Duality relation for the two-dimensional Ising model at finite lattice dimensions

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It is shown that the partition function of the two-dimensional Ising model on a dual lattice of finite dimensions with periodic boundary conditions can be expressed in terms of a certain combination of partition functions on the original lattice with different boundary conditions. A generalization to the inhomogeneous case is given, and a proof is presented for a weakly inhomogeneous distribution of the coupling constants when the system has arbitrary finite dimensions. © *1996 American Institute of Physics.* [S1063-7761(96)02603-5]

1. INTRODUCTION

The duality transformation for the two-dimensional Ising model was discovered in 1941 by Kramers and Wannier.[1] They succeeded in establishing the correspondence between the partition function of the model in the low-temperature phase and the partition function on the dual lattice in the high-temperature phase

$$(\sinh 2K)^{-N/2} Z(K)=(\sinh 2\tilde{K})^{-N/2}\tilde{Z}(\tilde{K}),$$

$$\sinh 2K\cdot\sinh 2\tilde{K}=1. \tag{1}$$

This property of self-duality made it possible, in particular, to determine the critical temperature even before an exact solution of the Onsager model[2] had been obtained.

In Ref. 3 the Kramers–Wannier duality relation (1) was generalized to the inhomogeneous case, in which the coupling constants are arbitrary functions of the lattice coordinates:

$$\prod_{r,i}(\sinh K_i(r))^{-1/4}Z[K_i(r)]$$

$$=\prod_{\tilde{r},i}(\sinh \tilde{K}_i(\tilde{r}))^{-1/4}\tilde{Z}[\tilde{K}_i(\tilde{r})], \tag{2}$$

$$\sinh 2K_1(r)\cdot\sinh 2\tilde{K}_2(\tilde{r})=1, \tag{3}$$

$$\sinh 2K_2(r)\cdot\sinh 2\tilde{K}_1(\tilde{r})=1.$$

The notation of the coordinates and the parameters on the original and dual lattices is precisely defined in the next section (see Fig. 1). The Kadanoff–Ceva ansatz (2) is extremely informative: it establishes an equality between functionals, rather than functions, in contrast to the Kramers–Wannier duality relation (1). For example, the duality relation (2) can be used to determine the disorder variable μ in a physically and mathematically correct manner, to derive expressions which relate the correlation functions on the original and dual lattices, to define the "mixed" correlation functions $\langle\sigma(r_i)\ldots\sigma(r_j),\mu(r_k)\ldots\mu(r_l)\rangle$, etc. Using (2), Kadanoff and Ceva[3] obtained the duality relation between the two-point correlation functions of the spin variables and the disorder variables on the original and dual lattices.

Nevertheless, as was noted by the authors themselves,[1,3] the relations (1) and (2) cannot be taken literally. For example, when (1) is derived by the method of comparing the high- and low-temperature expansions, it is difficult to take into account and compare graphs which include spins on boundaries (particularly those that contain loops that embrace a torus in the most "popular" periodic boundary conditions). On the basis of general physical arguments it can be assumed that the equality (1) is valid in the thermodynamic limit, i.e, for the specific free energy. However, in the inhomogeneous variant (2) the very procedure of going to the thermodynamic limit is quite indefinite. In either case it would be very useful to have exact equalities unlike (1) and (2), relating partition functions in mutually dual lattices when the system has finite dimensions. This is the subject of the present work.

The notation and a representation of an inhomogeneous Ising model in the form of a Grassmann functional integral are introduced in Sec. 2. An exact duality relation for the homogeneous case on a finite lattice with periodic boundary conditions is derived in Sec. 3. It is found that the partition function of the model on the original lattice can be expressed in terms of a certain combination of partition functions on the dual lattice with different boundary conditions. In Sec. 4 a generalization is made to the inhomogeneous case, and a proof is presented for a weakly inhomogeneous distribution of the coupling constants. An exact duality relation between pairwise correlation functions of the order and disorder variables on finite lattices is derived in Sec. 5.

2. THE MODEL

The partition function of the 2D Ising model with interaction between nearest neighbors on a rectangular lattice stretched over a torus with the dimension $N=n\times m$ has the form

$$2^N Z[K]=\sum_{[\sigma]}e^{-\beta H[\sigma]}. \tag{4}$$

$$-\beta H[\sigma]=\sum_r \sigma(r)(K_1(r)\nabla_x+K_2(r)\nabla_y)\sigma(r). \tag{5}$$

The Ising spin takes two values $\sigma(r)=\pm 1$; the coordinates of the lattice points $r=(x,y)$ run through the values (see Fig.

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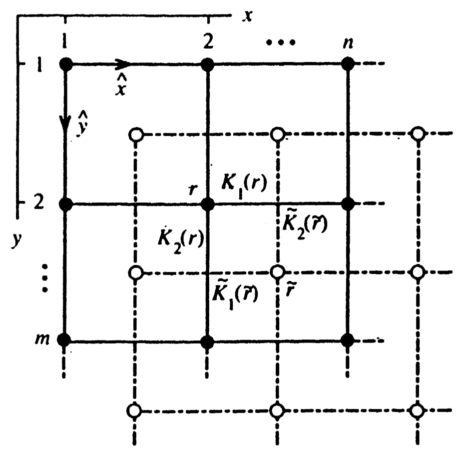

FIG. 1. Correspondence between the coordinates and coupling constants in mutually dual lattices. The lattice dimension is $N=n\times m$. The coordinates are $r=(x,y)$ and $\tilde{r}=r+(\hat{x}+\hat{y})/2$, where $\hat{x}$ and $\hat{y}$ are unit vectors.

1) $x=1,\ldots,n$ and $y=1,\ldots,m$; the coupling constants along the horizontal and vertical bonds $K_1(r)$ and $K_2(r)$ are arbitrary functions of the coordinates in the general case. The shift operators ∇_x and ∇_y act on $\sigma(r)$ in the following manner:

$$\nabla_x\sigma(r)=\sigma(r+\hat{x}),\quad \nabla_y\sigma(r)=\sigma(r+\hat{y}),$$

$$\nabla_x^T=\nabla_{-x},\quad \nabla_y^T=\nabla_{-y};$$

where $\hat{x}$ and $\hat{y}$ are unit vectors along the x and y axes. For boundary conditions which are periodic with respect to x and y

$$(\nabla_x^p)^n=1,\quad (\nabla_y^p)^m=1,$$

and for antiperiodic boundary conditions

$$(\nabla_x^a)^n=-1,(\nabla_y^a)^m=-1,$$

where the superscript p denotes periodic boundary conditions and the superscript a denotes antiperiodic boundary conditions.

Bearing in mind the four possible alternatives for the boundary conditions on a torus, we introduce the corresponding superscripts into the statistical sum $Z^{(\alpha,\beta)}[K]$: for example,

$$2^N Z^{(a,p)}[K]=\sum_{[\sigma]}\exp\left(\sum_r \sigma(r)(K_1(r)\nabla_x^a + K_2(r)\nabla_y^p)\sigma(r)\right).$$

Henceforth it will be more convenient for us to regard $Z^{(\alpha,\beta)}[K]$ as a four-component vector $\mathbf{Z}[K]$ with the components $Z_b[K]$, where $b=1, 2, 3, 4$

$$\mathbf{Z}[K]=(Z^{(p,p)},Z^{(p,a)},Z^{(a,p)},Z^{(a,a)}). \tag{6}$$

The coordinates, as well as the corresponding functions and functionals on the dual lattice, are marked with a tilde:

$$\tilde{r},\ \tilde{\sigma}(\tilde{r}),\ \tilde{K}_1(\tilde{r}),\ \tilde{H}[\tilde{\sigma}],\ \tilde{Z}[\tilde{K}],\ldots$$

The coordinates of the points of the dual lattice coincide with the coordinates of the centers of the plaquettes of the original lattice and vice versa (see Fig. 1):

$$\tilde{r}=r+(\hat{x}+\hat{y})/2.$$

In these notations the dual partition fiunction and Hamiltonian, unlike (4) and (5), have the form

$$2^N\tilde{Z}[\tilde{K}]=\sum_{[\tilde{\sigma}]} e^{-\beta\tilde{H}[\tilde{\sigma}]},$$

$$-\beta\tilde{H}[\tilde{\sigma}]=\sum_r \tilde{\sigma}(\tilde{r})(\tilde{K}_1(\tilde{r})\nabla_{-x}+\tilde{K}_2(\tilde{r})\nabla_{-y})\tilde{\sigma}(\tilde{r}).$$

The coupling constants $K_i(r)$ and $\tilde{K}_i(\tilde{r})$ are related to one another by the (local) duality condition (3).

The fact that the partition function of the 2D Ising model can be represented in the form of a functional integral with respect to a real fermion field has been known for a long time.[4,5] For the problem which we are considering it is important that such a representation can be rigorously derived for the case of a finite inhomogeneous lattice:[6–8]

$$Z^{(p,p)}[K]=\frac{1}{2}(-Q^{(p,p)}[K]+Q^{(p,a)}[K]+Q^{(a,p)}[K] +Q^{(a,a)}[K]). \tag{7}$$

Here $Q^{(\alpha,\beta)}[K]$ is a functional integral over a four-component Grassmann field with a Gaussian distribution:

$$Q^{(\alpha,\beta)}[K]=\left(\prod_{r,i}\cosh K_i(r)\right)\int \mathscr{D}\psi\, \exp(S^{(\alpha,\beta)}[\psi]), \tag{8}$$

where

$$\mathscr{D}\psi=\prod_r\prod_{j=1}^4 d\psi_j(r),$$

and the $\psi_j(r)$ are the Grassmann variables:

$$\{\psi_i(r),\psi_j(r')\}=0.$$

The action in (8) has the form

$$S^{(\alpha,\beta)}[\psi]=\sum_r (\mathscr{L}^{(0)}(\psi(r))+\mathscr{L}^{(\alpha,\beta)}(\psi(r))), \tag{9}$$

$$\mathscr{L}^{(0)}(\psi(r))=\sum_{1\le i<j}^4 \psi_i(r)\psi_j(r),$$

$$\mathscr{L}^{(\alpha,\beta)}(\psi(r))=-t_1(r)\psi_3(r)\nabla_x^\alpha\psi_1(r) +t_2(r)\psi_2(r)\nabla_y^\beta\psi_4(r),$$

$$t_i(r)\equiv \tanh K_i(r).$$

The representation (7) was written for boundary conditions which are periodic with respect to x and y in the Hamiltonian (5). Nevertheless, it is obvious that it is valid for any combination of (periodic or antiperiodic) shift operators in the Ising Hamiltonian. For example, $Z^{(a,p)}$ is distinguished from $Z^{(p,p)}$ only by changes in sign in all the coupling constants $K_1(r)$ of the horizontal bonds in the extreme right-hand column of the lattice, i.e.,

$$Z^{(p,p)} \to Z^{(a,p)} \quad \text{when} \quad K_1(n,y) \to -K_1(n,y),$$

and thus

$$t_1(n,y) \to -t_1(n,y). \tag{10}$$

However, the transition (10) is nothing but a change in the boundary conditions of the shift operator in the action (9)

$$\nabla_x^p \leftrightarrow \nabla_x^a.$$

Therefore, the expression for $Z^{(a,p)}$ differs from $Z^{(p,p)}$ only with respect to the arrangement of the plus and minus signs in front of the terms in (7):

$$Z^{(a,p)}[K] = \frac{1}{2}(Q^{(p,p)}[K] + Q^{(p,a)}[K] - Q^{(a,p)}[K] + Q^{(a,a)}[K]). \tag{11}$$

There are expressions similar to (7) and (11) for the other two possible alternatives of the boundary conditions on the torus ($Z^{(p,a)}$, $Z^{(a,a)}$). Using the definition (6), we can write all four representations for the partition functions with different boundary conditions in the following compact form

$$\mathbf{Z}[K] = \hat{R}\mathbf{Q}[K],$$

where, in analogy to (6), we introduced the vector

$$\mathbf{Q} = (Q^{(p,p)}, Q^{(p,a)}, Q^{(a,p)}, Q^{(a,a)})$$

and the matrix $\hat{R}$, which has the form

$$\hat{R} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad \hat{R}^2 = 1. \tag{12}$$

The integral (8) can be expressed in terms of the Pfaffian of an antisymmetric matrix D of dimensionality $4N \times 4N$, which assigns the anticommutative quadratic form (9) of $S(\psi)$

$$S^{(\alpha,\beta)}[\psi] = \frac{1}{2}(\psi, D^{(\alpha,\beta)}\psi),$$

where

$$D^{(\alpha,\beta)} = \begin{pmatrix} 0 & 1 & 1 + t_1(r-\hat{x})\nabla_{-x}^{\alpha} & 1 \\ -1 & 0 & 1 & 1 + t_2(r)\nabla_y^{\beta} \\ -1 - t_1(r)\nabla_x^{\alpha} & -1 & 0 & 1 \\ -1 & -1 - t_2(r-\hat{y})\nabla_{-y}^{\beta} & -1 & 0 \end{pmatrix}. \tag{13}$$

In these notations

$$Q^{(\alpha,\beta)}[K] = \left(\prod_{r,i} \cosh K_i(r)\right) \mathrm{Pf}(D^{(\alpha,\beta)}). \tag{14}$$

There are similar representations for the dual lattice

$$\tilde{\mathbf{Z}}[\tilde{K}] = \hat{R}\tilde{\mathbf{Q}}[\tilde{K}],$$

$$\tilde{Q}^{(\alpha,\beta)}[\tilde{K}] = \left(\prod_{\tilde{r},i} \cosh \tilde{K}_i(\tilde{r})\right) \mathrm{Pf}(\tilde{D}^{(\alpha,\beta)}),$$

where the matrix $\tilde{D}$, unlike (13), has the following form:

$$\tilde{D}^{(\alpha,\beta)} = \begin{pmatrix} 0 & 1 & 1 + \tilde{t}_1(\tilde{r})\nabla_{-x}^{\alpha} & 1 \\ -1 & 0 & 1 & 1 + \tilde{t}_2(\tilde{r}-\hat{y})\nabla_y^{\beta} \\ -1 - \tilde{t}_1(\tilde{r}+\hat{x})\nabla_x^{\alpha} & -1 & 0 & 1 \\ -1 & -1 - \tilde{t}_2(\tilde{r})\nabla_{-y}^{\beta} & -1 & 0 \end{pmatrix}, \tag{15}$$

$$\tilde{t}_i(\tilde{r}) \equiv \tanh \tilde{K}_i(\tilde{r}).$$

We note in concluding this section that the Pfaffians of the matrices (13) and (15) are polynomials of finite degree in $t_i(r)$ or $\tilde{t}_i(\tilde{r})$ when the lattice has finite dimensions. In the high-temperature limit $\mathrm{Pf}(D)$ is calculated by direct integration of (8), because the cross terms created by $\mathscr{S}^{(\alpha,\beta)}(\psi)$ vanish:

$$\mathrm{Pf}(D) = 1 \quad \text{when} \quad t_i(r) = 0. \tag{16}$$

In precisely the same manner in the low-temperature limit we have

$$\mathrm{Pf}(\tilde{D})=1 \quad \text{when} \quad \tilde{t}_i(\tilde{r})=0. \tag{17}$$

3. THE HOMOGENEOUS CASE

In the general case of arbitrary dimensions m and n and irregular distributions of the coupling constants $K_i(r)$, it is impossible to express either the Pfaffian or the determinant of the matrix D in any closed analytic form. Conversely, in the homogeneous case, in which

$$K_i(r)=K_i=\text{const},$$

this can easily be done, since the matrix D becomes translationally invariant, and its determinant can be calculated by means of Fourier transformation:

$$\det(D)=\prod_{\mathbf{p}} [(1+t_1^2)(1+t_2^2)-2t_1(1-t_2^2)\cos p_x - 2t_2(1-t_1^2)\cos p_y],$$

where $t_i \equiv \tanh K_i$. Here we have not written out the superscripts (α, β), implying that, depending on the boundary conditions for the shift operators ∇_x and ∇_y, the projections of the quasimomentum $\mathbf{p}=(p_x,p_y)$ run through integral or half-integral values, respectively, in units of $2\pi/n$ and $2\pi/m$.

Taking into account that

$$(\mathrm{Pf}(D))^2=\det(D), \tag{18}$$

for $Q[K]$ we find from (14) and (18)

$$Q^2(K_1,K_2)=\prod_{\mathbf{p}} (c_1c_2-s_1 \cos p_x - s_2 \cos p_y), \tag{19}$$

where

$$c_i \equiv \cosh 2K_i, \quad s_i \equiv \sinh 2K_i.$$

Similarly, for the dual lattice

$$\tilde{Q}^2(\tilde{K}_1,\tilde{K}_2)=\prod_{\mathbf{p}} (\tilde{c}_1\tilde{c}_2-\tilde{s}_1 \cos p_x - \tilde{s}_2 \cos p_y), \tag{20}$$

where

$$\tilde{c}_i \equiv \cosh 2\tilde{K}_i, \quad \tilde{s}_i \equiv \sinh 2\tilde{K}_i.$$

Comparing (19) and (20) to one another, we find that

$$(s_1s_2)^{-N/2}Q^2(K_1,K_2)=(\tilde{s}_1\tilde{s}_2)^{-N/2}\tilde{Q}^2(\tilde{K}_1,\tilde{K}_2), \tag{21}$$

if

$$s_1\tilde{s}_2=1, \quad s_2\tilde{s}_1=1.$$

However, (21) is none other than the duality relation for the squares of the functional integrals, in terms of which the individual terms can be expressed in the representation of the partition function (7), rather than the duality relation for the partition functions. Taking the square roots of both sides of the equality (21), we obtain

$$(s_1s_2)^{-N/4}Q^{(\alpha,\beta)}(K_1,K_2)=\pm(\tilde{s}_1\tilde{s}_2)^{-N/4}\tilde{Q}^{(\alpha,\beta)}(\tilde{K}_1,\tilde{K}_2). \tag{22}$$

We now show that the minus sign in (22) applies only to the functions $Q^{(p,p)}$ and that the plus sign applies to the remaining components of the vectors $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$, i.e.,

$$(s_1s_2)^{-N/4}\mathbf{Q}(K_1,K_2)=(\tilde{s}_1\tilde{s}_2)^{-N/4}\hat{g}\tilde{\mathbf{Q}}(\tilde{K}_1,\tilde{K}_2), \tag{23}$$

where the singular matrix $\hat{g}$ has the form

$$\hat{g}=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since, as was previously noted, $\mathrm{Pf}(D)$ is a polynomial in t_i, $\det(D)$ is the perfect square of that polynomial, as can easily be seen from the explicit expressions (19) and (20). In fact, all the multipliers appear in the product (19) in pairs according to the quasimomentum projections $\pm p_x$ and $\pm p_y$. The only exceptions are the multipliers corresponding to $p_x=p_y=\pi$ and $p_x=p_y=0$. We use q_π and q_o to denote them:

$$q_\pi=c_1c_2+s_1+s_2=(1+t_1+t_2-t_1t_2)^2 \cosh^2 K_1 \times \cosh^2 K_2,$$

$$q_0=c_1c_2-s_1-s_2=(1-t_1-t_2-t_1t_2)^2 \cosh^2 K_1 \times \cosh^2 K_2=(1-s_1s_2)^2/q_\pi.$$

Hence it is seen that all the multipliers in the products determining the functions $Q^{(\alpha,\beta)}$ (apart from $Q^{(p,p)}$) are sign-invariant in the range of variation of the parameters

$$s_1 \geqslant 0_1, \quad s_2 \geqslant 0_2,$$

and thus the functions themselves do not change sign. Conversely, the function $Q^{(p,p)}$, which contains the multiplier $(q_0)^{1/2} \sim (1-s_1s_2)$, changes sign upon passage through the critical line $s_1s_2=1$. We clearly arrive at the same conclusions regarding the dual functions $\tilde{Q}^{(\alpha,\beta)}$. Therefore, with consideration of the limiting values (16) and (17) for the Pfaffians in the high- and low-temperature limits, we find

$$\mathrm{Sgn}(Q^{(p,p)}(K_1,K_2))=\mathrm{Sgn}(1-s_1s_2),$$

$$\mathrm{Sgn}(\tilde{Q}^{(p,p)}(\tilde{K}_1,\tilde{K}_2))=\mathrm{Sgn}(1-\tilde{s}_1\tilde{s}_2)=-\mathrm{Sgn}(Q^{(p,p)} \times (K_1,K_2)),$$

thereby proving the equality (23).

Multiplying the right- and left-hand sides of (23) by the matrix $\hat{R}$ (12), we obtain the duality relation sought for the partition functions

$$(s_1s_2)^{-N/4}\mathbf{Z}(K_1,K_2)=(\tilde{s}_1\tilde{s}_2)^{-N/4}\hat{T}\tilde{\mathbf{Z}}(\tilde{K}_1,\tilde{K}_2), \tag{24}$$

where

$$\hat{T}=\hat{R}\hat{g}\hat{R}, \quad \hat{T}^2=1,$$

$$\hat{T}=\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \hat{R}^2=1. \tag{25}$$

It is seen from (24), for example, that the difference between the right- and left-hand sides of the Kramers–Wannier duality relation (1)

$$(s_1s_2)^{-N/4}Z^{(p,p)}(K_1,K_2)-(\tilde{s}_1\tilde{s}_2)^{-N/4}\tilde{Z}^{(p,p)}(\tilde{K}_1,\tilde{K}_2)=$$
$$=(\tilde{s}_1\tilde{s}_2)^{-N/4}\tilde{Q}^{(p,p)}(\tilde{K}_1,\tilde{K}_2)=-(s_1s_2)^{-N/4}Q^{(p,p)}$$
$$\times(K_1,K_2) \qquad (26)$$

does not vanish for any values of K_i, m, and n with the exception of the critical line

$$\sinh 2K_1\cdot \sinh 2K_2=1.$$

Moreover, away from this line (more precisely, outside the scaling region) the right- and left-hand sides of (26) become equal as the dimensions increase $(m,n\to\infty)$. Therefore, the duality relation in the form (1) should be understood only in the sense of a thermodynamic limit, i.e.,

$$\lim_{m,n\to\infty}\left(\frac{1}{N}\ln\left(\frac{Z(K_1,K_2)}{(s_1s_2)^{N/4}}\right)\right)=\lim_{m,n\to\infty}\left(\frac{1}{N}\ln\left(\frac{\tilde{Z}(\tilde{K}_1,\tilde{K}_2)}{(\tilde{s}_1\tilde{s}_2)^{N/4}}\right)\right).$$

The duality relation (24) was derived for the ferromagnetic sector $K_1\geqslant 0$, $K_2\geqslant 0$. However, since the partition functions $Z^{(\alpha,\beta)}$ are analytic functions of K_i (polynomials of finite degree in the exponential functions $e^{\pm K_i}$) when the lattice dimensions are finite, the entire nonanalytic dependence of the right-and left-hand sides of (24) can originate only from the multipliers $(s_1s_2)^{-N/4}$. Therefore, the relation (24) can easily be continued into the other three regions of values of the parameters ($K_1\geqslant 0, K_2<0$; $K_1<0$, $K_2\geqslant 0$; $K_1<0$, $K_2<0$) with proper observance of the rules for circumventing the branch points corresponding to $\sinh 2K_1=0$ and $\sinh 2K_2=0$.

4. THE INHOMOGENEOUS CASE

In the preceding section we showed how the Kramers–Wannier duality relation (1) should be modified so as to be transformed from a symbolic expression to an exact equality. It is clear that the Kadanoff–Ceva ansatz (2), for which (1) is a special case, is not an exact equality, and it must be treated circumspectly. For example, we cannot restrict ourselves, as in the homogeneous case, to a simple allusion to a thermodynamic limit without imposing some requirements on the class of functions assigning the distribution of coupling constants $K_i(r)$. This can be illustrated by a simple example, in which the sequence of the functions $K_i^{(N)}(r)$ for increasing values of N is chosen such that the coupling constants on the boundaries Γ of clusters of finite dimensions would become equal to zero: $K_i(r\in\Gamma)=0$. Then, clearly, as N increases we obtain a growing number of lattices of finite dimensions, which do not interact with one another. The situation is more reminiscent of a self-averaging regime for unordered systems than a thermodynamic limit in the conventional sense. Among other things, this is another argument in favor of attempting to formulate a duality relation for an inhomogeneous model, which would have the form of a strict equality, as it can be a good tool for theoretically analyzing systems with random bonds.

The covariant form for the exact result (24) in the homogenous case hints at an obvious procedure for generalization to the inhomogeneous case. For a lattice of finite dimensions stretched over a torus, the Kadanoff–Ceva ansatz (2) should be modified in the following manner:

$$\prod_{r,i}(\sinh 2K_i(r))^{-1/4}\mathbf{Z}[K]=\prod_{\tilde{r},i}(\sinh 2\tilde{K}_i(\tilde{r}))^{-1/4}\hat{T}\tilde{\mathbf{Z}}[\tilde{K}]. \qquad (27)$$

Unfortunately, we are unable to prove (27) in the general case of arbitrary lattice dimensions and distributions of the coupling constants. We note, however, that we tested this relation by means of direct computer calculations on lattices of small dimensions ($n,m=2$, 3, 4) using a program of analytic calculations and became convinced of its validity. Moreover, the duality relation (27) can be rigorously proved for the weakly inhomogeneous case, in which

$$K_i(r)=K_i+\delta K_i(r),\quad K_i=\text{const},\quad \delta K_i(r)\ll 1$$

at arbitrary finite n and m.

In first order with respect to $\delta K_i(r)$ we have

$$\prod_{\mathbf{r},i}(\sinh(2K_i+\delta K_i(r)))^{-1/4}=(s_1s_2)^{-N/4}\left[1-\frac{c_1}{2s_1}\sum_r\delta K_1(r)-\frac{c_2}{2s_2}\sum_r\delta K_2(r)\right], \qquad (28)$$

$$\mathbf{Z}[K]=\mathbf{Z}(K_1,K_2)\left[1+\sum_r(\langle\sigma(r)\sigma(r+\hat{x})\rangle\delta K_1(r)+\langle\sigma(r)\sigma(r+\hat{y})\rangle\delta K_1(r))\right]. \qquad (29)$$

Because of their translational invariance, the means $\langle\sigma(r)\sigma(r+\hat{x})\rangle$ and $\langle\sigma(r)\sigma(r+\hat{y})\rangle$ do not depend on r, but they are not equal to one another when K_1 and K_2 are not equal and (or) $n\neq m$:

$$\mathbf{Z}(K_1,K_2)\langle\sigma(r)\sigma(r+\hat{x})\rangle=\frac{1}{N}\hat{R}\frac{\partial\mathbf{Q}(K_1,K_2)}{\partial K_1},$$
$$\mathbf{Z}(K_1,K_2)\langle\sigma(r)\sigma(r+\hat{y})\rangle=\frac{1}{N}\hat{R}\frac{\partial\mathbf{Q}(K_1,K_2)}{\partial K_2}. \qquad (30)$$

With consideration of (28)–(30), the duality relation (27) takes on the following form in first order with respect to $\delta K_i(r)$ and $\delta\tilde{K}_i(\tilde{r})$:

$$(s_1s_2)^{-1/4}\left[\mathbf{Z}(K_1,K_2)+\hat{R}\left(\frac{1}{N}\frac{\partial\mathbf{Q}}{\partial K_1}-\frac{c_1}{2s_1}\mathbf{Q}\right)\sum_r\delta K_1(r)+\hat{R}\left(\frac{1}{N}\frac{\partial\mathbf{Q}}{\partial K_2}-\frac{c_2}{2s_2}\mathbf{Q}\right)\sum_r\delta K_2(r)\right]$$
$$=(\tilde{s}_1\tilde{s}_2)^{-1/4}\left[\tilde{\mathbf{Z}}(\tilde{K}_1,\tilde{K}_2)+\hat{R}\left(\frac{1}{N}\frac{\partial\tilde{\mathbf{Q}}}{\partial\tilde{K}_1}-\frac{\tilde{c}_1}{2\tilde{s}_1}\tilde{\mathbf{Q}}\right)\sum_r\delta\tilde{K}_1(\tilde{r})+\hat{R}\left(\frac{1}{N}\frac{\partial\tilde{\mathbf{Q}}}{\partial\tilde{K}_2}\right.$$

$$- \frac{\tilde{c}_2}{2\tilde{s}_2} \tilde{\mathbf{Q}} \bigg) \sum_r \delta \tilde{K}_2(\tilde{r}) \bigg]. \qquad (31)$$

For the first two terms on the left- and right-hand sides of (31) this equality is satisfied according to the homogeneous duality relation (24). We show that it is also satisfied for the terms which are linear in δK and $\delta \tilde{K}$. From (3) it follows that

$$\delta K_1(r) = -\frac{1}{\tilde{s}_2} \delta \tilde{K}_2(\tilde{r}), \quad \delta K_2(r) = -\frac{1}{\tilde{s}_1} \delta \tilde{K}_1(\tilde{r}),$$

$$\frac{\partial}{\partial K_1} = -\tilde{s}_2 \frac{\partial}{\partial \tilde{K}_2}, \quad \frac{\partial}{\partial K_2} = -\tilde{s}_1 \frac{\partial}{\partial \tilde{K}_1}, \qquad (32)$$

and from (23) it follows that

$$\mathbf{Q}(K_1, K_2) = (\tilde{s}_1 \tilde{s}_2)^{-N/2} \hat{g} \tilde{\mathbf{Q}}(\tilde{K}_1, \tilde{K}_2). \qquad (33)$$

Substituting (32) and (33) into the left-hand side of (31) and combining similar terms, we obtain

$$(\tilde{s}_1 \tilde{s}_2)^{-N/4} \bigg[\hat{T} \tilde{\mathbf{Z}}(\tilde{K}_1, \tilde{K}_2) + \hat{R} \hat{g} \bigg(\frac{1}{N} \frac{\partial \tilde{\mathbf{Q}}}{\partial \tilde{K}_2} - \frac{\tilde{c}_2}{2\tilde{s}_2} \tilde{\mathbf{Q}} \bigg) \sum_{\tilde{r}} \delta \tilde{K}_2(\tilde{r}) + \hat{R} \hat{g} \bigg(\frac{1}{N} \frac{\partial \tilde{\mathbf{Q}}}{\partial \tilde{K}_1} - \frac{\tilde{c}_1}{2\tilde{s}_1} \tilde{\mathbf{Q}} \bigg) \sum_{\tilde{r}} \delta \tilde{K}_1(\tilde{r}) \bigg]. \qquad (34)$$

Recalling (25) (whence follows $\hat{R}\hat{g} = \hat{T}\hat{R}$), we see that (34) coincides exactly with the right-hand side of (31). This also proves the duality relation (27) in the weakly inhomogeneous case for arbitrary m and n.

To conclude this section we note that a duality relation in a normalization different from (26) is more convenient, for example, for applications to the analysis of correlation functions. Using the equalities which are obvious from (3)

$$\frac{\cosh^2 2K_1(r)}{\sinh 2K_1(r)} = \frac{\cosh^2 2\tilde{K}_2(\tilde{r})}{\sinh 2\tilde{K}_2(\tilde{r})},$$

$$\frac{\cosh^2 2K_2(r)}{\sinh 2K_2(r)} = \frac{\cosh^2 2\tilde{K}_1(\tilde{r})}{\sinh 2\tilde{K}_1(\tilde{r})},$$

and introducing the notations from Ref. 3

$$\mathbf{Y}[K] = \prod_{r,i} (\cosh 2K_i(r))^{-1/2} \mathbf{Z}[K],$$

$$\tilde{\mathbf{Y}}[\tilde{K}] = \prod_{r,i} (\cosh 2\tilde{K}_i(\tilde{r}))^{-1/2} \tilde{\mathbf{Z}}[\tilde{K}], \qquad (35)$$

instead of (27) we obtain

$$\mathbf{Y}[K] = \hat{T} \tilde{\mathbf{Y}}[\tilde{K}]. \qquad (36)$$

5. DUALITY FOR CORRELATION FUNCTIONS

The duality relation for the inhomogeneous Ising model is useful for studying the properties of correlation functions. For this purpose it is convenient to use the representation of correlation functions in terms of magnetic dislocations,[3] i.e., extended defects on an Ising lattice. This representation is based on the trivial equality

$$e^{(K+i\pi/2)\sigma_1\sigma_2} = i\sigma_1\sigma_2 e^{K\sigma_1\sigma_2}.$$

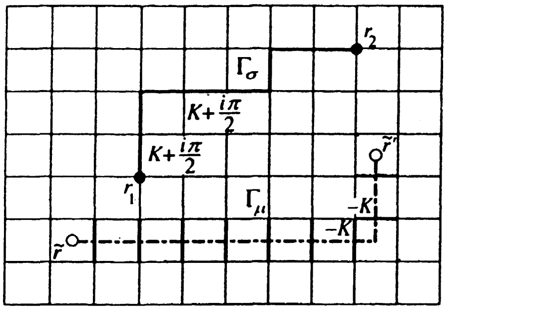

FIG. 2. Magnetic dislocations of two types. Dislocation Γ_σ: coupling constants on the line joining the points r_1 and r_2 are replaced by $K + i\pi/2$. Dislocation Γ_μ: coupling constants intersecting the line joining the points $\tilde{r}$ and $\tilde{r}'$ are replaced by $-K$.

Taking into account that

$$\sigma_1\sigma_n = (\sigma_1\sigma_2)(\sigma_2\sigma_3) \ldots (\sigma_{n-1}\sigma_n),$$

we can write

$$\sum_{[\sigma]} e^{-\beta H[\sigma]} \sigma(r)\sigma(r') = i^{-\gamma} \sum_{[\sigma]} e^{-\beta H'[\sigma]},$$

where the Hamiltonian with the defect $\beta H'[\sigma]$ can be distinguished from $\beta H[\sigma]$ by the fact that along the line of defects, i.e., the magnetic dislocation Γ_σ (see Fig. 2), which joins the points r and r', the coupling constants K are replaced by $K' = K + i\pi/2$, and γ is the length of this path, i.e., the number of "dislocated" bonds. Then, using the functionals (35) introduced in the preceding section, we can write the correlation function in the following manner (the subscripts specifying the boundary conditions are no longer written out):

$$G_\sigma(r,r') \equiv \langle \sigma(r)\sigma(r') \rangle = Y[K']/Y[K], \qquad (37)$$

with the distribution of coupling constants

$$K_i(r) = K,$$

$$K_i'(r) = \begin{cases} K + i\pi/2 & \text{in bonds belonging to } \Gamma_\sigma \\ K & \text{in other bonds.} \end{cases}$$

The correlation function of the disorder variable $\mu(\tilde{r})$ was introduced in Ref. 3. This parameter characterizes the degree of disorder in the vicinity of the point $\tilde{r}$ on the original lattice, and it can be regarded as the result of the duality transformation of the Ising variable $\sigma(r)$. The correlation function $\langle \mu(\tilde{r})\mu(r') \rangle$ is defined by analogy to (37) using the magnetic dislocation Γ_μ (see Fig. 2)

$$G_\mu(\tilde{r},\tilde{r}') \equiv \langle \mu(\tilde{r})\mu(\tilde{r}') \rangle = Y[K'']/Y[K],$$

where

$$K_i'' = \begin{cases} -K & \text{in bonds intersecting } \Gamma_\mu \\ K & \text{in other bonds.} \end{cases}$$

The duality relation for the correlation functions obtained in Ref. 3

$$\langle\tilde{\mu}(r)\tilde{\mu}(r')\rangle=\langle\sigma(r)\sigma(r')\rangle \tag{38}$$

follows from the equality (2) and the mapping of the magnetic dislocation Γ_σ on the original lattice onto the magnetic dislocation Γ_μ on the dual lattice, i.e.,

$$K_1(r)+i\pi/2\rightarrow\tilde{K}_2(\tilde{r})e^{-i\pi},$$

$$K_2(r)+i\pi/2\rightarrow\tilde{K}_1(\tilde{r})e^{-i\pi},$$

which follows from (3). The duality relation (27) for a lattice of finite dimensions differs from the Kadanoff–Ceva ansatz (2). Accordingly, the duality relation for the correlation functions takes a form which is more complex than (38). For example, using (36), for a dual lattice with periodic boundary conditions we obtain

$$\begin{aligned}\tilde{G}_{\tilde{\mu}}^{(p,p)}(r,r')&=\tilde{Y}^{(p,p)}[\tilde{K}'']/\tilde{Y}^{(p,p)}[\tilde{K}]\\&=(\hat{T}\mathbf{Y}[K'])^{(p,p)}/(\hat{T}\mathbf{Y}[K])^{(p,p)}\\&=[Z^{(p,p)}G_\sigma^{(p,p)}(r,r')+Z^{(p,a)}G_\sigma^{(p,a)}(r,r')\\&\quad+Z^{(a,p)}G_\sigma^{(a,p)}(r,r')+Z^{(a,a)}G_\sigma^{(a,a)}\\&\quad\times(r,r')]/[Z^{(p,p)}+Z^{(p,a)}+Z^{(a,p)}\\&\quad+Z^{(a,a)}].\end{aligned} \tag{39}$$

It is not difficult to see that (39) transforms into (38) only under the condition that the correlation length is small in comparison with the lattice dimensions, which occurs outside the scaling region and when m and n are sufficiently large. We note that the relation (39) is consistent with the relation obtained for the critical point in Ref. 9 by the methods of quantum conformal field theory.[10]

6. CONCLUSIONS

The duality relation for the inhomogeneous Ising model can have several useful applications. We touched upon one of them in the preceding section. This relation can also be used to correctly introduce mixed correlation functions of the form $\langle\sigma\mu\sigma'\mu'\rangle$ and to explicitly reveal their fermionic behavior. The ansatz (27) makes it possible, in principle, to construct a generating functional, which depends simultaneously on the external sources $J(r)$, $\tilde{J}(r)$, and $\chi(r)$, the first two of which are associated with fluctuations of the order and disorder variables, while $\chi(r)$ generates excitations of the fermion type.

The duality relation (27) was formulated for lattices stretched over a torus. Nevertheless, it is not difficult to obtain a duality relation for several other boundary conditions from it. In particular, it can be shown that a lattice with the dimensions $n\times m$ and free boundaries is dual to an $(n-1)\times(m-1)$ lattice with magnetic fields on its boundaries. Application of the inhomogeneous duality relation to the analysis of unordered systems seems attractive in the case of an Ising model with random bonds.

Thus, the ansatz (27) is interesting not only in itself, but also in the context of various applications and possible generalizations. However, unfortunately, we still do not have a proof for the general case of arbitrary distributions of the coupling constants and lattice dimensions. The duality relation (27) was rigorously proved in this paper for the homogeneous case and for the weakly inhomogeneous case in the first order (it would not be difficult to prove it in the second order). The relation (27) can also be proved for conditions under which a certain small number of coupling constants are chosen arbitrarily on a background of the remaining identical bonds on the lattice. A positive result from direct testing of the duality relation (27) on small lattices with $m,n=2$, 3, 4 would be an additional argument supporting its validity.

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[1] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252 (1941).
[2] L. Onsager, Phys. Rev. **65**, 117 (1944).
[3] L. P. Kadanoff and H. Ceva, Phys. Rev. B **3**, 3918 (1971).
[4] C. A. Hurst and H. S. Green, J. Chem. Phys. **33**, 1059 (1960).
[5] F. A. Berezin, Usp. Mat. Nauk **24**, 3 (1969).
[6] A. I. Bugriĭ, Preprint No. ITF-85-114R, Institute of Theoretical Physics, Kiev (1985).
[7] A. I. Bugriĭ and V. N. Shadura, Teor. Mat. Fiz. **103**, 388 (1995).
[8] V. S. Dotsenko, Usp. Fiz. Nauk **165**, 482 (1995) [Phys. Usp. **38**, (1995)].
[9] P. Di Francesco, H. Saleur, and J. B. Zuber, Preprint No. SPh T/87-097, Service de Physique Théorique de Saclay, Gif-sur Yvette CEDEX, France (1987).
[10] A. Belavin, A. Polyakov, and A. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).

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