\omega_M\omega_t^2 F \cos 2\theta \cos\theta + U_y[\Omega^2 - \omega_t^2 - \zeta\omega_M\omega_t^2 F \cos^2\theta] \\
&\quad + U_z i\zeta\omega_M\omega_t^2 F \cos 2\theta \sin\theta = 0, \\
&U_x \zeta\omega_M\omega_t^2 F \cos 2\theta \sin 2\theta + U_y i\zeta\omega_M\omega_t^2 F \sin 2\theta \cos\theta \\
&\quad - U_z[\Omega^2 - \omega_l^2 - \zeta\omega_M\omega_t^2 F \sin^2 2\theta] = 0.
\end{aligned} \tag{8}$$

Here

$$\zeta = (\Delta P)^2 \frac{M_0^2}{4Gv_t^2}.$$

In contrast to the case of $\theta = 0$, both the transverse and longitudinal components of the elastic displacement vector now participate in the interaction.

3. DISPERSION EQUATION AND EIGENVECTORS

The dispersion equation following from (8) has the form

$$\begin{aligned}
(\Omega^2 - \omega_t^2)\Big\{ &(\Omega^2 - \omega_t^2)(\Omega^2 - \omega_l^2) \\
&- \zeta\omega_M\omega_t^2 \int \frac{S(\mathbf{k}-\mathbf{q})}{\Omega - \omega_s(\mathbf{q})} d\mathbf{q}[(\Omega^2 - \omega_t^2)\sin^2 2\theta \\
&+ (\Omega^2 - \omega_l^2)(\cos^2\theta + \cos^2 2\theta)]\Big\} = 0.
\end{aligned} \tag{9}$$

The solution $\Omega = \omega_t$ of this equation corresponds to one of the transverse branches not participating in the interaction. The remaining solutions of this equation are sought by the method described in Ref. 1, separately in the vicinity of the resonances at the frequencies of the transverse and longitudinal phonons.

A correlation function of the form (4) yields the spectral density

$$S(k) = \frac{k_c}{\pi^2} \frac{1}{(k^2 + k_c^2)^2}. \tag{10}$$

Calculating the integral in (9) with this function, we bring the expression in curly brackets into the form

$$\begin{aligned}
(\Omega^2 - \omega_t^2)(\Omega^2 - \omega_l^2) &= \zeta\omega_M\omega_t^2 \\
&\times \frac{(\Omega^2 - \omega_t^2)\sin^2 2\theta + (\Omega^2 - \omega_l^2)(\cos^2\theta + \cos^2 2\theta)}{\Omega - \omega_s - 2\sqrt{\kappa_s(\omega_0 - \Omega)}},
\end{aligned} \tag{11}$$

where $\kappa_s = \alpha\omega_M k_c^2$.

We rewrite (11) in the form of two equivalent equations:

$$\begin{aligned}
&(\Omega^2 - \omega_l^2)(\Omega - \omega_s - i\Gamma_l) \\
&\quad = \zeta\omega_M\omega_t^2\left[\sin^2 2\theta + \frac{\Omega^2 - \omega_l^2}{\Omega^2 - \omega_t^2}(\cos^2\theta + \cos^2 2\theta)\right], \\
&(\Omega^2 - \omega_t^2)(\Omega - \omega_s - i\Gamma_t) \\
&\quad = \zeta\omega_M\omega_t^2\left[\sin^2 2\theta \frac{\Omega^2 - \omega_t^2}{\Omega^2 - \omega_l^2} + (\cos^2\theta + \cos^2 2\theta)\right].
\end{aligned} \tag{13}$$

These equations can be rewritten as

$$(\Omega^2 - \omega_{t,l})(\Omega - \omega_s - i\Gamma_{t,l}) = \frac{\lambda_{t,l}^2(\Omega)}{4}, \tag{13}$$

where we have introduced $\lambda_{t,l}$ and $\Gamma_{t,l}$, which are the effective interaction and relaxation parameters of the transverse and longitudinal waves, respectively.

The essential point of the method for investigating such equations, which was described in Ref. 1, is that, in order to find the solution for a particular branch, the value of the unperturbed frequency of that branch is plugged into the equations for λ and Γ. For example, to find the frequency corresponding to the coherent branch near the resonance at the transverse frequency, ω_t must be substituted into λ_t and Γ_t instead of Ω. Thereafter Eqs. (13) cease to be equivalent and describe the dispersion curves in the vicinity of the transverse [for the interaction parameter $\lambda_t(\omega_t)$] and longitudinal [$\lambda_l(\omega_l)$] resonances, respectively.

In this case the parameters $\Gamma_{t,l}$ and $\lambda_{t,l}$, which describe the relaxation properties of the fluctuational magnons and the effective constant of their interaction with the transverse (longitudinal) phonons, are specified by the equations

$$\lambda_t^2=\frac{\zeta\omega_M\omega_t^2(\cos^2 2\theta+\cos^2\theta)}{\omega_r^t},$$

$$\lambda_l^2=\frac{\zeta\omega_M\omega_t^2\sin^2 2\theta}{\omega_r^l}, \tag{14}$$

$$\Gamma_{t,l}=2\sqrt{\kappa_s(\omega_r^{t,l}-\omega_0)},$$

where $\omega_r^{t,l}$ are the resonant frequencies for the transverse and longitudinal phonons, respectively. Thus, in this case, too, we arrive at the equations of the standard model considered in Refs. 1 and 4 with the one difference that here the interaction parameter depends on the direction in which the wave propagates. Therefore, all the conclusions drawn in Refs. 1 and 4 can simply be carried over to the case under consideration.

The behavior of the dispersion curves and the damping in the vicinity of a resonance depends sensitively on the relationship between the interaction and relaxation parameters.

In the case $\lambda_{t,l}<\Gamma_{t,l}$ one of the solutions of Eq. (13) describes a weakly damped wave with a dispersion law which is slightly modified in comparison with the original law corresponding to longitudinal or transverse elastic waves. The second solution does not correspond to any propagating mode due to the strong damping in this case. In the opposite case of $\lambda_{t,l}>\Gamma_{t,l}$ the existence of two well defined solutions is possible in the vicinity of a resonance, whose dispersion curves are separated at the resonance point by a gap

$$\Delta_{t,l}=\sqrt{\lambda_{t,l}^2-\Gamma_{t,l}^2}, \tag{15}$$

and for which the damping is the same and is equal to $\Gamma_{t,l}/2$. As the distance from the resonance point increases, the damping of one of the modes decreases, and the solution transforms into the corresponding original law, while the damping of the other mode increases, and it ultimately becomes poorly defined. The vanishing of the damping for $\Omega>\omega_0$ in proportion to $\sqrt{\kappa_s(\Omega-\omega_0)}$ is also noteworthy.

When these results are applied to the case under consideration, the most important thing to note is the difference in the dependence of the effective interaction parameters λ_t and λ_l on the angle θ. At small angles λ_l decreases significantly, while λ_t remains practically unchanged. Therefore, at small θ the value of l_t must be smaller than Γ_l, while λ_t can be greater than Γ_t. In this situation there is only one slightly modified branch in the vicinity of the longitudinal resonance, while there can be two branches separated by a gap in the vicinity of the transverse resonance. When θ increases, the value of λ_l becomes of order Γ_l, and then a second solution corresponding to a propagating mode can appear in the vicinity of the longitudinal resonance.

Let us now consider the polarizations of the elastic waves.

The simplest polarization vector is that of a noninteracting wave with the dispersion law $\Omega=\omega_t$:

$$U_1=\begin{pmatrix}\cos\theta\\ i\cos 2\theta\\ 0\end{pmatrix}. \tag{16}$$

Such an eigenvector corresponds to a transverse, elliptically polarized wave with right-handed rotation.

The polarization vectors for resonantly interacting waves can be written in the form

$$U_{2,3}=\begin{pmatrix}\cos 2\theta(\Omega^2-\omega_l^2)\\ -i\cos\theta(\Omega^2-\omega_l^2)\\ \sin 2\theta(\Omega^2-\omega_t^2)\end{pmatrix}, \tag{17}$$

where Ω must be replaced by the corresponding solution of the dispersion equation.

Since the longitudinal component of the elastic displacement vector participates in the interaction, the polarization of the wave (except in the cases of $\theta=0$ and $\theta=\pi/2$) differs from purely transverse or purely longitudinal polarization. For this reason, we shall henceforth refer to the wave which becomes transverse in the absence of an interaction as the quasitransverse wave and to a wave which becomes longitudinal at $\zeta=0$ as quasilongitudinal wave.

Mention should be made here of a certain peculiarity stemming from the fact that in the case of an open gap in the vicinity of, say, the transverse resonance, a given value of k corresponds to two branches of the quasitransverse wave, viz., a coherent and a fluctuational branch, with different frequency values and thus with different polarization vectors [according to (17)]. Although the following analysis is valid for both branches, i.e., both the quasitransverse and the quasilongitudinal waves, to be specific we shall refer to the coherent branch of the quasitransverse wave.

After representing the corresponding solution of the dispersion equation in the form $\Omega=\omega+i\xi$ (the expressions for ω and ξ were presented in Ref. 4), we write the time dependence of the mean components of the elastic displacement vector:

$$U_x=\cos 2\theta\cos\omega t,$$

$$U_y=\cos\theta\sin\omega t, \tag{18}$$

$$U_z=\frac{2\sin 2\theta}{\omega_l^2-\omega_t^2}[\omega_t(\omega-\omega_t)\cos\omega t-\omega\xi\sin\omega t].$$

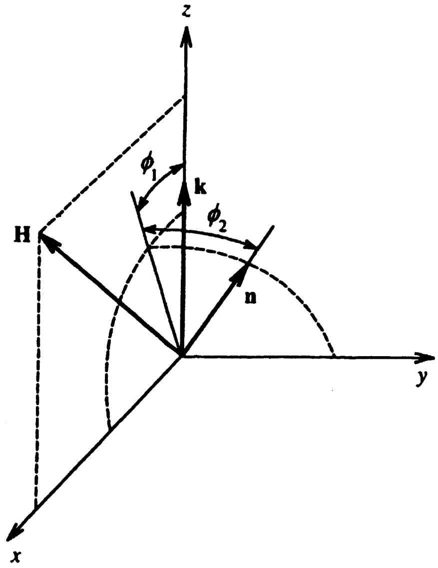

FIG. 2. Orientation of the plane of the polarization ellipse of a fundamental wave in space.

In writing (18) we have discarded the exponential multiplier common to all three components, which describes damping, but does not influence the polarization properties.

The vector (18) has the same normalization as (16) and is orthogonal to within terms which are quadratic in $\min(\zeta, \Gamma_t/\omega_r)$. The significant difference in (18) is the presence of a longitudinal component, which determines all the features of the polarization properties of the resonantly interacting wave.

The transformation of the coordinates of the ellipse described by (18) can easily be brought into canonical form. If we apply the rotations $R_y(\phi_1)R_x(\phi_2)R_z(\phi_3)$, where $R_i(\phi_j)$ is a rotation around the axis i through the angle ϕ_j, to (18), we obtain the vector

$$U_x = a \cos \omega t,$$
$$U_y = b \sin \omega t, \tag{19}$$
$$U_z = 0.$$

The angles ϕ_1 and ϕ_2 specify the position in space of the plane of the polarization ellipse: ϕ_1 is the angle between the projection of the normal **n** to the plane of the polarization ellipse onto the (**k**,**H**) plane and the direction of wave propagation **k**, and ϕ_2 is the angle between **n** and the (**k**,**H**) plane (Fig. 2). The angle ϕ_3 assigns the orientation of the polarization ellipse in that plane.

In the general case the expressions for ϕ_1 and ϕ_2 can be represented in the form

$$\tan \phi_1 = 2\frac{\omega_t \xi}{\omega_l^2 - \omega_t^2} \tan 2\theta, \tag{20}$$
$$\tan \phi_2 = 4\frac{(\omega - \omega_t)\omega_t}{\omega_l^2 - \omega_t^2} \cos \phi_1 \sin \theta.$$

Since the expression for ϕ_3 is fairly involved, we shall not present it.

As is seen from (20), the dependence of ϕ_1 on k actually retraces the dependence of the damping ξ normalized to ω_t, repeating all the features of its behavior in the case of an open or closed gap in the spectrum of the quasitransverse elastic wave.

For its part, ϕ_2 mimics the dependence on k of a modification of the dispersion law for $\omega - \omega_t$ normalized to ω_t. This accounts for the feature in the behavior of ϕ_2 observed upon passage through the resonance. If we move along the coherent branch, ϕ_2 changes sign abruptly in the case of an open gap or continuously in the opposite case.

We note that in the absence of damping (for $\omega < \omega_0$), $\phi_1 = 0$ holds and ϕ_2 is always small, so that the plane of the polarization ellipse practically coincides with the xy plane in this case.

When θ is so far from $\pi/4$ that $\omega_t \xi \tan 2\theta/(\omega_l^2 - \omega_t^2) \ll 1$ holds, the following approximate expressions are valid for ϕ_1 and ϕ_2:

$$\phi_1 \approx 2\frac{\omega_t \xi}{\omega_l^2 - \omega_t^2} \tan 2\theta, \tag{21}$$
$$\phi_2 \approx 4\frac{(\omega - \omega_t)\omega_t}{\omega_l^2 - \omega_t^2} \sin \theta.$$

It is seen that in this case the angles ϕ_1 and ϕ_2 are small, so that the polarization ellipse lies in a plane deviating slightly from the xy plane, which is perpendicular to the wave vector **k**.

For θ so close to $\pi/4$ that $\omega_t \xi \tan 2\theta/(\omega_l^2 - \omega_t^2) \gg 1$, the orientation of the polarization ellipse is described by different expressions:

$$\phi_1 \approx \frac{\pi}{2} - \frac{\omega_l^2 - \omega_t^2}{\omega_t \xi} \cot 2\theta, \tag{22}$$
$$\phi_2 \approx \frac{(\omega - \omega_t)\cos 2\theta}{\xi \cos \theta}.$$

Here ϕ_1 is close to $\pi/2$, and ϕ_2 remains small (because of the small value of $\cos 2\theta$). In this case the polarization ellipse can be assumed in an approximation to be oriented in the zy plane, which is parallel to **k**. This is because the x-component of the elastic displacement vector vanishes at $\theta = \pi/4$.

The characteristic dependence of ϕ_1 on θ is represented in the form of a peak with half-width ϵ, which is determined by the damping:

$$\epsilon \approx \frac{1}{4}\frac{\omega_t \xi}{\omega_l^2 - \omega_t^2}. \tag{23}$$

Apart from the orientation, the ratio η between the semiaxes of the polarization ellipse is an important characteristic. The dependence of η on the wave number is caused only by the longitudinal component, since the amplitude of the transverse modes does not depend on k. Therefore, we present the expression for the ratio between the semiaxes for $\theta = \pi/4$, for which the dependence on k is strongest:

$$\eta = 2\sqrt{2}\frac{v_t}{(v_l^2 - v_t^2)k} \xi(k,H). \tag{24}$$

In this case the polarization ellipse is oriented in a plane parallel to **k**.

We also note, first, that in the absence of damping $(\omega<\omega_0)$, the ratio of the semiaxes vanishes, i.e., in this case (for $\theta=\pi/4$) the wave is linearly polarized. Second, η, like ϕ_1, retraces the dependence on k and H of the damping ξ normalized to ω_t.

4. WAVE PROPAGATION IN AN INHOMOGENEOUS MEDIUM

Let a plane elastic wave, which has an assigned polarization state in the $z=0$ plane, propagate along the z axis at an angle θ to the magnetic field in a inhomogeneous medium.

As we have already noted, this situation differs from the case already examined in that when the dispersion equation (9) is solved, the frequency should be considered real and the wave should be considered complex.

In the vicinity of the transverse and longitudinal resonances for the quasitransverse and quasilongitudinal waves, it is more convenient to write this equation in the form

$$(k^2-q_{t,l}^2)(k^2-q_m^2)=\delta_{t,l}^2k^2, \tag{25}$$

where

$$q_{t,l}=\frac{\Omega}{v_{t,l}},\qquad q_m=\sqrt{\frac{\Omega-\omega_0}{\alpha\omega_M}}-ik_c,$$

$$\delta_t^2=\frac{\zeta}{\alpha}(\cos^2\theta+\cos^2 2\theta),\qquad \delta_t^2=\frac{\zeta}{\alpha}\left(\frac{v_t}{v_l}\right)^2\sin^2 2\theta.$$

To be specific, we analyze this equation in the vicinity of the transverse resonance only.

From the exact solution of (25)

$$k^2=\frac{q_t^2+q_m^2+\delta_t^2}{2}\pm\frac{1}{2}\sqrt{(q_t^2+q_m^2+\delta_t^2)^2-4q_t^2q_m^2} \tag{26}$$

it is seen that the dependence of $k(\Omega)$ consists of two branches, which can either cross or anticross. The condition for the absence of anticrossing, which corresponds, at resonance, to equality of the real part of the root in (26) to zero, can be written in the form

$$\tilde{\zeta}<\alpha k_c^2, \tag{27}$$

where $\tilde{\zeta}=\zeta(\cos^2\theta+\cos^2 2\theta)$. This condition differs significantly from the analogous condition obtained above from the analysis of the inverse dependence of $\Omega(k)$. A simple numerical evaluation reveals that anticrossing of the branches does not occur for real materials; therefore, perturbation theory can be used to find $k(\Omega)$. Because of the absence of anticrossing, one of the branches is always the coherent branch, and the other is the fluctuational branch. The dependence of the real and imaginary parts of the coherent branch on the frequency in first-order perturbation theory has the following form:

$$k_t-il_t^{-1}$$

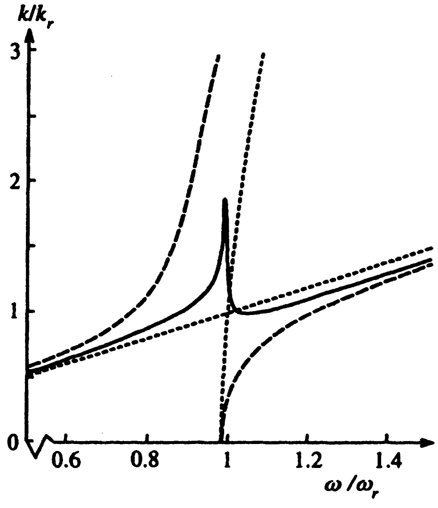

FIG. 3. Dependence of $k(\omega)$ for the coherent branch in the case of the existence of "anomalous" dispersion (solid line). The original dispersion curves (dotted lines) and the curves corresponding to $k_c=0$ (dashed lines) are also shown.

$$=\frac{\Omega}{v_t}\left[1-\tilde{\zeta}\omega_M\frac{1}{\Omega-\omega_0-\alpha\omega_M\Omega^2v_t^{-2}-2\sqrt{\kappa_s(\omega_0-\Omega)}}\right], \tag{28}$$

where k_t and l_t are the wave vector and the mean free path, respectively.

The fluctuational branch is poorly defined everywhere: its mean free path is of the order of the correlation radius; therefore, we shall not examine it.

The behavior of $k(\Omega)$ as a function of ζ and k_c differs qualitatively from the behavior of the dispersion curves, i.e., $\Omega(k)$ (as is seen from a comparison of Fig. 3 with the corresponding figures in Ref. 4). First, as we have already noted, the condition for the appearance of anticrossing of the solutions of $k(\Omega)$ is far more rigid than the condition for the dispersion curves, and, second, under certain conditions a segment of "anomalous" dispersion, where the sign of the derivative changes on the plot of $k(\Omega)$, appears on the coherent branch. The condition for the appearance of this "anomalous" dispersion can be obtained, if we find the values of ζ and k_c at which the inequality $(dk/d\Omega)|_{\Omega=\omega_r}\leq 0$ holds.

It is interesting to note that the condition

$$\tilde{\zeta}\omega_M\omega_r>8(\omega_M\alpha k_ck_r)^2 \tag{29}$$

coincides in order of magnitude with the condition for anticrossing of the dispersion curves in resonance.[4] We note that $k(\Omega)$ is also described well by the solution (28) obtained by perturbation theory when there is "anomalous" dispersion.

Let us now consider the damping of the wave. When $\Omega<\omega_0$ holds, the wave propagates without damping in the approximation under consideration, and when $\Omega>\omega_0$, holds the mean free path is given by the expression

$$l_t^{-1}=2\frac{\Omega}{v_t}\tilde{\zeta}\omega_M$$

$$\times\frac{\sqrt{\kappa_s(\Omega-\omega_0)}}{(\Omega-\omega_0-\alpha\omega_M\Omega^2 v_t^{-2}-\kappa_s)^2+4\kappa_s(\Omega-\omega_0)}. \quad (30)$$

The dependence of the reciprocal of the mean free path on the frequency has a pronounced resonant character and coincides qualitatively with the wave-vector dependence of the imaginary part of the frequency of the coherent eigenmodes in the case of an open gap on the dispersion curves.[4] At the same time, the following circumstance must be noted. When there is "anomalous" dispersion in the vicinity of the resonance at the frequencies $\omega_r \pm \Delta$, where Δ coincides with the magnitude of the gap between the dispersion curves in resonance (15), the damping is given by the expression

$$l_t^{-1} \sim v_t^{-1}\alpha\omega_M k_c\sqrt{k_r^2+k_c^2}, \quad (31)$$

which does not depend on the magnitude of the interaction. This expression coincides with the estimate of the mean free path obtained in Ref. 1 for states in the vicinity of a resonance when there is gap on the dispersion curve of the coherent elastic waves. In this sense, it can be stated that the appearance of the "anomalous" dispersion on $k(\Omega)$ corresponds to the case of an open gap on $\Omega(k)$.

Let us proceed to an examination of the evolution of the polarization state. A wave with polarization assigned at the origin of coordinates is the result of the superposition of all three fundamental waves with the specific weights determined by (16) and (17), where now, in contrast to Sec. 3, Ω is assumed to be assigned, and k must be replaced by the solutions of (25) corresponding to a particular branch.

Since the expressions for the general case of the initial polarization are cumbersome, we shall consider the special cases of a linearly polarized wave (with a polarization plane oriented parallel to the x axis at $z=0$), a circularly polarized wave, and a longitudinally polarized wave.

In the case of a wave which is linearly polarized from the onset, the expressions for the mean components of the elastic displacement vector have the following form:

$$U_x=\cos^2\theta e^{-iq_t z}+\cos^2 2\theta e^{-ik_2 z},$$

$$U_y=i\cos\theta\cos 2\theta(e^{-iq_t z}-e^{-ik_2 z}), \quad (32)$$

$$U_z=\sin 2\theta\cos 2\theta\frac{v_t^2}{v_l^2-v_t^2}\frac{\Delta k_t+l_t^{-1}}{q_t}(e^{-ik_2 z}-e^{-ik_3 z}),$$

where $k_2=k_t-il_t^{-1}$ and $k_3=k_l-il_l^{-1}$ are the wave vectors of the quasitransverse and quasilongitudinal waves, respectively, and Δk_t is a modification of the corresponding primal dispersion law. In writing (32) and below we omit the multiplier $e^{i\Omega t}$, which is common to all the components. Also, U_x and U_y represent the transverse component of the wave, and U_z represents the longitudinal component.

The wave is elliptically polarized in a plane perpendicular to **k**. The corresponding ellipse is characterized by an angle ϕ describing its orientation relative to the x axis and by the ratio between the semiaxes η. To determine them it is convenient to introduce, according to Ref. 5, the polarization coefficient p:

$$p=\frac{U_z}{U_y}=|p|e^{i\delta}. \quad (33)$$

Using it, we can write the following expressions for the characteristics of the ellipse:

$$\tan 2\phi=\frac{2|p|}{1-|p|^2}\cos\delta, \qquad \eta^2=\frac{|p|^2 2-\tan^2\phi}{1-|p|^2\tan^2\phi}. \quad (34)$$

At small distances the orientation of the ellipse varies in accordance with the usual law for the Faraday effect:

$$\phi=\frac{\cos\theta\cos 2\theta}{\cos^2\theta+\cos^2 2\theta}\Delta k_t z. \quad (35)$$

As is seen from (35), the specific Faraday rotation is determined by the modification Δk_t of the primal dispersion law $k=\Omega/\omega_t$; therefore, the observation of this effect can, in principle, enable us to measure this modification. At the sufficiently low frequencies $\Omega<\omega_0$ the dependence of the wave number of the wave interacting with the magnetic subsystem remains nearly linear with, however, a renormalized velocity $\tilde{v}_t$, which depends on the magnetic field H. The magnitude of this renormalization $\tilde{v}_t-v_t$ is given by the expression obtained in Ref. 4 for the case of $\theta=0$ following replacement of the interaction parameter ζ by the angle-dependent analog $\tilde{\zeta}$.

An experimental investigation of the Faraday effect makes it possible, in principle, to measure $\tilde{v}_t-v_t$ and its dependence on the magnetic field, which, as was pointed out in Ref. 4, makes it possible, in turn, to determine both the rms fluctuation and the correlation radius of the inhomogeneities of the magnetostriction parameter.

Because of the damping of the interacting component, the linear transverse component is transformed into an elliptic component with a ratio between its semiaxes

$$\eta=\frac{\cos\theta\cos 2\theta}{\cos^2\theta+\cos^2 2\theta}l_t^{-1}z. \quad (36)$$

At large distances the interacting components (including the longitudinal component) undergo exponentially rapid damping, and the polarization consequently coincides with the polarization of the noninteracting component described by (16).

It should, however, be noted that in this case a significant role is played by the fluctuational component of the coherence matrix, which we have neglected in the present work. Therefore, the noninteracting component with the polarization (16) will propagate superposed on this fluctuational component.

While the polarization properties of the transverse components of the wave have the standard character for ordinary magneto-optics, the fact that an elastic wave has a longitudinal component, for which there is no analog in magneto-optics, produces some new effects.

At distances much smaller than the mean free paths of the quasilongitudinal and quasitransverse waves the expression for the real part U_z can be rewritten in the form

$$U_z\sim\frac{\sqrt{\Delta k_t^2+l_t^{-2}}}{q_t}\sin\left(\frac{k_2-k_3}{2}z\right)\cos\times\left(\omega t-\frac{k_t+k_l}{2}z+\psi\right), \quad (37)$$

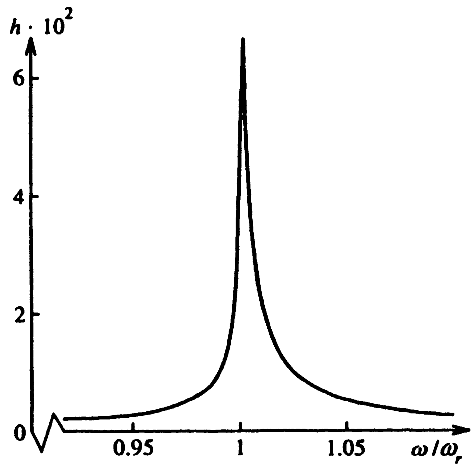

FIG. 4. Frequency dependence of h, which determines the ellipticity (40) and the amplitude of the longitudinal component (38) when a transverse wave propagates. This dependence has a qualitatively similar form for both normal and "anomalous" dispersion.

where $\tan\psi=-(l_t\Delta k_t)^{-1}$ (we have omitted the insignificant angular multipliers).

Thus, the longitudinal component propagates in this case with a velocity equal to the average of the velocities of the quasitransverse and quasilongitudinal components, and its phase is shifted relative to the phase of the transverse components by ψ, which is determined by the modification of k and the damping of the transverse waves. Beyond the resonance point, at which $\Delta k_t=0$ holds, the phase shift ψ varies abruptly by π.

The behavior of the amplitude of the longitudinal component, which exhibits a peculiar interference effect, is more interesting. As the coordinate z varies, this amplitude undergoes oscillations with a spatial period $L=\pi/(k_t-k_l)$, i.e, at certain distances the wave again becomes purely transverse, but, for the reasons stated above, it has nonlinear polarization. The maximum value, which it attains at $z_n=\pi(2n+1)/2L$, equals

$$U_z\sim\frac{\sqrt{\Delta k_t^2+l_t^{-2}}}{q_t}. \tag{38}$$

The characteristic dependence of $h=\sqrt{\Delta k_t^2+l_t^{-2}}/k_r$ on ω in the vicinity of the magnetoelastic resonance is presented in Fig. 4.

The resultant polarization of the wave is elliptical at a given z. The orientation of the polarization ellipse is specified by the angles ϕ_1, ϕ_2, and ϕ_3 (see Sec. 3), which depend in a complex manner on z, but since ϕ_1 and ϕ_2 are small (due to the smallness of the longitudinal component), qualitatively the behavior of the polarization ellipse is as follows: as z varies, the ellipse rotates around the direction of wave propagation, and, at the same time, it tilts itself around an axis which is perpendicular to $\mathbf{k}$ and turns together with the transverse component. We note that the rates of rotation of the ellipse and variation of its incline are not commensurate.

In the case of a wave which is circularly polarized at $z=0$, the expressions for the mean components of the elastic displacement vector have the form

$$\begin{aligned}U_x&=\cos\theta(\cos\theta+\cos 2\theta)e^{-iq_tz}+\cos 2\theta(\cos\theta\\&\quad-\cos 2\theta)e^{-ik_2z},\\U_y&=i\cos 2\theta(\cos\theta+\cos 2\theta)e^{-iq_tz}-i\cos\theta(\cos\theta\\&\quad-\cos 2\theta)e^{-ik_2z},\\U_z&=2\sin\theta\cos 2\theta(\cos 2\theta\\&\quad-\cos\theta)\frac{v_t^2}{v_l^2-v_t^2}\frac{\Delta k_t+il_t^{-1}}{q_t}(e^{-ik_2z}-e^{-ik_3z}).\end{aligned} \tag{39}$$

As we see, the longitudinal component behaves precisely as in the preceding case.

The transverse component also has the form of an ellipse, but in this case its characteristics depend differently on z. The corresponding expressions contain complicated terms which depend on θ; therefore, we present them only for the case of $\theta=\pi/2$. The variation of the ellipticity is confined to the deviation from circular polarization

$$\eta=1-z\sqrt{l_t^{-2}+\Delta k_t^2} \tag{40}$$

and is determined by the same coefficient as is the amplitude of the longitudinal modes in (38).

We note that although the variation of the ellipticity is associated both with the existence of damping and with a velocity difference, nevertheless, as is seen from Fig. 4, the ellipticity varies most rapidly at resonance, where the velocity difference equals zero because of the crossing of the dispersion curves of the transverse and quasitransverse waves. However, far from resonance, where l_t^{-1} is small, the main role is played by Δk_t, while in the vicinity of the resonance Δk_t is small and the resonant character of $\eta(\Omega)$ is specified by the corresponding dependence of the damping.

Although the transverse component has the form of a circle with an undetermined orientation at $z=0$, as the wave propagates the ratio between the semiaxes varies, and the angle of orientation is then given by the expression

$$\tan 2\phi=-2\frac{\exp(-z/l_t)}{1-\exp(-2z/l_t)}\sin\Delta k_tz. \tag{41}$$

A wave which is longitudinal at $z=0$ has the opposite type of behavior in a certain sense. The mean components of the elastic displacement vector for it have the following dependence on position:

$$\begin{aligned}U_x&=2\frac{\cos 2\theta}{\sin 2\theta}\frac{v_t^2}{v_l^2}\frac{\Delta k_l+il_l^{-1}}{q_l}(e^{-ik_2z}-e^{-ik_3z}),\\U_y&=i\frac{1}{\sin 2\theta}\frac{v_t^2}{v_l^2}\frac{\Delta k_l+il_l^{-1}}{q_l}(e^{-ik_2z}-e^{-ik_3z}),\\U_z&=\frac{v_l^2-v_t^2}{v_t^2}e^{-ik_3z}.\end{aligned} \tag{42}$$

The longitudinal component has the form of a wave which propagates with a velocity v_l and is damped in proportion to $\exp(-z/l_l)$.

The transverse component has the form of an ellipse with a ratio between its semiaxes equal to $\cos\theta/\cos 2\theta$ and a major semiaxis oriented parallel to the y axis.

Now the amplitude of the transverse modes (for $l_t^{-1} \ll k_r$ and $l_l^{-1} \ll k_r$) is

$$U_x \sim \frac{\sqrt{\Delta k_l^2 + l_l^{-2}}}{q_t} \sin\left(\frac{k_2 - k_3}{2} z\right), \tag{43}$$

i.e., it exhibits an interference effect similar to (38), which is also caused by the difference between the velocities of the quasitransverse and quasilongitudinal waves. The resultant polarization has the form of an ellipse with an orientation and parameters which depend in a complicated manner on z, but in view of the small magnitude (at short distances) of the transverse component, qualitatively the polarization is longitudinal with some tilting around the direction of wave propagation.

5. CONCLUSIONS

The properties of elastic waves in a ferromagnet with a random zero-mean magnetostriction parameter have been considered in this paper for an arbitrary angle θ between the wave vector $\mathbf{k}$ and the applied magnetic field. The presence of two types of resonance points, which correspond to crossing of the underlying dispersion curves of the transverse and longitudinal elastic waves with the dispersion curve of the spin waves, is new in comparison with the case of $\theta = 0$ considered in Ref. 4.

We analyzed the special features of the behavior of the modified dispersion curves in the vicinity of the longitudinal and transverse resonances and showed that these curves behave in qualitatively different manners as the angle θ varies. If the parameters of the material permit the appearance of a gap on the dispersion curve and two well defined branches of coherent transverse elastic modes for $\theta = 0$, the dispersion curve maintains the same form when $\theta \neq 0$. Conversely, the gap on the dispersion curve of the longitudinal waves for θ close to zero and $\pi/2$ is always "closed," and its appearance is possible only for θ close to $\pi/4$.

For $\theta \neq 0$ the question of the polarization properties of elastic waves in such a medium also becomes nontrivial. We analyzed two aspects of this problem (with neglect of the functional contribution to the polarization density matrix, which is permissible at distances much smaller than the mean free path). First, we investigated the fundamental polarization states of the coherent (averaged) elastic waves with an assigned wave vector k and a complex frequency Ω obtained from the solution of the dispersion equation, and, second, we investigated the evolution of a given initial polarization state as a wave with a given (real) frequency Ω propagates in a layer of the medium under consideration. Here the corresponding wave vector is determined from the dispersion equation and acquires an imaginary part, which determines the mean free path in the medium. Linear and circular transverse and longitudinal polarizations were considered as the initial polarizations. The first two cases correspond to the classical Faraday and Cotton–Mouton effects known in magneto-optics.

This formulation of the problem is of intereset because, as is well known in magneto-optics, polarization methods for measuring various effects are frequently far more sensitive than, for example, a direct investigation of dispersion curves, especially under the conditions of strong damping. Therefore, there is hope that polarization measurements of elastic waves in unordered ferromagnets with compensated magnetostriction will make it possible to more reliably determine the modification of the dispersion laws and the damping coefficients of elastic waves in such media. Such measurements are important because, as was noted in Ref. 4, knowledge of the magnetic-field and frequency dependences of these parameters will enable us to determine the principal characteristics of the magnetostriction in such materials: the rms fluctuation and the correlation radius of the magnetostriction parameter.

The relationship between the polarization and dispersion properties of coherent elastic waves has been established in the present work. We note that the presence of the longitudinal component resulted in an interesting modification of the Faraday and Cotton–Mouton effects: the polarization ellipse not only exhibits the standard behavior for these effects in the plane transverse to the direction of wave propagation, but also performs small oscillations about that plane, whose period is determined by the difference between the velocities of the longitudinal and transverse waves and whose amplitude is determined by the mean free path and the modification of the dispersion law [see (38)].

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