Multiple interaction of solitary waves in $\lambda\phi^4$ theory and Arnold diffusion

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Resonant structures formed when a kink is scattered by an antikink ($K\bar{K}$) in the range of initial kink velocities $0.18<V<0.26$ were previously detected in the classical scalar $\lambda\phi^4$ field theory in $1+1$ dimensions. In addition, it was shown that $K\bar{K}$ scattering has a "quasifractal" character in the space of initial kink velocities. A bound state for kink–antikink–kink ($K\bar{K}K$) scattering was also found in a numerical experiment. In the present work the range of initial kink velocities $0.005<V<0.9$ is investigated numerically for antisymmetric $K\bar{K}K$ systems. It is found that bound states of a $K\bar{K}K$ system exist for initial kink velocities $V<0.72$, and resonant structures like those obtained in $K\bar{K}$ scattering are detected in the $0.72<V<0.764$ range. This points out the possibility of the existence of a special solution in $\lambda\phi^4$ theory at the critical point $V_{cr}\approx 0.26$ for the scattering of a kink by an antikink or at the point $V_{cr}\approx 0.764$ for $K\bar{K}K$ scattering. The emission accompanying the interaction of a $K\bar{K}K$ system is also recorded. Arnold diffusion, which randomizes the process of bound-state formation, occurs in the case of asymmetric systems or for four or more colliding kinks (or antikinks). Three- and four-soliton interactions are considered within $\lambda|\phi|^4$ theory in the $(1+1)$-dimensional case, and the existence of a critical velocity and the formation of a three-soliton (SSS) bound state is noted. © *1996 American Institute of Physics.* [S1063-7761(96)03103-X]

1. INTRODUCTION

The problem of the interaction of the solitary waves (kinks) of the scalar $\lambda(\phi^2-1)^2$ field theory arose for the first time in the cosmological problem of the creation of domain walls.[1] The interaction of domain walls in solid-state physics and the interaction of vacancies in *trans*-polyacetylene chains[2] can also be cited as additional, physically interesting consequences of kink–antikink ($K\bar{K}$) interactions. Some numerical and theoretical results of the interaction of a kink with an antikink were considered in Refs. 3 and 4 within the classical $(1+1)$-dimensional $\lambda\phi^4$ theory. Consideration of the internal excitation of nonlinear solitary waves resulted in a resonance picture of $K\bar{K}$ interaction.[5,6]

As further investigations demonstrated,[7,8] whether the process ends with the formation of a bound state or the kink and antikink fly apart after the interaction depends "quasifractally" on the initial velocity of the colliding kinks. Further details of the already classic problem of the interaction of solitary waves in nonlinear equations which are not integrable by the inverse scattering problem method can be found in Ref. 9.

There is special interest in the interactions of a large number of solitary waves [kinks in the scalar $\lambda\phi^4$ theory, solitons in $\lambda|\phi|^n$ theory ($n=4$, 6), etc.], since they offer one possible way to obtain special solutions, if they exist.

2. SCALAR $\lambda\phi^4$ THEORY

The present work is a continuation of the systematic investigations of the multiple resonant interaction of solitary waves (kinks) in the nonintegrable scalar $\lambda\phi^4$ field theory. The model considered here is described by a Lagrangian of the form

$$\mathscr{L}(x,t)=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{\lambda}{4}(\phi^2-1)^2. \quad (1)$$

As we know (see, for example, Ref. 9), among the solutions $\phi(x,t)$ of the Euler equation for the Lagrangian (1) ($\lambda=1$),

$$\phi_{tt}-\phi_{xx}-\phi+\phi^3=0, \quad (2)$$

there are two vacuum solutions $\phi_\pm=\pm 1$ and solutions in the form of a solitary wave, i.e, a kink (K) or an antikink ($\bar{K}$):

$$\phi_{K(\bar{K})}=\pm\tanh\frac{x-x_0}{\sqrt{2}}, \quad (3)$$

both kinks being topologically stable. The multiple interaction of kinks (the system must always represent an alternating sequence of kinks and antikinks) is prescribed by the following initial conditions:

$$\phi(x,0)=\tanh[(x-x_0)\beta]-\tanh\frac{x}{\sqrt{2}}+\tanh[(x+x_0)\beta],$$
$$\phi_t(x,0)=-V\beta\{\tanh^2[(x-x_0)\beta]-\tanh^2[(x+x_0)\beta]\}, \quad (4)$$

where V is the velocity of the kink at $x\to\infty$ (in units of the velocity of light) and is a parameter in the calculations ($\beta\equiv 1/\sqrt{2(1-V^2)}$) for the interaction in a $K\bar{K}K$ system (Fig. 1a); for a $K\bar{K}K\bar{K}$ system we have the initial condition

$$\phi(x,0)=\tanh[(x-x_1)\beta_1]-\tanh[(x-x_0)\beta_0]+\tanh[(x+x_0)\beta_0]-\tanh[(x+x_1)\beta_1]+1. \quad (5)$$

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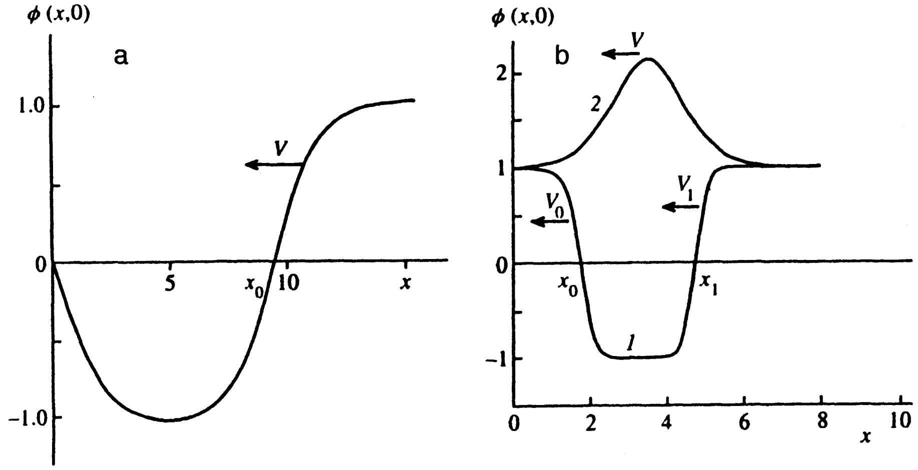

FIG. 1. Initial conditions for $K\bar{K}K$ (a) and $K\bar{K}K\bar{K}$ (b) systems. The right-hand part of each system is shown. The left-hand part is the antisymmetric (a) or symmetric (b) continuation on the $x<0$ semiaxis. Curve *1* in panel b corresponds to the initial condition (5), and curve *2* corresponds to an initial condition in the form of the bound state of the $K\bar{K}$ system.

Here $\beta_0 \equiv 1/\sqrt{2(1-V_0^2)}$, $\beta_1 \equiv 1/\sqrt{2(1-V_1^2)}$, V_0 and V_1, i.e., the initial velocities, and x_0 and x_1, i.e., the initial positions of the kinks and antikinks, are parameters in the calculations (Fig. 1b, curve *1*).

Problems (2), (4) and (2), (5) conserve the energy integral

$$E=\int_{-\infty}^{\infty} dx W(x,t)=\int_{-\infty}^{\infty} dx\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{1}{4}(\phi^2-1)^2\right], \tag{6}$$

which provides a good check of numerical accuracy.

The preceding research and investigations of the bound state in the $K\bar{K}K$ system showed[4] that such a state exists in the range of initial kink velocities $V=0.75\pm0.03$ (in units of the velocity of light). In the present work the range of initial velocities $0.05<V<0.9$ was numerically investigated with a spacing $\Delta V=10^{-4}$ for the $K\bar{K}K$ system, and it was found that a $K\bar{K}K$ bound state exists for $V<0.72$. In the range $0.72<V<0.764$ there are resonant structures like those detected in the $K\bar{K}$ interaction in Refs. 5 and 6.

Equation (2) was investigated numerically by the method of characteristics[10] with the initial conditions (4); a similar procedure was previously employed in Ref. 11. The results obtained for the dependence of the energy flux F on the initial velocity V are presented in Fig. 2. The energy flux through the plane $x_f=15$ during the time interval $\Delta t_f=150$ was calculated in the following manner:

$$F(V)|_{x_f=15}=-\int_0^{t_f} dt\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}. \tag{7}$$

The kink rest mass is given by

$$M_K=\int_{-\infty}^{\infty} dx\left[\frac{1}{2}\left(\frac{\partial\phi_K}{\partial x}\right)^2+\frac{1}{4}(\phi_K^2-1)^2\right]=\frac{2\sqrt{2}}{3}.$$

It is seen in Fig. 2 that the energy flux during the time $\Delta t_f=150$ is less than the kink rest mass plus the kink kinetic energy for initial kink velocities $V<0.72$ (a kink passing through the plane x_f at infinity was detected at this initial velocity for the first time in our problem). Thus we have a bound $K\bar{K}K$ state with an easily detectable flux. In the range of initial kink velocities $0.72<V<0.764$ there is no longer a monotonic dependence of the energy flux on V (part of this range is shown in Fig. 3). Here we have resonant structures like those detected in Ref. 5 and investigated in Ref. 6 (see

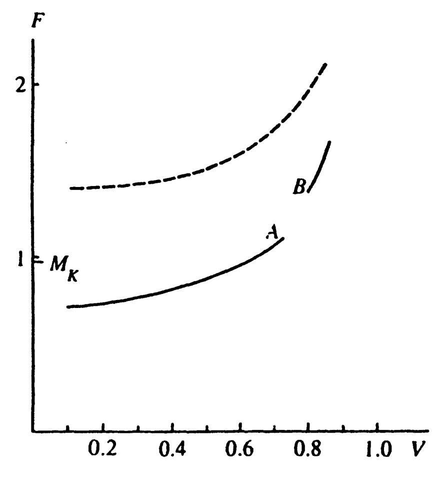

FIG. 2. Energy flux through the plane $x_f=15$ during the time $\Delta t_f=150$ for a $K\bar{K}K$ system as a function of the initial kink velocity V. Between points A and B there is a region of resonant structures. The dashed line is a plot of the total energy of the $K\bar{K}K$ system on the $x>0$ semiaxis calculated from Eq. (6) for $t=0$.

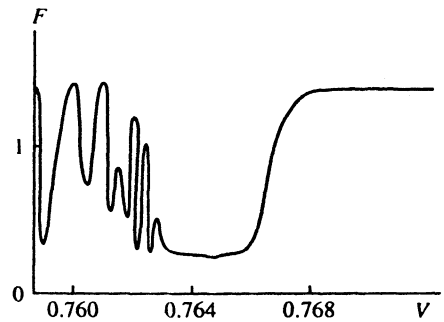

FIG. 3. Energy flux similar to that shown in Fig. 2 for the region near the critical velocity, which is a limit point for resonances.

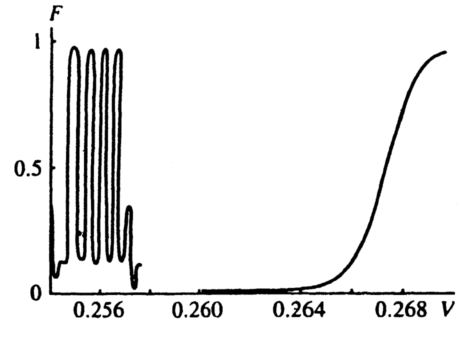

FIG. 4. Energy flux through the plane $x_f = 10$ plane during the time $\Delta t_f = 200$ for a $K\bar{K}$ system as a function of the initial kink velocity V. The region near the resonance limit point at $V_{cr} \simeq 0.2598$ is shown (since the resonant "windows" are extremely narrow, they have been partially omitted).

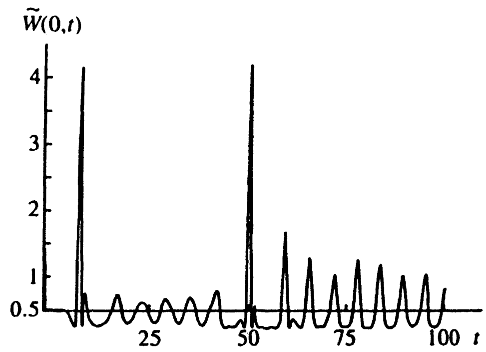

FIG. 6. Energy density of a $K(K+\bar{K}+1)$ system as a function of time for $V_K = -V_{\bar{K}} = 0.7600$. The resonant interaction in such a system is similar to the interaction in the $K\bar{K}K$ system (see Fig. 5b).

Fig. 4, which presents the region of initial kink velocities in the $K\bar{K}$ interaction near the critical velocity). "Quasifractal" structures are also represented in the $K\bar{K}K$ interaction, as in the $K\bar{K}$ interaction[7] (a comparison of Figs. 3 and 4 leads to such a conclusion). Moreover, the interaction specified by the initial conditions (4) for a $K\bar{K}K$ system with a distance between the kinks and antikinks tending to infinity is similar to scattering in the $K(K+\bar{K}+1)$ system, which is represented at $V = 0.7600$ in Fig. 5 for the former case and in Fig. 6 for the latter case. This phenomenon can be called "quasisupersymmetry," if a "fermion charge" equal to

$$\xi_K = \frac{1}{4}[\phi_K(+\infty) - \phi_K(-\infty)] \tag{8}$$

is ascribed to a kink. Then the $K\bar{K}K$ interaction will be treated as a "three-fermion" interaction, and the $K(K+\bar{K}+1)$ interaction will be treated as a "two-fermion" interaction in the field of the third kink, but with a distance between kinks tending to infinity. The escape of a kink to infinity can be determined in Fig. 7a from the nonmonotonic behavior of the energy flux in the $K\bar{K}K$ interaction: if a kink escapes, a jump (in the plane x_f) equal to (or greater than) the kink rest mass is observed in the energy flux. Here (see Figs. 5 and 7) the behavior of a resonant $K\bar{K}K$ system with a number of energy density oscillations at the center $n_\nu = 7$ and an initial kink velocity $V = 0.7600$ in the case of resonance (Fig. 5) and $V = 0.7590$ for the formation of a bound state (Fig. 7) is presented for comparison; the difference in the energy flux through $x_f = 15$ is clearly seen.

A bound state was previously observed[4] at an initial kink velocity $V \simeq 0.75 \pm 0.03$. It can be assumed from our calculations that one of the long-lived resonant structures, such as, for example, the one presented in Fig. 8 for the initial velocity $V = 0.7646$, was observed there. It is seen that a highly excited kink leaves the interaction region.

A typical example of a $K\bar{K}K\bar{K}$ interaction is presented in Fig. 9. A $K\bar{K}$ bound state (Fig. 1b, curve 2) is taken as the initial state, which then collides with a similar state. Here the initial velocity of the system is $V_{K\bar{K}} = 0.9$. A bound state is not observed in the $K\bar{K}K\bar{K}$ system in our numerical calculations, since it becomes stochastically unstable due to the appearance of an additional degree of freedom not present in a $K\bar{K}$ or $K\bar{K}K$ system (the same explanation also applies to a larger number of interacting kinks and antikinks).

In this case the lifetime of a bound state with a large number of degrees of freedom can be determined by utilizing the Arnold diffusion formula[12]

$$\tau_D \sim \omega_0^{-1} \varepsilon^{-1} \exp(\varepsilon^{-a}). \tag{9}$$

Here a is a function of the number of degrees of freedom N ($a = 2/(12\zeta + 3N + 14)$, where $\zeta \geq N(N-1)/2$), ω_0 is the unperturbed oscillation frequency (here it is assumed that it is equal to the oscillation frequency in the bound $K\bar{K}$, $K\bar{K}K$, or, if they exist, multi-$K\bar{K}$ systems), and ε is a parameter which appears in the Hamiltonian when there is a perturbation (the ratio of the internal perturbation of the kink to the total energy of the interacting kinks can be used here). The bound state is stochastically stable, if the number of degrees of freedom is at most two and the resonances do not

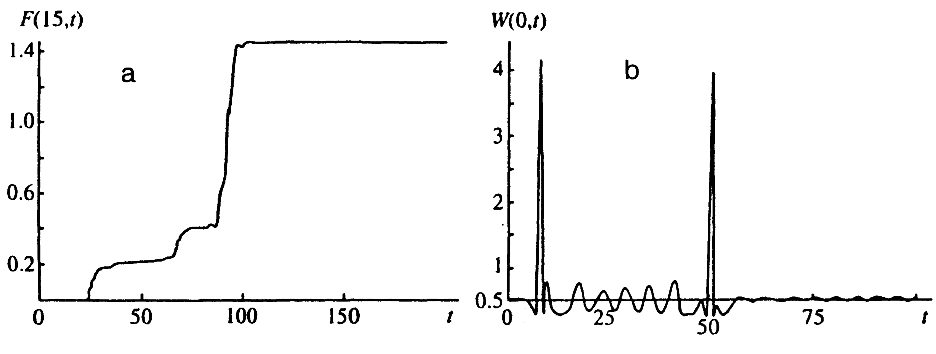

FIG. 5. a) Energy flux as a function of time for a $K\bar{K}K$ interaction with an initial kink velocity $V = 0.7600$. A jump in the energy flux of the system greater than the kink rest mass M is clearly visible. b) Energy density of a $K\bar{K}K$ system as a function of time for $V = 0.7600$ at $x = 0$ and a resonant structure with a number of internal oscillations of the field of the $K\bar{K}K$ system $n_\nu = 7$.

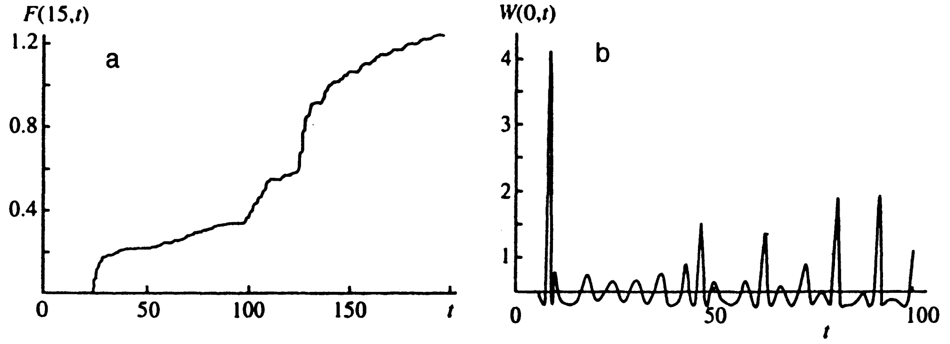

FIG. 7. a) Energy flux as a function of time for a $K\bar{K}K$ system when $V=0.7590$. The bound $K\bar{K}K$ state formed after the first collision is shown. b) Energy density of a $K\bar{K}K$ system as a function of time for $V=0.7590$ at $x=0$ (the center of mass of the system). The high peaks correspond to collisions of kinks with an antikink.

overlap.[12] This condition holds for a $K\bar{K}$ system (with neglect of the internal perturbations) far from the critical velocity $V_{cr}\approx 0.259$ (see Fig. 4) and for an antisymmetric $K\bar{K}K$ system far from $V_{cr}\approx 0.764$ (see Fig. 3). However, if a $K\bar{K}K\bar{K}$ system is considered, there are always at least three degrees of freedom; therefore, even having "frozen" the internal oscillations of the kinks, we obtain stochastic instability for the system. (It should be noted here that, for example, crystal-like structures will be stochastically stable, since they can be described by a dynamical system with a small number of parameters, which can be the lattice constants of the crystal in certain cases.)

3. COMPLEX $\lambda\phi^4$ THEORY

Solitary waves in scalar charge fields have been discussed repeatedly in the literature. For example, in Ref. 13 they were called Q balls, in Ref. 14 they were called solitons, in Ref. 15 they were called Q lumps, and in Ref. 16 they were called nontopological solitons. Since charge is conserved in such theories, the existence of stable solitons can be expected in both 3+1 (Ref. 17) and 1+1 dimensions (Ref. 18). The interaction of solitons of the Q ball type with the formation of a bound state was considered in Ref. 19.

Here we consider the nonlinear equations for $\lambda|\phi|^n$ theory ($n=4$, 6), for which a stochastically stable bound state can be obtained, if the Arnold diffusion is taken into account. Let us consider the nonlinear Klein–Gordon equation for $\lambda|\phi|^4$ theory in Ref. 13 (here $\lambda\equiv\mu^2$) in 1+1 dimensions:

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+m^2\phi-\mu^2\phi|\phi|^2=0. \tag{10}$$

The soliton solution of this equation is well known:[13]

$$\phi_S=\frac{\sqrt{2}}{\mu}\,\frac{\sqrt{m^2-\omega^2}e^{i\omega t}}{\cosh(x\sqrt{m^2-\omega^2})}, \tag{11}$$

where ω is the frequency of the complex field, m and μ are parameters of the problem, and the stable solutions are confined to the range $m/\sqrt{2}\leqslant\omega\leqslant m$ (Refs. 17 and 18). The soliton charge for the solution (11) equals

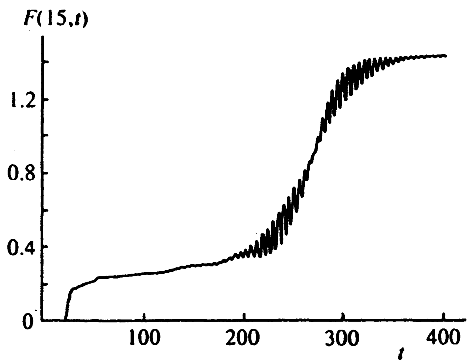

FIG. 8. Energy flux through the plane $x_f=15$ plane as a function of time for a long-lived resonant structure. The field of the escaping kink is strongly perturbed, as is seen from the flux oscillations. The initial kink velocity was $V=0.7646$.

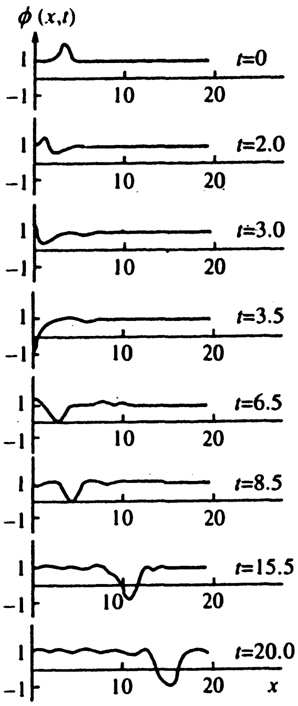

FIG. 9. Temporal evolution of a $K\bar{K}K\bar{K}$ system. Two bound $K\bar{K}$ states were selected as the initial conditions. The initial velocity of the $K\bar{K}$ system was equal to 0.9. The small oscillations of the field near the vacuum ($\phi=1$) are a result of the emission accompanying the collision of $K\bar{K}$ systems. The figure is symmetric about $x=0$.

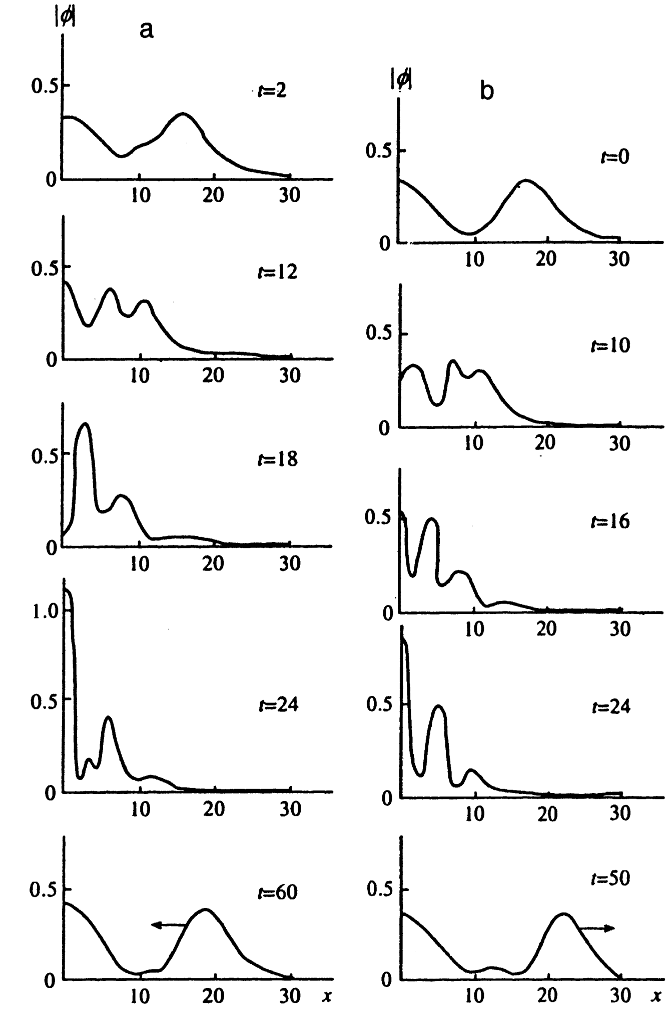

FIG. 10. a) Formation of a three-soliton bound state for an initial soliton velocity $V_{S1}=0.70$ and $\omega_1=\omega_2=\omega_3=0.972$; b) absence of a bound SSS state for $V_{S1}=0.78$. The solitons leave the interaction region (calculations for $\lambda|\phi|^4$ theory; here $\lambda\equiv\mu^2=1$, $m=1$, $x_{01}=-x_{03}=18$ is the initial position of a soliton). The arrow shows the beginning of secondary collapse (a) or escape of a soliton (b). The figures are symmetric about $x=0$.

$$Q_S=\frac{8\omega}{\mu^2}\sqrt{m^2-\omega^2}. \tag{12}$$

If we change the sign of the frequency ($\omega\rightarrow-\omega$), the sign of the soliton charge is reversed, i.e., we have an antisoliton with charge

$$Q_A=-Q_S.$$

Thus, both multisoliton interactions and soliton–antisoliton interactions can be considered. As was shown in Ref. 19, the bound state of two solitons (SS) in $\lambda|\phi|^4$ theory exists in a symmetric system with $\omega_1=\omega_2=0.95$ when the initial velocities are $V_1=-V_2=0.25$. In the present work we show that a bound state (or, more likely, a resonant structure) exists in a system of three solitons (SSS) when $\omega_1=\omega_2=\omega_3=0.972$, $V_{S1}=-V_{S3}=0.70$, $V_{S2}=0$, and $m=\mu=1$ (see Fig. 10a). For comparison, Fig. 10b shows the temporal evolution of a similar system with initial velocities $V_{S1}=-V_{S3}=0.78$ and $V_{S2}=0$, and it is seen that the solitons leave the interaction region after the first collision. All the solitons were chosen from the stability region.[18] We have as yet detected no four-soliton bound systems, even for symmetric systems up to $V=0.9$, where V is the velocity of the $K\bar{K}$ system at $t=0$ (calculations with large values of V require a great deal of computer time). However, it stands to reason that in this case, too, the addition of an extra degree of freedom would render the system stochastically unstable.

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