Coherent states of anharmonic oscillators with a quasiequidistant spectrum

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We introduce operators for the time-dependent Schrödinger equation that link the solutions of equations with different potentials. For steady-state solutions these reduce to Darboux transformations. With these transformations we build coherent states for Hamiltonians whose spectrum differs from that of harmonic oscillators in that between the ground-state level and the other part of the spectrum there is a forbidden band consisting of a finite number of levels.
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1. INTRODUCTION

We interpret a quasiequidistant spectrum as an equidistant spectrum with lacunae, or gaps, i.e., an equidistant spectrum with a finite number of "deleted" levels. A characteristic feature of potentials with such a spectrum is that any solution of the time-dependent Schrödinger equation with such a potential is a periodic function of time, which excludes the spreading of wave packets.[1] In the absence of lacunae we have an equidistant spectrum. The properties of potentials with an equidistant spectrum have been extensively studied (see, e.g., Refs. 2 and 3).

To build potentials with a specified spectrum one can use the methods of the inverse problem of quantum scattering theory,[4,5] which have been developed into a qualitative theory of spectrum control[6] and allow for obtaining potentials with preassigned spectra and for inserting or deleting levels in the spectrum. Note, however, that the simplest potential, whose spectrum consists of an equidistant part and individual ground-state levels separated from the other part of the spectrum by a gap consisting of two levels, was obtained from other considerations.[1] Dubov *et al.*,[7] remarked that the same potential could be obtained from the harmonic oscillator potential by a Darboux transformation and built more complicated examples of potentials that lead to a wider gap and to a gap above the two lower levels. The work of Adler[9] should also be noted, since the potential was obtained through a superposition of two Darboux transformations. A detailed analysis of the solutions of the Schrödinger equation for a potential whose ground-state level is separated from the equidistant part of the spectrum by four levels is given in Ref. 10, while the bifurcation of gap creation and annihilation in an equidistant spectrum and the analysis of the relation between the approach and the spectrum-shift Fock operators are examined in Ref. 11. We also note Ref. 12, which gives the general expressions for potentials with a quasiequidistant spectrum obtained by the Darboux transformation, and Ref. 13, where such potentials are studied.

The Darboux transformation and the integral transformation method of the inverse problem of quantum scattering theory constitute particular cases of more general constructions known from Delsart's paper[14] as transformation operators (see Ref. 5 for details). An analysis of the results of Ref. 7 from the standpoint of first-order differential transformation operators, of which the Darboux transformations are an example, is done in Ref. 15. The transformation operators are used to "dress" the Schrödinger operator (see, e.g., Refs. 16 and 17), a procedure that for a superposition of differential transformation operators leads to "dressing chains."[18] Anharmonic potentials that are regular in R^1 and the spectra that emerge because of different ways of dressing the harmonic-oscillator Hamiltonian are analyzed in Refs. 2, 11, and 18. The relationship between the method of the dressing chain for the Hamiltonian operator and the theory of the fourth and fifth Painlevé equations is analyzed in Ref. 18, and so are the spectral properties of dressed Hamiltonians.

We also note that the problem of classifying the exactly solvable problems of one-dimensional quantum mechanics by employing quadratic algebras is discussed in Refs. 19 and 20, and by employing the SU(1.1) algebra in Ref. 21.

The Hamiltonians with equidistant[3] and quasiequidistant[1,7,10,12,13] spectra have no continuous spectrum in the $L_2(R^1)$ space of functions that are square integrable in R^1. On the other hand, continuous base sets in $L_2(R^1)$ of the harmonic-oscillator coherent states are well-studied,[22] and so are the coherent states of varying-frequency oscillators with an external variable force acting on them,[23] oscillators that allow for a group-theoretic approach. The group-theoretic methods used in Ref. 23 cannot be directly employed in building a system of coherent states for the new Hamiltonians, since the differential spectrum-shift operators do not form a closed algebra in this case. Fernandez *et al.*,[24,25] were, apparently, the first to use different methods for obtaining coherent states for the family of isospectral potentials with an equidistant spectrum.

In the present paper we give the general expressions for potentials whose spectrum consists of an equidistant part and a ground-state level separated from the other part by a gap consisting of an arbitrary even number of levels. The coherent states are the nonstationary solutions of the time-dependent Schrödinger equation, and the potentials are obtained through a Darboux transformation from the harmonic-oscillator potential. Hence, if for the time-dependent

Schrödinger equation we introduce a transformation that is similar to the Darboux transformation and coincides with it in the case of stationary solutions, we can use it to obtain solutions in the form of coherent states for the new potentials. For potentials with a quasiequidistant spectrum such states are nonspreading wave packets. Two types of transformation are given in the paper, differential and integral transformations, and they coincide with the Darboux transformation in the case of stationary solutions of the time-dependent Schrödinger equation. When applied to nonstationary solutions of the time-dependent Schrödinger equation, they lead to various nonstationary states of the new Hamiltonians. This property is used to build a system of coherent states for Hamiltonians with a quasiequidistant spectrum.

2. DIFFERENTIAL TRANSFORMATIONS

We take the time-dependent Schrödinger equation describing the motion of a particle in a field with potential energy $-V_0(x,t)$:

$$[i\partial_t+\partial_x^2+V_0(x,t)]\psi(x,t)=0,$$
$$\partial_t=\partial/\partial t, \quad \partial_x^2=\partial_x\partial_x. \tag{1}$$

We call a particular solution $u(x,t)$ of this equation the transformation function and impose an addition condition on $u(x,t)$:

$$\left(\ln\frac{u}{u^*}\right)_{xxx}=0. \tag{2}$$

Here and in what follows $(\ldots)_x$ stands for the partial derivative with respect to x, the complex conjugate of u is denoted by u^*, and all arguments of the functions are dropped for the sake of brevity (except to avoid ambiguity). With steady-state potentials ($V_{0t}=0$) Eq. (1) always has solutions satisfying condition (2) (for instance, stationary states).

If condition (2) is met, the function

$$L_1=\exp\left[2\int dt\ \mathrm{Im}(\ln u)_{xx}\right] \tag{3}$$

depends only on the variable t, i.e., $L_1=L_1(t)$. For each solution of Eq. (1) $\psi(x,t)\neq u(x,t)$ we build a function $\varphi(x,t)$ via the first-order differential operator L:

$$\varphi(x,t)=L\psi(x,t),$$
$$L=L_1u^{-1}\begin{vmatrix} u & 1 \\ u_x & \partial_x \end{vmatrix}=L_1\left(-\frac{u_x}{u}+\partial_x\right). \tag{4}$$

Here and in what follows the operator determinants are interpreted as differential operators obtained by expanding the determinant in the last column with the functional coefficient placed before (to the left of) the operator for finding the derivative.

Simple calculations show that the function $\varphi(x,t)$ is the solution of a new Schrödinger equation,

$$[i\partial_t+\partial_x^2+V_1(x,t)]\varphi(x,t)=0, \tag{5}$$

whose potential $V_1(x,t)$ differs for that of the initial equation (1) by the function $A_1(x,t)=V_1(x,t)-V_0(x,t)$ defined in terms of the transformation function u as follows:

$$A_1(x,t)=(\ln|u|^2)_{xx}. \tag{6}$$

For $(\ln(u/u^*))_x=0$ and $L_1=1$ Eqs. (4) and (6) become the well-known Darboux transformation[8] for the steady-state Schrödinger equation. It is therefore natural to call transformation (4) the Darboux transformation for the time-dependent Schrödinger equation.

Formula (4) does not give a nontrivial solution of Eq. (5) for $\psi(x,t)=u(x,t)$. However, by analogy with the Darboux transformation, the solution of Eq. (5) in this case is the function

$$\varphi(x,t)=\frac{1}{L_1(t)u^*(x,t)}. \tag{7}$$

If this function is used as the transformation function for transforming the solutions of Eq. (5), the new potential difference differs from the initial difference (6) only in sign, with the result that we are back at the initial equation (1). The transformation operator in this case is $L^+=-L_1(u_x^*/u^*+\partial_x)$. It is also clear that successive application of the operators L^+ and L generally transforms one solution of Eq. (1) into another. Therefore, the operator L^+L is the symmetry operator of Eq. (1) and the operator LL^+ is the symmetry operator of Eq. (5). For stationary states, when the transformation function is an eigenfunction of the system Hamiltonian, these operators coincide with the Hamiltonian to within a constant term. This property of the Darboux transformation forms the basis for the well-known factorization method.[26]

Applying transformation (4) N times leads to a generalization of the Crum–Kreĭn formula[27,28] to the nonstationary case:

$$L^{(N)}=L_N(t)W^{-1}(u_1,\ldots,u_N)\times\begin{vmatrix} u_1 & u_2 & \cdots & 1 \\ u_{1x} & u_{2x} & \cdots & \partial_x \\ \vdots & \vdots & \ddots & \vdots \\ u_{1x}^{(N)} & u_{2x}^{(N)} & \cdots & \partial_x^{(N)} \end{vmatrix}, \tag{8}$$

where $u_{1x}^{(N)}=\partial^N u_1/\partial x^N$, $\partial_x^{(N)}=\partial^N/\partial x^N$, and $W(u_1,\ldots,u_N)$ is the Wronskian of the $u_1,\ldots,u_N$. The function $\varphi(x,t)=L^{(N)}\psi(x,t)$ is the solution of the new Schrödinger equation with the potential $V_N(x,t)=V_0(x,t)+A_N(x,t)$, where

$$A_N=(\ln|W(u_1,\ldots,u_N)|^2)_{xx}, \tag{9}$$

if the transformation functions $u_1,\ldots,u_N$, the linearly independent solutions of Eq. (1), are such that

$$\left(\ln\frac{W(u_1,\ldots,u_N)}{W^*(u_1,\ldots,u_N)}\right)_{xxx}=0. \tag{10}$$

The function $L_N(t)$ is given here by the formula

$$L_N(t)=\exp\left(2\int dt\ \mathrm{Im}[\ln W(u_1,\ldots,u_N)]_{xx}\right). \tag{11}$$

When the operator (8) is applied to any of the functions $u_1,\ldots,u_N$, the result is zero. However, Eq. (7) can be generalized to the case of N transformation functions. The functions

$$\varphi^{(k)}(x,t)=\frac{W^{(k)*}(u_1,\ldots,u_N)}{L_N(t)W^*(u_1,\ldots,u_N)},$$

where $W^{(k)}(u_1,\ldots,u_N)$ is the Wronskian of order $N-1$ built with the functions $u_1,\ldots,u_N$ with the exception of u_k, are the linearly independent solutions of the new Schrödinger equation.

The majority of potentials $V_0(x,t)$ of interest from the standpoint of physics are such that a Hilbert-space structure can be introduced using the solutions of Eq. (1). This usually means that Eq. (1) has an integral of motion (not necessarily the total energy) that is a self-adjoint second-order differential operator with a discrete spectrum. The set of the discrete-spectrum eigenfunctions $\psi_i(x,t)$ of this operator forms the base of the Hilbert space $L_2^0(R)$ of solutions of Eq. (1). By $R=[a,b]$ we denote the interval of variation of the variable x in Eq. (1), which may amount to the entire real axis. Usually the functions $\psi_i(x,t)$ are ordered in such a way that i is the number of their zeros.

Another requirement is that the potential difference $A_N(x,t)$ be a regular function in $[a,b]$. The new potential $V_N(x,t)$ in this case has no additional singularities in comparison to those of the initial potential $V_0(x,t)$. For the transformation (4) the potential difference is regular if the transformation u has no zeros in $[a,b]$. In the space $L_2^0(R)$ only the function $\psi_0(x,t)$ satisfies this condition. However, more possibilities for choosing the transformation functions exist outside this space.

A single transformation with the function $\psi_i(x,t)\in L_2^0(R)$ generates a potential difference with i singularities. We found that a repeated transformation with the function $\psi_{i+1}(x,t)\in L_2^0(R)$ removes all singularities and leads to a regular potential difference. For the steady-state Schrödinger equation this property was formulated by Kreĭn[28] and has been recently discussed in Ref. 9. Here we can interpret it as the property of the Wronskian built with the eigenfunctions $u_{k_i}=\psi_{k_i}(x,t)$: the Wronskian $W(u_{k_1},u_{k_2},\ldots,u_{k_N})$ preserves its sign in $[a,b]$ if $(k-k_1)(k-k_2)\cdots(k-k_N)\geq 0$ for all $k=0,1,2,\ldots$. This is true, in particular, if the functions u_{ki} are pairwise adjacent.

We also note that for stationary transformation functions the reality condition (10) is satisfied and the formulas (8) and (9) become the well-known formulas for the Darboux transformation applied consecutively N times.[27,28]

3. INTEGRAL TRANSFORMATIONS

Integral transformations for the steady-state Schrödinger equation have been thoroughly studied in the inverse problem of scattering theory (see, e.g. Refs. 4, 5, and 16), as have been their applications to the solution of nonlinear equations.[29] The relation of these transformations to the Darboux transformation is currently under discussion.[30,31] The first to note the possibility of using integral formulations in the Darboux method was, apparently Faddeev.[4] This is possible because the Wronskian of two solutions of the steady-state Schrödinger equation can be calculated in integral form as

$$W(\psi_1,\psi_2)=(E_2-E_1)\int^x \psi_1\psi_2 dx+C.$$

These transformations also allow for a generalization to the time-dependent Schrödinger equation. One such generalization was done in Ref. 32. In this section we introduce integral transformations of the time-dependent Schrödinger equation of a more general type.

Let the functions $u(x,t)$ and $\psi(x,t)$ satisfy Eq. (1). We define the function $w(u^*,\psi)$ by the following relationships:

$$w_x(u^*,\psi)=u^*\psi, \tag{12}$$

$$w_t(u^*,\psi)=-i(u_x^*\psi-u^*\psi_x). \tag{13}$$

Using Eq. (1), we establish that $w_{xt}=w_{tx}$, a property proving the consistency of Eqs. (12) and (13). There are two ways in which w can be calculated:

$$w=-i\int_{t_0}^{t}(u_x^*\psi-u^*\psi_x)dt+C_1(x), \tag{14}$$

$$w=\int_{x_0}^{x}u^*\psi\,dx+C_2(t). \tag{15}$$

To find the functions $C_1(x)$ and $C_2(t)$ we employ the relationships

$$C_1'(x)=(u^*\psi)_{t=t_0}, \tag{16}$$

$$C_2'(t)=-i(u_x^*\psi-u^*\psi_x)_{x=x_0}, \tag{17}$$

which are corollaries of Eqs. (12) and (13).

Let the transformation function u be a fixed solution of Eq. (1). Then, obviously, the set of solutions $\psi(x,t)$ for which either $C_1'(x)=0$ or $C_2'(t)=0$ holds forms a linear space. On this space we can define a linear operator M by the following relationship:

$$\varphi(x,t)=M\psi(x,t)=\frac{w(u^*,\psi)}{L_1(t)u^*}. \tag{18}$$

One can easily see that if the transformation function u meets the reality condition (2), the function (18) is a solution of the Schrödinger equation (5) with the potential difference specified by (6). The function $L_1(t)$ is calculated by employing (3). Note that the function $\varphi(x,t)$ defined in (7) is also a solution of Eq. (5).

Now we take N transformation functions $u_1,\ldots,u_N$ satisfying Eq. (1). The solution of the new Schrödinger equation (5) in this case is

$$\varphi(x,t)=\frac{w(\psi,u_1,\ldots,u_N)}{L_N(t)w(u_1,\ldots,u_N)}. \tag{19}$$

The definition of w depends on whether N is even or odd. For N even,

$$w(u_1,\ldots,u_N)=\begin{vmatrix} w(u_2^*,u_1) & w(u_4^*,u_1) & \cdots & w(u_N^*,u_1) \\ w(u_2^*,u_3) & w(u_4^*,u_3) & \cdots & w(u_N^*,u_3) \\ \vdots & \vdots & \ddots & \vdots \\ w(u_2^*,u_{N-1}) & w(u_4^*,u_{N-1}) & \cdots & w(u_N^*,u_{N-1}) \end{vmatrix}, \tag{20}$$

$$w(\psi,u_1,\ldots,u_N)=\begin{vmatrix} \psi & w(u_2^*,\psi) & w(u_4^*,\psi) & \cdots & w(u_N^*,\psi) \\ u_1 & w(u_2^*,u_1) & w(u_4^*,u_1) & \cdots & w(u_N^*,u_1) \\ u_3 & w(u_2^*,u_3) & w(u_4^*,u_3) & \cdots & w(u_N^*,u_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{N-1} & w(u_2^*,u_{N-1}) & w(u_4^*,u_{N-1}) & \cdots & w(u_N^*,u_{N-1}) \end{vmatrix}. \tag{21}$$

For N odd,

$$w(u_1,\ldots,u_N)=\begin{vmatrix} u_1^* & w(u_1^*,u_2) & w(u_1^*,u_4) & \cdots & w(u_1^*,u_{N-1}) \\ u_3^* & w(u_3^*,u_2) & w(u_3^*,u_4) & \cdots & w(u_3^*,u_{N-1}) \\ u_5^* & w(u_5^*,u_2) & w(u_5^*,u_4) & \cdots & w(u_5^*,u_{N-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_N^* & w(u_N^*,u_2) & w(u_N^*,u_4) & \cdots & w(u_N^*,u_{N-1}) \end{vmatrix}, \tag{22}$$

$$w(\psi,u_1,\ldots,u_N)=\begin{vmatrix} w(u_1^*,\psi) & w(u_3^*,\psi) & \cdots & w(u_N^*,\psi) \\ w(u_1^*,u_2) & w(u_3^*,u_2) & \cdots & w(u_N^*,u_2) \\ \vdots & \vdots & \ddots & \vdots \\ w(u_1^*,u_{N-1}) & w(u_3^*,u_{N-1}) & \cdots & w(u_N^*,u_{N-1}) \end{vmatrix}. \tag{23}$$

The expressions for the function $L_N(t)$ and the potential difference $A_N(x,t)$ and the reality condition coincide with (11), (9), and (10) with the function W replaced by w.

The functions $w(u_i^*,u_j)$ used to build the determinants (20)–(23) and defined in (14) and (15) depend on constants, C_{ij}. These constants can be used to ensure the regularity of the potential difference A. Similar constants in the function $w(u_i,\psi)$ can be set to zero without loss of generality, since another set of solutions of Eq. (5) is specified by the functions

$$\varphi^{(k)}(x,t)=\frac{L_N(t)w^{(k)}(u_1,\ldots,u_N)}{w(u_1,\ldots,u_N)},$$

where the $w^{(k)}(u_1,\ldots,u_N)$ are calculated by Eq. (22) for even N and by Eq. (20) for odd N with the set of functions $u_1,\ldots,u_N$ from which u_k is excluded.

For even N there is always the possibility of selecting the transformation functions so that $u(u_1,\ldots,u_N)$ is a real function. To do this we must put $u_{2i}=u_{2i-1}$, $i=1,\ldots,N/2$, for arbitrary functions u_{2i-1} and select the constants C_{ij} so that the matrix of the determinant (20) is Hermitian. Then its determinant is real. If the transformation functions satisfy the condition $w_i(a,t)=0$ and the $u_{ix}'(a,t)$ are bounded functions, $C_2'(t)=0$ holds for all transformation functions u_i and the functions $\psi\in L_2^0(R)$ at $x_0=a$. In this case Eqs. (15) and (19)–(21) determine the transformation studied in Ref. 32.

Now let us discuss a particular case of these transformations when only stationary functions of the type $\psi(x,t)\equiv\psi_E(x,t)=e^{-iEt}\psi_E(x)$ are involved in solving Eq. (1). Obviously, if $\psi_E\in L_2^0(R)$, for $u_\alpha\notin L_2^0(R)$, $\alpha<E_{\min}$, and $E-\alpha>0$ the condition $C_1'(x)=0$ is met for all ψ_E with $t_0=-i\infty$. Equations (14) and (18) define the integral transformation operator for all $\psi\in L_2^0(R)$. The function w defined in (14) is proportional to the Wronskian of the functions $u(x)$ and $\psi(x)$ and Eq. (18) defines, to within a constant factor, the same Darboux transformation as the differential operator (4). Equations (19)–(20) in this case can be used to obtain integral representations, known from the inverse problem of quantum scattering theory,[4,5,16,29] that link the solutions of two steady-state Schrödinger equations. This problem, however, lies outside the scope of the present article.

4. HARMONIC-OSCILLATOR COHERENT STATES

Below we give information about the coherent states of a harmonic oscillator $V_0(x)=-x^2/4$.

The creation and annihilation operators $\hat{a}^+(t)$ and $\hat{a}(t)$, which are integrals of motion for the harmonic oscillator, belong to the Lie algebra of first-order symmetry differential operators isomorphic to the Lie algebra of the Schrödinger equation of a free particle:

$$\begin{aligned} &\hat{a}(t)=e^{it}(\partial_x+x/2), \\ &\hat{a}^+(t)=e^{-it}(-\partial_x+x/2), \\ &\hat{a}(t)\hat{a}^+(t)-\hat{a}^+(t)\hat{a}(t)=1. \end{aligned} \tag{24}$$

Coherent states defined as the eigenfunctions of the annihilation operator are solutions in separated variables of the time-dependent Schrödinger equation:

$$\psi_z(x,t)=(2\pi)^{-1/4}\exp\left(-\frac{1}{4}x^2-\frac{1}{2}it+zx-\frac{1}{2}z^2-\frac{1}{2}zz^*\right),$$
$$\hat{a}(t)\psi_z(x,t)=\zeta\psi_z(x,t),\quad z=\zeta e^{-it}. \tag{25}$$

The transition to an oscillator with an arbitrary frequency ω can be achieved by substituting $2\omega x^2$ for x^2 and $2\omega t$ for t. The functions $\psi_z(x,t)$ can be expanded in the complete set (in $L_2(R)$) of the orthonormalized stationary oscillator states $|n\rangle$, which in the coordinate representation have the form

$$\psi_n(x,t)=(2\pi)^{-1/4}(n!)^{-1/2}\times\exp\left[-\frac{x^2}{4}-i\left(n+\frac{1}{2}\right)t\right]\mathrm{He}_n(x),$$
$$\mathrm{He}_n(x)=2^{-n/2}H_n\left(\frac{x}{\sqrt{2}}\right), \tag{26}$$

where $H_n(z)$ are the Hermite polynomials,[33] and

$$|z;t\rangle=\exp\left(-\frac{zz^*}{2}-\frac{it}{2}\right)\sum_{n=0}^{\infty}\frac{z^n}{n!}|n\rangle. \tag{27}$$

It is easy to calculate the average coordinate and momentum and their dispersion in the states $|z;t\rangle$. For instance, the average energy $\langle E\rangle$ is equal to $zz^*+\frac{1}{2}$. In many problems the Fock–Bargmann representation proves useful (see, e.g., Ref. 23).

5. ANHARMONIC OSCILLATORS WITH A QUASIEQUIDISTANT SPECTRUM AND THEIR COHERENT STATES

For the transformation function we take the following solution of Eq. (1) with $V_0(x,t)=-x^2/4$:

$$u(x,t)=P_{2k}(x)\exp\left[\frac{x^2}{4}+i\left(2k+\frac{1}{2}\right)t\right],$$
$$P_k(x)=(-i)^k\,\mathrm{He}_k(ix),\quad k=0,1,2,\ldots. \tag{28}$$

Note that $H_0u(x,t)=-(2k+\frac{1}{2})u(x,t)$, where $H_0=\partial_x^2+\frac{1}{4}x^2$ is the harmonic-oscillator Hamiltonian, and the functions (28) do not belong to the space $L_2^0(R)$. For the function $C_2'(t)$ calculated by Eq. (17) it proves impossible to select a point x_0 such that $C_2'(t)=0$ holds for all the functions $\psi_n(x,t)$. On the other hand, $C_1'(x)\equiv 0$ holds for all $\psi_n(x,t)\in L_2^0(R)$ if we put $t_0=-i\infty$. Hence Eqs. (18) and (14) at $t_0=-i\infty$ determine the integral operator M in $L_2^0(R)$.

The transformation function (28) generates a family of exactly solvable stationary potentials obtained earlier in Ref. 12:

$$V_1(x,t)\equiv V^{(2k)}(x,t)=-\frac{x^2}{4}+4k(2k-1)\frac{P_{2k-2}(x)}{P_{2k}(x)}-8k^2\left[\frac{P_{2k-1}(x)}{P_{2k}(x)}\right]^2. \tag{29}$$

The wave functions of the stationary states of these potentials can be obtained through both differential (4) and integral (18) transformations, whose action on the functions $\psi_n(x,t)$ differs by an unimportant (in this case) factor

$$\varphi_{n+1}(x,t)=L\psi_n(x,t)=(2\pi)^{-1/4}(n!)^{-1/2}\times\exp\left[-\frac{x^2}{4}-i\left(n+\frac{1}{2}\right)t\right]\times\left[\mathrm{He}_n(x)\frac{P_{2k+1}(x)}{P_{2k}(x)}-n\,\mathrm{He}_{n-1}(x)\right]. \tag{30}$$

We find the ground-state function of the new Hamiltonian $H^{(2k)}=H_0+V^{(2k)}(x,t)$ from (7):

$$\varphi_0(x,t)=\exp\left[-\frac{x^2}{4}+i\left(2k+\frac{1}{2}\right)t\right]P_{2k}^{-1}(x). \tag{31}$$

The symmetry operator L^+L coincides, to within a constant term, with the harmonic-oscillator Hamiltonian:

$$L^+L=H_0+2k+1/2,$$

and LL^+ with the Hamiltonian of an anharmonic oscillator with the potential (29):

$$LL^+=H^{(2k)}+2k+1/2.$$

This property makes it possible to easily calculate the normalization integral for the function (30):

$$\langle\varphi_{n+1}|\varphi_{n+1}\rangle=\langle\psi_n|L^+L|\psi_n\rangle=2k+n+1. \tag{32}$$

The normalization integral for the ground state can be calculated by direct integration:

$$\langle\varphi_0|\varphi_0\rangle=\frac{\sqrt{2\pi}}{(2k)!}. \tag{33}$$

The spectrum of the Hamiltonian $H^{(2k)}$ consists of an equidistant part coinciding with the harmonic-oscillator spectrum $E_{n+1}=n+1/2$, $n=0,1,2,\ldots$, and a separately positioned ground-state level $E_0^{(2k)}=-2k-1/2$.

For $k=1$ these functions have been thoroughly studied in Refs. 1 and 7. We believe, however, that Eq. (4.19) of Ref. 1 has an error, which resulted from a mistake in the expression for the normalization of the functions involved. Note that Dubov *et al.*,[7] give another expression for the normalization coefficient, which coincides with our value. For $k=2$ the functions (30) and (31) have been studied in Ref. 10.

We denote the functions (normalized to unity) of the stationary states of the Hamiltonian $H^{(2k)}$ by Θ_n, $n=0,1,2,\ldots$. These functions form an orthonormalized base of the space $L_2^1(R)$ of the solutions of Eq. (5).

Since the operators L [Eq. (4)] and M [Eq. (18)] link the solutions of two Schrödinger equations, the following relation holds:

$$K[i\partial_t+\partial_x^2+V_0(x,t)]=[i\partial_t+\partial_x^2+V_1(x,t)]K, \quad K=L,M.$$

Their reciprocity in the respective spaces can also be established. The inverse of operator (4) has the form of (18) and vice versa, and in both cases we must substitute $1/u^*$ for u. Hence, if g is the symmetry operator for Eq. (1), we can build two independent symmetry operators for Eq. (5), $\bar{h}=LgL^{-1}$ and $\tilde{h}=MgM^{-1}$. It is also evident that if the operator sets $\{g\}$, $\{\hat{h}\}$, and $\{\tilde{h}\}$ form algebras G, $\bar{H}$, and $\tilde{H}$, all are isomorphic to each other. But if, say, G is a Lie algebra of differential operators, then $\bar{H}$ and $\tilde{H}$ are integro-differential representations of the same algebra.

Thus, for the creation and annihilation operators $\hat{a}^+$ and $\hat{a}$ specified by (24), which are integrals of motion for the harmonic oscillator, we can set up the following operators:

$$\hat{b}^+=L\hat{a}^+L^+, \quad \hat{b}=L\hat{a}L^+, \quad \bar{b}^+=L\hat{a}^+L^{-1},$$

$$\bar{b}=L\hat{a}L^{-1}, \quad \tilde{b}^+=M\hat{a}^+M^{-1}, \quad \tilde{b}=M\hat{a}M^{-1}.$$

Since we have $L^+L=H_0+2k+\frac{1}{2}$, the differential operators $\hat{b}^+$ and $\hat{b}$ are spectrum-shift operators for the Hamiltonian $H^{(2k)}$ but cannot be interpreted as creation and annihilation operators since their commutator is not equal to unity. On the other hand, $\bar{b}$, $\bar{b}^+$, $\tilde{b}$, and $\tilde{b}^+$ are integro-differential operators that possess all the properties of annihilation and creation operators, respectively. It is also clear that the functions $\varphi^-{}_a=L\psi_a$ and $\tilde{\varphi}_a=M\psi_a$ are eigenfunctions of, respectively, the operators $\bar{b}$ and $\tilde{b}$ and can therefore be interpreted as different coherent states of the new Hamiltonians.

Let us examine the integral transformation (18) in greater detail. Applying the operator M to the coherent states (25), we obtain, to within a constant factor, the function

$$\tilde{\varphi}_z(x,t)\equiv\varphi_z(x,t)=\exp\left[-\frac{x^2}{4}-\frac{zz^*}{2}-i\left(2k+\frac{1}{2}\right)t\right]$$
$$\times\left[\frac{P'_{2k}(x)}{P_{2k}(x)}+x-\frac{\partial}{\partial x}\right]\frac{\partial^{2k}}{\partial x^{2k}}I(z,x),$$

$$I(z,x)=\sqrt{\frac{\pi}{2}}e^{x^2/2}\operatorname{erf}\left(\frac{z-x}{\sqrt{2}}\right), \tag{34}$$

which describes the coherent states of the Hamiltonian $H^{(2k)}$. The expansion (27) makes it possible to obtain an expansion of the function (34) in the system of states $\{\Theta_i\}$:

$$\varphi_z(x,t)=(2\pi)^{1/4}e^{-zz^*/2}\left[e^{-it/2}\zeta^{2k+1}\right.$$
$$\left.\times\sum_{s=0}^{\infty}\frac{z^s}{\sqrt{(2k+s+1)s!}}\Theta_{s+1}+\sqrt{(2k)!}\Theta_0\right]. \tag{35}$$

Note that at $\zeta=0$ this function coincides with the ground-state function Θ_0. Using the expansion (35), we can calculate the normalization integral for the function φ_z,

$$\langle\varphi_z|\varphi_z\rangle=\sqrt{2\pi}(2k)!\sum_{s=0}^{2k}\frac{(-1)^s}{s!}(zz^*)^s, \tag{36}$$

and the average energy,

$$\langle\varphi_z|H^{(2k)}|\varphi_z\rangle=\sqrt{2\pi}(zz^*)^{2k+1}-(2k+\tfrac{1}{2})\langle\varphi_z|\varphi_z\rangle. \tag{37}$$

The function (34) is expressed in terms of the error function $\operatorname{erf}(z)$. The explicit expressions for $\varphi_z(x,t)$, however, contain only elementary functions. In particular, at $k=0$ we have a harmonic oscillator whose energy origin is shifted upward by unity, and the formulas (29)–(31) and (34)–(37) become the respective oscillator formulas; for instance, from (37) we get

$$\langle\varphi_z|H^{(0)}|\varphi_z\rangle=\sqrt{2\pi}\left(zz^*-\frac{1}{2}\right).$$

At $k=1$ the function (34) has the form

$$\varphi_z(x,t)=N_z\left[z^2+\frac{2(1-zx)}{1+x^2}\right]\exp\left(-\frac{1}{4}x^2+zx-\frac{1}{2}zz^*\right.$$
$$\left.-\frac{1}{2}z^2+\frac{5}{2}it,\right. \tag{38}$$

where

$$N_z=\frac{1}{(2\pi)^{1/4}\sqrt{1+(1-zz^*)^2}}.$$

The average values of the coordinate and momentum in the states (38) are expressed in terms of the complementary error function $\operatorname{erfc}(z)$: for instance,

$$\langle x\rangle=z+z^*+\sqrt{2\pi}\,\frac{1}{1+(1-zz^*)^2}\exp\left[\frac{1}{2}-\frac{1}{2}(z\right.$$
$$\left.+z^*)^2\right]\operatorname{Re}\left[e^{-i(z+z^*)}(i-z-z^*\right.$$
$$\left.-izz^*)\operatorname{erfc}\frac{1-iz-iz^*}{\sqrt{2}}\right].$$

The differential transformation leads to another system of coherent states; for instance, for $k=1$ we have the states

$$\bar{\varphi}_z(x,t)=L\psi_z(x,t)=(2\pi)^{-1/4}\left(z-x-\frac{2x}{1+x^2}\right)$$
$$\times\exp\left(-\frac{1}{4}x^2+zx-\frac{1}{2}zz^*-\frac{1}{2}z^2-\frac{1}{2}it\right),$$

which at $\zeta=0$ coincide with the function of the first excited state of the Hamiltonian $H^{(2)}$ with the property $\bar{b}\Theta_1=0$.

6. CONCLUSION

We found that by employing the more general formulas (8) and (20)–(23) it is possible to obtain exactly solvable Hamiltonians with a more complicated quasiequidistant spectrum and to build systems of coherent states for such Hamiltonians.

The transformations introduced in this paper can be used to construct coherent states of other systems; for instance, isospectral Hamiltonians generated from the oscillator Hamiltonian by an integral transformation.[3] The transforma-

tions can also be used to derive time-dependent potentials and build exact solutions of the Schrödinger equations with such potentials.

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