Nonlinear periodic waves in stimulated Raman scattering of light and the creation of solitons at the leading edge of a pulse

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(Submitted 15 September 1995)
Zh. Éksp. Teor. Fiz. **109**, 786–804 (March 1996)

We obtain an efficient general periodic solution to the equations of Raman light scattering. Based on this solution, we develop a theory for the creation of solitons at the leading edge of a long pulse within the framework of the Whitham method. © *1996 American Institute of Physics.* [S1063-7761(96)00603-2]

1. INTRODUCTION

As is well known, the observation of solitons in stimulated Raman scattering (SRS) is accompanied by major experimental difficulties, since these solitons propagate on top of a longer Raman-wave pulse and disappear at one of its ends. Increasing the intensity and duration of the pump pulses leads to the formation of a periodic nonlinear wave.[1] Periodic oscillations as the pump pulse depletes were also observed in the experiments of Ref. 2. This makes a description of the creation of a sequence of solitons at the leading edge of the pump pulse problematic. In this case, solitons are produced by the modulation instability of a constant-amplitude wave, which transforms the pulse front into an ever-widening region of nonlinear oscillations. This process can be treated by using the Whitham method, which has been used previously to deal with waves described by the nonlinear Schrödinger equation,[3,4] waves in one-dimensional magnets,[5] and self-induced transparency waves (STW).[6,7] However, the nonuniform state that appears is a modulated nonlinear periodic wave with time-dependent and coordinate-dependent parameters. For a complete description of this state, it is first necessary to find a way to cast this periodic solution in some efficient form. In this paper this problem is solved by using a modification of the well-known method of finite-band integration[8] for the case where the Lax-pair operators of the corresponding integrable equations are not self-adjoint. This modification was proposed in Ref. 9 and has been applied to a rather large number of equations of immediate physical interest.[5,7,10–12] In this paper we apply this method to the SRS equations.[1)]

2. PERIODIC SOLUTIONS OF THE SRS EQUATIONS

The SRS equations describe the propagation of two waves with frequencies ω_1 and ω_2 and electric field envelopes E_1 and E_2 in a medium with a resonance at the difference frequency $\omega_1-\omega_2$. The derivation of these equations has been discussed many times in the literature (see, e.g., the review Ref. 15), so we will not pause to discuss it here. As shown by Steudel,[16,17] the equations acquire a symmetric form if we construct a vector **S** from the amplitudes E_1 and E_2 with Cartesian components

$$S_1+E_1E_2^*+E_2E_1^*, \quad S_2=i(E_1E_2^*-E_2E_1^*),$$
$$S_3=E_1E_1^*-E_2E_2^* \tag{1}$$

and pass from the retarded time $t'=t-x/c$ (where x is the coordinate along which the waves propagate and c is their group velocity in the medium) to the variable

$$\tau=k\int_{t_0}^{t} I(t')dt', \tag{2}$$

where $I(t) = E_1E_1^* + E_2E_2^*$ is the total field intensity, and k is a constant that measures the dipole interaction of these waves with the medium. Introducing the corresponding dimensionless coordinate ξ along which the wave propagates, and the Bloch vector **R** that describes the state of the medium ($R_\pm=R_1\pm iR_2$ corresponds to the off-diagonal elements of the density matrix of the two-level medium, R_3 to the population difference between the upper and lower levels), we are led to the SRS equations in the following form[15–17]:

$$\frac{\partial R_+}{\partial\tau}=i(\Delta R_+S_3+R_3S_+), \quad \frac{\partial R_3}{\partial t}=\frac{i}{2}(R_+S_--R_-S_+),$$
$$\frac{\partial S_+}{\partial\xi}=i(\Delta S_+R_3+S_3R_+), \quad \frac{\partial S_3}{\partial\xi}=\frac{i}{2}(S_+R_--S_-R_+), \tag{3}$$

where $S_\pm=S_1\pm iS_2$ and Δ is a parameter that characterizes the dynamic Stark effect. The vectors **R** and **S** are here normalized to unit length:

$$R_1^2+R_2^2+R_3^2=1, \quad S_1^2+S_2^2+S_3^2=1. \tag{4}$$

In Ref. 17 it was shown that the system of SRS Eqs. (3) is integrable by the inverse scattering method, which enables us to study both single-soliton and multisoliton solutions.[8] The specifics of formulating initial conditions for this system, which arise from the nontrivial connection[2] between the physical time t and the "time" coordinate τ, were discussed in Ref. 19. In this paper we find a periodic solution to the system (3).

The integrability of the SRS Eqs. (3) by the method of inverse scattering is based on the possibility of expressing these equations in the form of compatibility conditions for the two linear systems[17,19]

 1063-7761/96/030424-10$10.00

$$\frac{\partial\psi}{\partial\tau}=\begin{pmatrix} F & G \\ H & -F \end{pmatrix}\psi, \quad \frac{\partial\psi}{\partial\xi}=\begin{pmatrix} A & B \\ C & -A \end{pmatrix}\psi, \tag{5}$$

where $\psi=(\psi_1,\psi_2)^T$ is a two-component "spinor" of solutions to system (5). The general Ablowitz–Kaup–Newell–Segur (AKNS) scheme[20] leads to Eqs. (3) if we take the following quantities as the coefficients of the system (5)[17]:

$$F=-i\lambda S_3, \quad G=(i\sigma+\lambda)S_+, \quad H=(i\sigma-\lambda)S_-, \tag{6}$$

$$A=\frac{i}{2}\left(\Delta-\frac{1}{2\lambda+\Delta}\right)R_3, \quad B=\frac{i\sigma+\lambda}{2\lambda+\Delta}R_+,$$

$$C=\frac{i\sigma-\lambda}{2\lambda+\Delta}R_-, \tag{7}$$

where the parameter σ is related to Δ by

$$\sigma^2=\frac{1}{4}(1-\Delta^2), \tag{8}$$

and λ is an arbitrary spectral parameter.

The system (5) has two basis solutions (ψ_1,ψ_2) and (ϕ_1,ϕ_2), so that we can construct a "vector" out of them with the spherical components

$$f=-\frac{i}{2}(\psi_1\varphi_2+\psi_2\varphi_1), \quad g=\psi_1\varphi_1, \quad h=-\psi_2\varphi_2, \tag{9}$$

which satisfy the linear systems

$$\frac{\partial f}{\partial\tau}=-iHg+iGh, \quad \frac{\partial f}{\partial\xi}=-iCg+iBh,$$

$$\frac{\partial g}{\partial\tau}=2iGf+2Fg, \quad \frac{\partial g}{\partial\xi}=2iBf+2Ag,$$

$$\frac{\partial h}{\partial\tau}=-2iHf-2Fh, \quad \frac{\partial h}{\partial\xi}=-2iCf-2Ah. \tag{10}$$

It is easy to verify that the length of the vector (9)

$$f^2-gh=P(\lambda) \tag{11}$$

does not depend on τ or ξ. Periodic solutions are identified by the condition that $P(\lambda)$ be a polynomial[21–24] in λ. For our purposes it is sufficient to know the simplest nontrivial single-phase solution, where $P(\lambda)$ is a fourth-degree polynomial

$$P(\lambda)=\prod_{i=1}^{4}(\lambda-\lambda_i)=\lambda^4-s_1\lambda^3+s_2\lambda^2-s_3\lambda+s_4. \tag{12}$$

Then from the system (10) with coefficients (6), (7) we find easily that the functions (9) have the form

$$f=S_3\lambda^2-f_1\lambda+f_2, \quad g=(i\sigma+\lambda)S_+(\lambda-\mu),$$

$$h=(i\sigma-\lambda)S_-(\lambda-\mu^*), \tag{13}$$

where f_1, f_2, μ, and μ^* by virtue of (11), (12) must satisfy

$$2f_1S_3+(1-S_3^2)(\mu+\mu^*)=s_1,$$

$$2f_1f_2+(1-S_3^2)\sigma^2(\mu+\mu^*)=s_3,$$

$$f_1^2+2f_2S_3+(1-S_3^2)(\sigma^2+\mu\mu^*)=s_2,$$

$$f_2^2+(1-S_3^2)\sigma^2\mu\mu^*=s_4, \tag{14}$$

and, in addition,

$$\frac{\partial S_3}{\partial\tau}=-i(1-S_3^2)(\mu-\mu^*),$$

$$\frac{\partial S_+}{\partial\tau}=-2i(f_1-\mu S_3)S_+, \tag{15}$$

$$R_+S_-\left(\mu^*+\frac{\Delta}{2}\right)=R_-S_+\left(\mu+\frac{\Delta}{2}\right),$$

$$f\left(-\frac{\Delta}{2}\right)R_++\frac{1}{2}\left(\mu+\frac{\Delta}{2}\right)R_3S_+=0. \tag{16}$$

Setting the spectral parameter λ equal to μ in Eqs. (10) for g, we obtain the equation of motion for μ:

$$\frac{\partial\mu}{\partial\tau}=-2if(\mu)=-2i\sqrt{P(\mu)},$$

$$\frac{\partial\mu}{\partial\xi}=\frac{R_+}{(2\mu+\Delta)S_+}\frac{\partial\mu}{\partial\tau}. \tag{17}$$

Let us rewrite Eq. (16) in the form

$$\frac{R_+}{(\mu+\Delta/2)S_+}=\frac{R_-}{(\mu^*+\Delta/2)S_-}=-\frac{R_3}{2f(-\Delta/2)}=-\frac{2}{V}, \tag{18}$$

where V, as we will now verify, is the nonlinear phase velocity of the wave. From (18),

$$\frac{1-R_3^2}{(1-S_3^2)(\mu+\Delta/2)(\mu^*+\Delta/2)}=\frac{R_3^2}{4f^2(-\Delta/2)}=\frac{4}{V^2}.$$

Substituting $\lambda=-\Delta/2$ into (11), and taking into account (8) and (13), we obtain

$$(1-S_3^2)\left(\mu+\frac{\Delta}{2}\right)\left(\mu^*+\frac{\Delta}{2}\right)=4\left[P\left(-\frac{\Delta}{2}\right)-f^2\left(-\frac{\Delta}{2}\right)\right],$$

so that from the previous equation,

$$V=4\sqrt{P\left(-\frac{\Delta}{2}\right)}. \tag{19}$$

Note that this kind of relation is very general in character: the phase velocity of the nonlinear wave is determined by the value of the polynomial that specifies this wave at the point where the coefficients of the Lax pair have a pole as a function of the spectral parameter λ. (This question is discussed in more detail in the Appendix.)

Thus, as follows from (17)–(19), the parameter μ depends only on the phase W:

$$W=\tau-\frac{\xi}{V}, \quad \frac{d\mu}{dW}=-2i\sqrt{P(\mu)}. \tag{20}$$

The last equation of system (3) can also be transformed with the help of (15) and (16):

$$\frac{\partial S_3}{\partial\xi}=-\frac{1}{V}\frac{\partial S_3}{\partial\tau},$$

i.e., S_3 also depends only on the phase W.

As W varies, the parameter μ describes some curve in the complex plane; the initial data for Eq. (20) should be chosen such that (11) is satisfied. Therefore, it is convenient to introduce a coordinate on this curve in such a way that (11) is satisfied automatically (see Ref. 9). From the system of Eqs. (14) it follows that it is convenient to take the component S_3 as the coordinate that parametrizes the points of the curve along which μ moves, so that μ is expressed as a function of S_3. The system (14) coincides, save for changes in notation, with the analogous system of Ref. 11, where periodic waves were studied in a magnet with uniaxial anisotropy, so that we can make use of the solutions found there. Then for the coefficients f_1 and f_2 we have

$$f_1^2=\frac{1}{2\sigma^2}\left[\sqrt{P_2(\sigma^2)}-\sigma^4+s_2\sigma^2-s_4\right], \tag{21}$$

$$f_2=\frac{s_3-s_1\sigma^2}{2f_1}+\sigma^2 S_3, \tag{22}$$

where

$$P_2(\sigma^2)=\prod_{i=1}^{4}(\sigma^2+\lambda_i^2).$$

The sign of f_1 that must be chosen in extracting the root from (21) is determined by the condition for a stable solution, $S_3=R_3=-1$ (see Ref. 17). As we will confirm, we obtain the required stable solution when the negative sign is chosen, i.e., $f_1=-\sqrt{f_1^2}$.

The equation of motion (3) for S_+ gives, instead of (15), (16), and (22),

$$\frac{\partial S_+}{\partial\xi}=-\frac{2i}{V}\left[4f_1\sigma^2-\frac{(s_3-s_1\sigma^2)\Delta}{f_1}\right]S_+-\frac{1}{V}\frac{\partial S_+}{\partial\tau},$$

i.e.,

$$S_+=\exp\left\{-\frac{2i}{V}\left[4f_1\sigma^2-\frac{(s_3-s_1\sigma^2)\Delta}{f_1}\right]\xi\right\}\widetilde{S}_+, \tag{23}$$

where $\widetilde{S}_+$ depends only on the phase W and is determined by the equation

$$\frac{d\widetilde{S}_+}{dW}=-2i(f_1-\mu S_3)\widetilde{S}_+. \tag{24}$$

The parameter μ can be expressed in terms of S_3 (see Ref. 11) as follows:

$$\mu=\frac{s_1-2f_1S_3+2i\sqrt{\sigma^2R(S_3)}}{2(1-S_3^2)}, \tag{25}$$

where

$$R(\nu)=\nu^4+\frac{s_3-s_1\sigma^2}{f_1\sigma^2}\nu^2-\frac{s_2}{\sigma^2}\nu^2+\left(\frac{s_1f_1}{\sigma^2}-\frac{s_3-s_1\sigma^2}{f_1\sigma^2}\right)\nu+\frac{4s_2-4f_1^2-s_1^2-4\sigma^2}{4\sigma^2} \tag{26}$$

is a polynomial that is the algebraic resolvent of the original polynomial $P(\lambda)$. Its zeroes ν_i $(i=1,2,3,4)$ are related to the zeroes λ_i $(i=1,2,3,4)$ of the polynomial $P(\lambda)$ by rather complicated symmetric expressions obtained in Ref. 11:

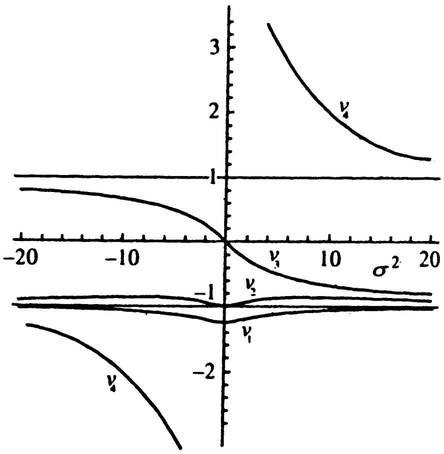

FIG. 1. Dependence of the zeroes of the resolvent ν_i $(i=1,2,3,4)$ on the parameter σ^2 (which, according to the definition (8) can be either positive or negative). The figure is schematic because the scale of the curves is changed in order that these zeroes might be plotted on a single figure. The symmetry of the curves with respect to the replacement $\sigma^2\to-\sigma^2(\nu_1,\nu_2)$ or relative to the coordinate origin (ν_3,ν_4) is due to the fact that the chosen values of λ_i lie on bisectrices of the first and fourth quadrants: $\lambda_1=\lambda_3^*=1+i$ and $\lambda_2=\lambda_4^*=2+2i$. For other values of λ_i the curves are deformed; however, their ordering with respect to magnitude (see (31) and the text) is the same as that shown in the figure.

$$\begin{aligned}\nu_1=\frac{1}{4f_1\sigma^2}&[(\lambda_1-\lambda_3)(\lambda_2'-\lambda_4')+(\lambda_2-\lambda_4)\\&\times(\lambda_1'-\lambda_3')]^{-1}\{(\lambda_1-\lambda_3)[2(\lambda_1+\lambda_3)(\lambda_2'-\lambda_4')\sigma^2\\&+(\lambda_2\lambda_4'-\lambda_4\lambda_2')((\lambda_1+\lambda_3)^2-(\lambda_1'-\lambda_3')^2)]\\&+(\lambda_2-\lambda_4)[2(\lambda_2+\lambda_4)(\lambda_1'-\lambda_3')\sigma^2\\&+(\lambda_1\lambda_3'-\lambda_3\lambda_1')((\lambda_2+\lambda_4)^2-(\lambda_2'-\lambda_4')^2)]\},\end{aligned} \tag{27}$$

where

$$\lambda_i'=\sqrt{\lambda_i^2+\sigma^2},$$

ν_2 and ν_3 are obtained from ν_1 by exchanging the indices $3\leftrightarrow4$ and $3\leftrightarrow2$ respectively, while

$$\nu_4=\frac{s_1\sigma^2-s_3}{f_1\sigma^2}-(\nu_1+\nu_2+\nu_3). \tag{28}$$

From Eqs. (15) and (25) we find an equation for S_3:

$$\frac{dS_3}{d(2W)}=\sqrt{\sigma^2R(S_3)}. \tag{29}$$

The variable S_3 is real, and by virtue of (4) it can oscillate between the two zeroes of the resolvent, which lie between -1 and 1. We can verify that the zeroes ν_i are real if the zeroes λ_i of the polynomial $P(\lambda)$ separate into two complex-conjugate pairs

$$\lambda_1=\alpha+i\gamma,\quad \lambda_2=\beta+i\delta,\quad \lambda_3=\alpha-i\gamma,\quad \lambda_4=\beta-i\delta. \tag{30}$$

As an example, Fig. 1 shows schematically the dependence of the zeroes on σ^2 (if the plots were truly to scale, ν_1 and ν_2 would be too small) for $\lambda_1=1+i$, $\lambda_2=2+2i$. As we see, when $\sigma^2>0$ the zeroes of the resolvent are ordered according to the inequalities $\nu_1<-1<\nu_2<\nu_3<1<\nu_4$, and S_3 oscillates within the interval

$$-1<\nu_2\leqslant S_3\leqslant \nu_3<1, \tag{31}$$

where $R(S_3)\geqslant 0$. For $\sigma^2<0$ we have $\nu_4<\nu_1<-1<\nu_2<\nu_3<1$ and S_3 oscillates within the same interval (31), where now $R(S_3)\geqslant 0$; however, the radicand in (29) remains positive.

Equations (20) and (29) enable us to compute the period of oscillations T of the nonlinear wave by two methods:

$$T=\frac{1}{2}\oint\frac{d\mu}{\sqrt{-P(\mu)}}=\int_{\nu_2}^{\nu_3}\frac{d\nu}{\sqrt{\sigma^2R(\nu)}},$$

which leads to equality of the parameters of the elliptic integrals in terms of which T is expressed in both cases:

$$m=\frac{(\nu_2-\nu_3)(\nu_1-\nu_4)}{(\nu_1-\nu_3)(\nu_2-\nu_4)}=\frac{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)}, \tag{32}$$

and to the useful relations

$$\sigma^2(\nu_1-\nu_3)(\nu_2-\nu_4)=(\lambda_1-\lambda_4)(\lambda_3-\lambda_2)=(\alpha-\beta)^2+(\gamma+\delta)^2,$$

$$\sigma^2(\nu_1-\nu_4)(\nu_2-\nu_3)=(\lambda_1-\lambda_3)(\lambda_4-\lambda_2)=4\gamma\delta. \tag{33}$$

The periodic solution to Eq. (29) yields the required expression for S_3:

$$S_3=\frac{(\nu_2-\nu_4)\nu_3-(\nu_2-\nu_3)\nu_4\ \mathrm{sn}^2(\sqrt{\sigma^2(\nu_1-\nu_3)(\nu_2-\nu_4)}W,m)}{\nu_2-\nu_4-(\nu_2-\nu_3)\mathrm{sn}^2(\sqrt{\sigma^2(\nu_1-\nu_3)(\nu_2-\nu_4)}W,m)}, \tag{34}$$

where the initial value of the phase has been set equal to zero.

Let us now compute S_+. Substituting (25) and (29) into (24), we find

$$\tilde{S}_+=\sqrt{1-S_3^2}\exp\left[i\int_0^W\frac{s_1S_3-2f_1}{1-S_3^2}\,dW\right]. \tag{35}$$

In order to calculate the integral, it is now convenient to transform to the Weierstrass elliptic functions

$$\mathrm{sn}^2(\sqrt{\sigma^2(\nu_1-\nu_3)(\nu_2-\nu_4)}W,m)=\frac{e_1-e_3}{\wp(W)-e_3},$$

where

$$e_1=-s_2/3-\sigma^2(\nu_1\nu_4+\nu_2\nu_3),$$

$$e_2=-s_2/3-\sigma^2(\nu_1\nu_3+\nu_2\nu_4),$$

$$e_3=-s_2/3-\sigma^2(\nu_1\nu_2+\nu_3\nu_4). \tag{36}$$

The integrand in (35) can be written in the form

$$\frac{s_1S_3-2f_1}{1-S_3^2}=\frac{s_1-2f_1}{2(1-\nu_3)}\frac{\wp(W)-\wp(\rho)}{\wp(W)-\wp(\kappa)}-\frac{s_1+2f_1}{2(1+\nu_3)}\frac{\wp(W)-\wp(\rho)}{\wp(W)-\wp(\tilde{\kappa})},$$

ρ, κ, and where the parameters $\tilde{\kappa}$ are determined by the expressions

$$\wp(\rho)=e_3+\sigma^2(\nu_1-\nu_3)(\nu_2-\nu_3),$$

$$\wp(\kappa)=e_3+\frac{\sigma^2(\nu_1-\nu_3)(\nu_2-\nu_3)(1-\nu_4)}{1-\nu_3},$$

$$\wp(\tilde{\kappa})=e_3+\frac{\sigma^2(\nu_1-\nu_3)(\nu_2-\nu_3)(1+\nu_4)}{1+\nu_3}. \tag{37}$$

The integration can be carried out with the help of the expression

$$\int_0^W\frac{\wp(W)-\wp(\rho)}{\wp(W)-\wp(\kappa)}\,dW=W+\frac{\wp(\rho)-\wp(\kappa)}{\wp'(\kappa)}\left[\ln\frac{\sigma(\kappa+W)}{\sigma(\kappa-W)}-2\zeta(\kappa)W\right],$$

where $\zeta(\kappa)$ and $\sigma(\kappa)$ are Weierstrass functions. As a result, we find after some simple calculations

$$S_+=-\sqrt{1-\nu_3^2}\exp\left\{-\frac{2i}{V}\left[4f_1\sigma^2-\frac{(s_3-s_1\sigma^2)\Delta}{f_1}\right]\xi+\frac{i(s_1\nu_3-2f_1)}{1-\nu_3^2}W-(\zeta(\kappa)+\zeta(\tilde{\kappa}))W\right\}\times\frac{\sigma(W+\kappa)\sigma(W+\tilde{\kappa})\sigma^2(\rho)}{\sigma(\kappa)\sigma(\tilde{\kappa})\sigma(W+\rho)\sigma(W-\rho)},$$

$$W=\tau-\frac{\xi}{V},\quad V=4\sqrt{P\left(-\frac{\Delta}{2}\right)}=4\sqrt{\left[\left(\alpha+\frac{\Delta}{2}\right)^2+\gamma^2\right]\left[\left(\beta+\frac{\Delta}{2}\right)^2+\delta^2\right]}. \tag{38}$$

Equations (34) and (38) yield expressions for the vector **S** in the general periodic solution to the SRS equations. The components of **R** can be obtained by using Eq. (18); in particular,

$$R_3=\frac{4}{V}f\left(-\frac{\Delta}{2}\right)=\frac{1}{V}\left[S_3+2f_1\Delta+\frac{2}{f_1}(s_3-s_1\sigma^2)\right]. \tag{39}$$

Let us discuss the soliton limit of these solutions as the wavelength goes to infinity, i.e., when

$$\lambda_1=\lambda_2=\alpha+i\gamma,\quad \lambda_3=\lambda_4=\alpha-i\gamma.$$

In this case $s_1=4\alpha$, $s_3=4\alpha(\alpha^2+\gamma^2)$, $f_1=-2\alpha$, and Eq. (39) yields

$$1+R_3=\frac{1}{V}(1+S_3), \tag{40}$$

where the velocity of the soliton is now

$$V=4\left[\left(\alpha+\frac{\Delta}{2}\right)^2+\gamma^2\right]. \tag{41}$$

The general solutions (27) and (28) for the zeroes of the resolvent lead to

$$\nu_1=\nu_2=-1,\quad \nu_3=-\frac{\lambda\lambda'+\lambda^*\lambda'^*}{\lambda'\lambda^*+\lambda\lambda'^*},$$

$$\nu_4=\frac{1}{\sigma^2}(\lambda\lambda^*+\lambda'\lambda'^*). \tag{42}$$

Taking into account (see (35)) that

$$(1+\nu_3)(1+\nu_4)=\frac{4\gamma^2}{\sigma^2}$$

and

$$\sum\nu_i=-2+\nu_3+\nu_4=-\frac{s_3-s_1\sigma^2}{f_1\sigma^2}=2\frac{\alpha^2+\gamma^2}{\sigma^2}-2,$$

we find that $1+\nu_3$ and $1+\nu_4$ are the roots of a simple quadratic equation, which leads to the expressions

$$\nu_3=\frac{1}{\sigma^2}[\alpha^2+\gamma^2-\sqrt{(\alpha^2+\gamma^2+\sigma^2)^2-4\gamma^2\sigma^2}],$$

$$\nu_4=\frac{1}{\sigma^2}[\alpha^2+\gamma^2+\sqrt{(\alpha^2+\gamma^2+\sigma^2)^2-4\gamma^2\sigma^2}] \tag{43}$$

in agreement with (42).

Expression (34) in the soliton limit becomes

$$S_3=\frac{\nu_4(1+\nu_3)+(\nu_3-\nu_4)\cosh^2(2\gamma W)}{1+\nu_3-(\nu_3-\nu_4)\cosh^2(2\gamma W)},$$

which yields

$$1+S_3=2\frac{(1+\nu_3)(1+\nu_4)/(\nu_4-\nu_3)}{\cosh(4\gamma W)+(\nu_3+\nu_4-2)/(\nu_4-\nu_3)}. \tag{44}$$

If $\sigma^2>0$, we can introduce a parameter θ such that

$$\tanh 2\theta=\frac{2\gamma\sigma}{\alpha^2+\gamma^2+\sigma^2}, \tag{45}$$

and the soliton solution then takes the form

$$1+S_3=V(1+R_3)=\frac{2\gamma}{\sigma}\frac{\sinh 2\theta}{\cosh(4\gamma W)+\cosh 2\theta}. \tag{46}$$

For $\sigma^2=-\varphi^2$ we introduce the parameter ϑ such that

$$\tan 2\vartheta=\frac{2\varphi\gamma}{\alpha^2+\gamma^2-\varphi^2}, \tag{47}$$

so that

$$1+S_3=V(1+R_3)=\frac{2\gamma}{\varphi}\frac{\sin 2\vartheta}{\cosh(4\gamma W)+\cos 2\vartheta}. \tag{48}$$

Expressions (45)–(48) coincide (up to notation) with the single-soliton solution of Steudel.[17]

As yet another limiting case we discuss a wave with $\delta=0$. The character of the solution in this case depends on the parameter σ^2, which characterizes the dynamic Stark effect (see (8)). Analytically it is easiest to investigate this function by setting $\alpha=0$, so that $\lambda_1=\lambda_3^*=i\gamma$, $\lambda_2=\lambda_4=3$. Then the zeroes of the resolvent are given by simple expressions. For $\sigma^2>\gamma^2$ we have

$$\nu_1=-\nu_4=\frac{\sqrt{\sigma^2+\beta^2}}{\sigma},\quad \nu_2=\nu_3=-\frac{\sqrt{\sigma^2-\gamma^2}}{\sigma}, \tag{49a}$$

for $-\beta^2<\sigma^2<\gamma^2$,

$$\nu_1=\frac{\beta\sqrt{\gamma^2-\sigma^2}+\gamma\sqrt{\beta^2+\sigma^2}}{\sigma^2},\quad \nu_2=\nu_3=0,$$

$$\nu_4=\frac{\beta\sqrt{\gamma^2-\sigma^2}-\gamma\sqrt{\beta^2+\sigma^2}}{\sigma^2} \tag{49b}$$

and for $\sigma^2<-\beta^2$,

$$\nu_1=\nu_4=-\sqrt{\frac{\sigma^2-\gamma^2}{\sigma^2}},\quad \nu_2=-\nu_3=-\sqrt{\frac{\beta^2+\sigma^2}{\sigma^2}}. \tag{49c}$$

As we see, when $\sigma^2>-\beta^2$ the zeroes ν_2 and ν_3 coincide, which after substitution into the general solution leads to a plane wave with constant amplitude. Here we write out the solution (49a) for $\sigma^2>\gamma^2$:

$$S_3=-\frac{1}{\sigma}\sqrt{\sigma^2-\gamma^2},\quad S_+=\frac{\gamma}{\sigma}\exp\left(i\frac{2\sigma\sqrt{\sigma^2-\gamma^2}}{\sqrt{\Delta^2/4+\gamma^2}}\xi\right),$$

$$R_3=-\frac{\Delta\sqrt{\sigma^2-\gamma^2}}{2\sigma\sqrt{\Delta^2/4+\gamma^2}},$$

$$R_+=\frac{\gamma}{2\sigma\sqrt{\Delta^2/4+\gamma^2}}\exp\left(i\frac{2\sigma\sqrt{\sigma^2-\gamma^2}}{\sqrt{\Delta^2/4+\gamma^2}}\xi\right). \tag{50}$$

However, for $\sigma^2<-\beta^2$ zeroes ν_1 and ν_4 coincide, so that although the parameter of the elliptic functions vanishes, the solution still remains a periodic wave with variable amplitude:

$$S_3=\sqrt{\frac{J-\beta^2}{J}}\frac{\sqrt{J+\gamma^2}\cos(2\sqrt{\beta^2+\gamma^2}W)-\sqrt{J-\beta^2}}{\sqrt{J+\gamma^2}-\sqrt{J-\beta^2}\cos(2\sqrt{\beta^2+\gamma^2}W)},$$

$$J=-\sigma^2>\beta^2,$$

where we have denoted the negative quantity σ^2 by $-J(J>0)$ to avoid misunderstanding. For $\alpha\neq 0$ the character of the solutions remains as before. This is confirmed by Fig. 2,

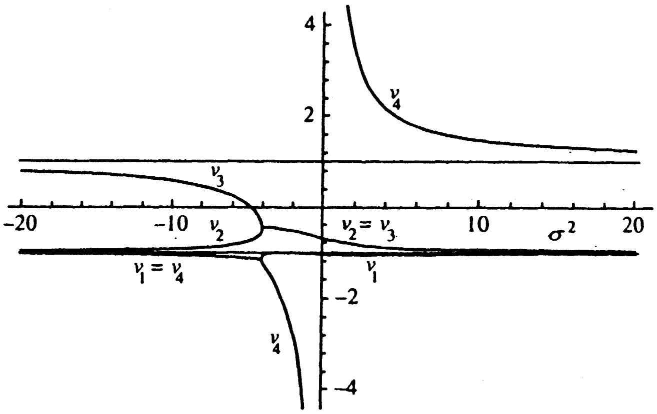

FIG. 2. Dependence of zeroes of the resolvent ν_i $(i=1,2,3,4)$ on the parameter σ^2 for $\lambda_1=\lambda_3^*=1+i$ and $\lambda_2=\lambda_4=2$. The curves are deformed in a way analogous to the curves in Fig. 1 as we go from $\delta=2$ to $\delta=0$.

which shows the dependence of the zeroes of the resolvent on σ^2 for $\alpha=\gamma=1$, $\beta=2$, $\delta=0$. These curves may be viewed as distortions of the curves in Fig. 1 as we go from $\delta=2$ to $\delta=0$. As we see, the zeroes ν_2 and ν_3 coincide once more for $\sigma^2>-4$ (i.e., $\beta^2=4$), while for $\sigma^2<-4$ the zeroes ν_1 and ν_4 coincide. In the first range of σ^2, the periodic solutions become a plane wave with constant amplitude, while in the second case they become a periodic solution in which the elliptic functions in the limit $m=0$ become elementary functions. It is important that the parameters of the wave in both cases are expressed in terms of the spectrum λ_i, whose complex nature leads to modulation instability of these solutions. This question will be discussed in the next section.

3. CREATION OF SOLITONS AT THE PULSE LEADING EDGE

The modulation of the periodic solutions we have found is described by Whitham's theory,[5] which leads, as is well known, to a diagonal form of the Whitham equations for the Riemann invariants λ_i $(i=1,2,3,4)$. Their complex nature implies modulation instability of the solution. In the special case of solution (50), we can confirm this directly by limiting ourselves to the linear approximation. In this case, let us modulate solution (50):

$$S_3=-\frac{1}{\sigma}\sqrt{\sigma^2-\gamma^2}+s[e^{i(K\xi-\Omega\tau)}+e^{-i(K\xi-\Omega\tau)}],$$

$$S_+=\left[\frac{\gamma}{\sigma}+s_+e^{i(K\xi-\Omega\tau)}+s_-e^{-i(K\xi-\Omega\tau)}\right]\times\exp\left(i\,\frac{2\sigma\sqrt{\sigma^2-\gamma^2}}{\sqrt{\Delta^2/4+\gamma^2}}\,\xi\right),$$

$$R_3=-\frac{\Delta\sqrt{\sigma^2-\gamma^2}}{2\sigma\sqrt{\Delta^2/4+\gamma^2}}+r[e^{i(K\xi-\Omega\tau)}+e^{-i(K\xi-\Omega\tau)}],$$

$$R_+=\left[\frac{\gamma}{2\sigma\sqrt{\Delta^2/4+\gamma^2}}+r_+e^{i(K\xi-\Omega\tau)}+r_-e^{-i(K\xi-\Omega\tau)}\right]\exp\left(i\,\frac{2\sigma\sqrt{\sigma^2-\gamma^2}}{\sqrt{\Delta^2/4+\gamma^2}}\,\xi\right).$$

Substituting these expressions into system (3), we linearize the latter with respect to the amplitudes $s, s_\pm, r, r_\pm$. Assuming these amplitudes are small, we obtain a linear homogeneous algebraic system of equations for them. The requirement that the determinant of this system vanish leads to the dispersion relation for the modulation waves, which after some rather lengthy calculations can be written in the form

$$K(\Omega)=\frac{\Omega(\sqrt{\Omega^2-4\gamma^2}-\Delta)}{\sqrt{\Delta^2+4\gamma^2[\Omega^2-(\Delta^2+4\gamma^2)]}}. \tag{51}$$

As we see, solution (50) is unstable against modulations with frequency $\Omega<2\gamma$, which is completely analogous to the case of the nonlinear Schrödinger equation,[3] the uniaxial magnet,[5] the AB system,[7] and the STW equations.[6]

The modulation instability of these waves leads to growth of any perturbation with harmonics such that $\Omega<2\gamma$. In particular, the leading edge of the pulse is transformed into an inhomogeneously broadened region, one end of which corresponds to solitons and the other end to modulation waves traveling along the pulse with a certain group velocity. The entire region can be viewed as the modulated nonlinear periodic wave we found in the previous section, with parameters λ_i $(i=1,2,3,4)$ that are slow functions of the spatial coordinates and time. Averaging over the fast oscillations of the wave leads to the Whitham equations for λ_i, which turn out to be Riemann invariants of these equations. Derivation of the Whitham equations is completely analo-

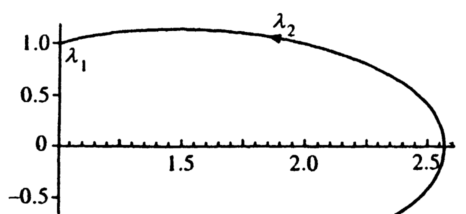

FIG. 3. Trajectory of the Riemann invariants λ_2 and λ_4 in the complex λ plane corresponding to the self-similar solution to the problem of soliton generation at the pulse leading edge.

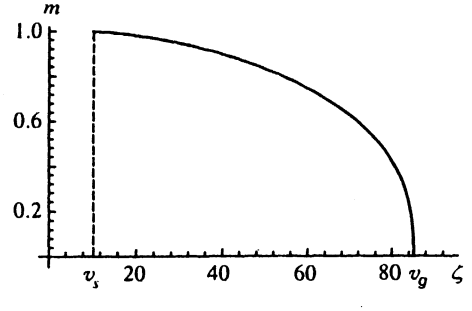

FIG. 4. Dependence of the parameter of the elliptic functions m on $\zeta=\xi/\tau$ for $\alpha=\gamma=1$, $\Delta=0.4$. The minimum velocity $m\to1$ corresponds to the soliton velocity (64) ($v_s=9.76$ for the parameter values chosen) and the maximum velocity $m\to0$ corresponds to the group velocity $v_g=d\Omega/dK$ for propagation of small modulations ($v_g=85.15$ in our case).

gous to the derivation used in Refs. 5, 7, 12, and 26 (see, also, Refs. 13, 14). Therefore, we will state only the result here. The Whitham equations for λ_i have the diagonal form

$$\frac{\partial\lambda_i}{\partial\xi}+\frac{1}{v_i}\frac{\partial\lambda_i}{\partial\tau}=0,\quad i=1,\,2,\,3,\,4, \tag{52}$$

where the Whitham group velocities v_i are given by

$$\frac{1}{v_i}=\left(1-\frac{T}{\partial_i T}\partial_i\right)\frac{1}{V},\quad \partial_i\equiv\frac{\partial}{\partial\lambda_i},\quad i=1,\,2,\,3,\,4, \tag{53}$$

the period T is determined from the expression

$$T=\frac{1}{2}\oint\frac{d\mu}{\sqrt{-P(\mu)}}=\frac{2\mathrm{K}(m)}{\sqrt{(\lambda_1-\lambda_4)(\lambda_3-\lambda_2)}}, \tag{54}$$

$\mathrm{K}(m)$ is the complete elliptic integral of the first kind, and V is defined in (38). (Note that these equations were obtained from the STW equations by replacing Δ by $-\Delta/2$, which corresponds to changing the position of the Lax-pair singularities in the complex λ plane; see also the Appendix.)

As an application of these equations, let us consider the evolution of a step-function pulse at $t=0$:

$$S_3=\begin{cases}\nu_3, & \xi\geqslant 0,\\ -1, & \xi<0,\end{cases} \tag{55}$$

where ν_3 corresponds to the values $\lambda_1=\lambda_3^*=\alpha+i\gamma$, $\lambda_2=\lambda_4=\beta$, i.e., the limit of zero modulation ($\delta=0$) of a plane wave (we assume that σ^2 is sufficiently small in absolute value that for $\delta=0$, the zeroes ν_2 and ν_3 of the resolvent coincide). Since in this problem there is no characteristic length, all the parameters λ_i depend only on the self-similar variable $\zeta=\xi/\tau$. By virtue of the complex conjugation relations $\lambda_3=\lambda_1^*$ and $\lambda_4=\lambda_2^*$, it is sufficient to consider only two of the Whitham equations (52), which in our self-similar case take the form

$$\frac{d\lambda_1}{d\zeta}(v_1-\zeta)=0,\quad \frac{d\lambda_2}{d\zeta}(v_2-\zeta)=0. \tag{56}$$

As we will verify, the initial conditions (55) correspond to the solution $\lambda_1=\mathrm{const}$, $v_2=\zeta=\xi/\tau$, or

$$\alpha+i\gamma=\mathrm{const}, \tag{57}$$

$$\left\{4\sqrt{\left[\left(\alpha+\frac{\Delta}{2}\right)^2+\gamma^2\right]\left[\left(\beta+\frac{\Delta}{2}\right)^2+\delta^2\right]}\right\}^{-1}\left\{1-\frac{1}{\beta-\delta+i\delta}\times\frac{2i\delta[\alpha-\beta+i(\gamma-\delta)]\mathrm{K}(m)}{[\alpha-\beta+i(\gamma-\delta)]\mathrm{K}(m)-[\alpha-\beta+i(\gamma+\delta)]\mathrm{E}(m)}\right\}=\frac{\tau}{\xi}, \tag{58}$$

where and $\mathrm{E}(m)$ is the complete elliptic integral of the second kind. Expanding the real and imaginary parts in (58), we obtain

$$\frac{\mathrm{E}(m)}{\mathrm{K}(m)}=\frac{\beta(\alpha^2+\beta^2+\gamma^2+\delta^2)-2\beta(\alpha\beta+\gamma\delta)+(\Delta/2)[(\alpha-\beta)^2+(\gamma-\delta)^2]}{\beta(\alpha^2+\beta^2+\gamma^2+\delta^2)-2\alpha(\beta^2+\delta^2)+(\Delta/2)[(\alpha-\beta)^2+\gamma^2-\delta^2]}, \tag{59}$$

$$-\left\{4\sqrt{\left[\left(\alpha+\frac{\Delta}{2}\right)^2+\gamma^2\right]\left[\left(\beta+\frac{\Delta}{2}\right)^2+\delta^2\right]}\right\}^{-1}\frac{\alpha(\beta^2+\delta^2)-\beta(\alpha^2+\gamma^2)-(\Delta/2)(\alpha^2-\beta^2+\gamma^2-\delta^2)-(\Delta^2/4)(\alpha-\beta)}{(\alpha-\beta)\left[\left(\beta+\frac{\Delta}{2}\right)^2+\delta^2\right]}=\frac{\tau}{\xi}, \tag{60}$$

where, together with $\alpha=\mathrm{const}$, $\gamma=\mathrm{const}$, and (32), we specify the dependence of β and δ on $\zeta=\xi/\tau$ implicitly.

As the authors of Ref. 6 have already noted, in the limit $\delta\to 0$ Eqs. (59) and (60) become the solution[7] of the AB system, while for $\delta\to\infty$ we recover, after suitable transformation of the space and time variables, the solution for the focusing nonuniform Schrödinger equation and for one-dimensional magnets.[3,5]

It is convenient to express β and δ in this solution as functions of m (see Refs. 3, 7, and 8):

$$\beta=-\frac{\Delta}{2}+\frac{\alpha+\Delta/2}{(\alpha+\Delta/2)^2+\gamma^2m^2A^2(m)}\left\{\left(\alpha+\frac{\Delta}{2}\right)^2+(2-m)\gamma^2A(m)+\gamma\sqrt{4\left(\alpha+\frac{\Delta}{2}\right)^2A(m)+4\gamma^2A^2(m)(1-m)-\left(\alpha-\frac{\Delta}{2}\right)^2[1+mA(m)]^2}\right\}, \tag{61}$$

$$\delta=\frac{\gamma}{\alpha+\Delta/2}mA(m)\left(\beta+\frac{\Delta}{2}\right), \tag{62}$$

where

$$A(m)=\frac{(2-m)\mathrm{E}(m)-2(1-m)\mathrm{K}(m)}{m^2\mathrm{E}(m)}. \tag{63}$$

The trajectories of the Riemann invariants λ_2 and λ_4 are shown in Fig. 3. for values of the parameters $\alpha=1$, $\gamma=1$, $\Delta=0.4$. The pair of complex-conjugate Riemann invariants starts at $\lambda_2=\lambda_4=2.57$ (for $m=0$, $\delta=0$) on the real axis, after which they move in the complex λ plane as a function of m and merge with $\lambda_1=1+i$ and $\lambda_3=1-i$ respectively when $m=1$. Substituting these expressions for β and δ into (60) gives us the dependence of $\zeta=\xi/\tau$ on m. An example of such

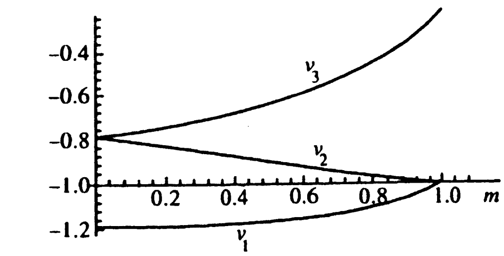

FIG. 5. Dependence of the zeroes ν_1, ν_2, ν_3 of the resolvent on m (the zero $\nu_4>1$ is much higher up, and is not shown in the figure). Merging with the plane wave and with the soliton front corresponds to merging of the pair of zeroes ν_2 and ν_3 at the boundary with the plane wave, and the pair ν_1 and ν_2 at the boundary with the region $S_3=-1$.

functions, Fig. 4, shows the behavior for our values of α and γ. Let us investigate this region of rapid oscillations at both of its ends. As $m\rightarrow 1$ we have

$$\beta+\frac{\Delta}{2}\simeq\left(\alpha+\frac{\Delta}{2}\right)\left(1+\frac{2\gamma\sqrt{1-m}}{\sqrt{(\alpha+\Delta/2)^2+\gamma^2}}\right),$$

$$\delta\simeq\gamma\left(1+\frac{2\gamma\sqrt{1-m}}{\sqrt{(\alpha+\Delta/2)^2+\gamma^2}}\right),$$

so that according to (60) this point moves with the soliton velocity

$$v_s=\frac{\xi}{\tau}\bigg|_{m\rightarrow 1}=4\left[\left(\alpha+\frac{\Delta}{2}\right)^2+\gamma^2\right]. \tag{64}$$

If $m\rightarrow 0$, then β and δ reduce to the values

$$\beta=-\frac{\Delta}{2}+\left(\alpha+\frac{\Delta}{2}\right)\left[1+\frac{3\gamma^2}{4(\alpha+\Delta/2)^2}\right.$$
$$\left.\times\left(1+\sqrt{1+\frac{8(\alpha+\Delta/2)^2}{9\gamma^2}}\right)\right],\quad \delta=0$$

and (60) takes the form

$$\frac{1}{v}=\frac{\tau}{\xi}=\frac{\alpha^2+\gamma^2-\alpha\beta+(\Delta/2)(\alpha-\beta)}{4(\beta+\Delta/2)^2(\alpha-\beta)\sqrt{(\alpha+\Delta/2)^2+\gamma^2}}. \tag{65}$$

In this limit of small modulation, the Whitham theory should reproduce the linear approximation, i.e., v should coincide with the corresponding group velocity of the modulation waves. From the general periodic solution (34), (38) we know that the phase of the modulation wave when $\delta=0$, taking (33) into account, has the form

$$\sqrt{(\alpha-\beta)^2+\gamma^2}\left[\tau-\frac{\xi}{4(\beta+\Delta/2)\sqrt{(\alpha+\Delta/2)^2+\gamma^2}}\right],$$

i.e., the frequency Ω and wave number K are expressed in terms of the parameters α, β and γ as follows:

$$\Omega=2\sqrt{(\alpha-\beta)^2+\gamma^2},$$

$$K=\frac{\Omega}{4(\beta+\Delta/2)\sqrt{(\alpha+\Delta/2)^2+\gamma^2}}. \tag{66}$$

It is easy to verify that in the special case $\alpha=0$ these values of Ω and K satisfy the dispersion relation (51). From (66) we can obtain the dispersion relation for the modulation waves:

$$K(\Omega)=\frac{\Omega[\sqrt{\Omega^2-4\gamma^2}-2(\alpha+\Delta/2)]}{2\sqrt{\left(\alpha+\frac{\Delta}{2}\right)^2+\gamma^2\{\Omega^2-4[(\alpha+\Delta/2)^2+\gamma^2]\}}}, \tag{67}$$

which generalizes (51) to the case of nonzero α. Calculation of the group velocity $v_g=(dK/d\Omega)^{-1}$ for values of Ω from (66), leads, as we might expect, to the solution (65) of the Whitham equations in the limit of weak modulation. We can show that $v_g>v_s$ for all values of α and γ. The dependence of the zeroes ν_1, ν_2, ν_3, on m in the region of oscillations is shown in Fig. 5. For $m=0$ (the end with the plane wave) the zeroes ν_2 and ν_3 merge, while for $m=1$ (the boundary with the region where $S_3=-1$) zeroes ν_1 and ν_2 merge. The degree to which this figure agrees with the behavior of the real Riemann invariants in problems of the Gurevich–Pitaevskiĭ type is especially noteworthy.[8,27–29] In the very strong dynamic Stark effect, when $\sigma^2<-\beta^2$, where β is the point of origin of the pair of complex-conjugate Riemann invariants λ_2 and λ_4 for prescribed values of the parameters α, γ, σ^2, the modulated wave for $m=0$ matches the limiting periodic solution discussed at the end of the previous section of the article.

As we see, a sharp front evolves into a broadening region of field oscillations. The slow edge of this region moves with the soliton velocity and consists of a sequence of solitons. The fast edge propagates along the pulse with the group velocity of the weak modulation wave. The entire region can be described as a modulated nonlinear periodic SRS wave. In Fig. 6 we show the region of oscillations for two values of the "time" τ. The transformation to physical time is accomplished by using Eq. (2). In these figures we see clearly the process of soliton creation at the trailing edge of a long

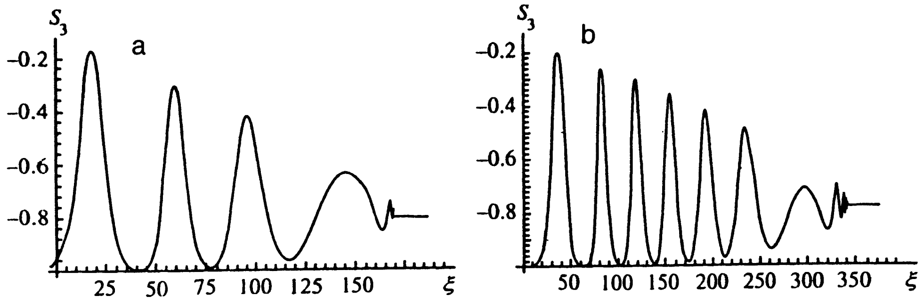

FIG. 6. Dependence of the variable S_3, which describes the field, on the coordinate ξ at two time points: a) $\tau=2$, b) $\tau=4$. The calculations were made using Eq. (34), where $\alpha=1$, $\gamma=1$, β and δ depend on the parameter m according to (61) and (62) and the dependence of ζ on m is determined by Eq. (60).

pulse. Note the relative narrowness of the region of small modulation. We expect that in experiments with pulses of finite width, this region will disappear after reaching the leading edge of the pulse, so that the pulse will be transformed into a finite sequence of solitons.[1,2]

4. CONCLUSION

The case we have discussed in this paper is in some sense a generalization of problems discussed previously.[3–7,9–12] In fact, the resolvent, which appears in our description of the periodic solution, depends on the parameter Δ in the same way as for a uniaxial ferromagnet.[11] As $\Delta\to 0$ we return to the simple cubic resolvent of the isotropic Heisenberg model,[5] which is gauge-equivalent to the nonlinear Schrödinger equation[9] with a different form of cubic resolvent, which also arises in the theory of STW.[12] On the other hand, the Whitham equations that describe the modulation of SRS waves coincide in essence with the corresponding equations for the STW case, which is due to the identity of the pole behavior of the coefficients of the Lax pair (see the Appendix). As $\Delta\to\infty$ these Whitham equations recover the modulation theory of periodic solutions of the nonlinear Schrödinger equation, while for $\Delta\to 0$ they recover the modulation theory for the AB system.[7] Thus, the theory set forth in this paper provides the machinery for solving a very wide class of problems described by integrable equations in the AKNS scheme.[20]

I am grateful to F. Zhinovar, A. L. Chernyakov, and H. Steudel for useful discussions.

5. APPENDIX

As was noted above, Eq. (19) for the phase velocity is general in character. Its importance stems from the fact that expressions (53) for the Whitham velocities also hold in the general case, since in essence they express the conservation of the "wave numbers" of the modulated wave.[30,31] Therefore, if we know how to express the phase velocity V of the nonlinear wave in terms of the Riemann invariants λ_i, we immediately obtain the Whitham modulation equations. Here we show how the derivation of Eq. (19) can be generalized to a wide class of equations described by the AKNS scheme.

Let the coefficients in A,B,C in the AKNS problem (5) have a pole as a function of λ at the point ρ, i.e.,

$$B(\lambda)=\frac{b(\lambda)}{\lambda-\rho},\quad C(\lambda)=\frac{c(\lambda)}{\lambda-\rho},\quad A(\lambda)|_{\rm pole}=\frac{a(\rho)}{\lambda-\rho},\tag{A1}$$

where we write out only the singular part of $A(\lambda)$ required here. We assume that the solution of Eq. (10) has the form

$$g(\lambda)=\phi(\lambda)(\lambda-\mu),\quad h(\lambda)=-\tilde{\phi}(\lambda)(\lambda-\mu^*),\tag{A2}$$

and $f(\lambda)$ is a second-order polynomial in λ. Substituting (A.2) into (10) for $\lambda=\mu$ gives an equation for $\mu(x,t)$ (compare with (17)):

$$\frac{\partial\mu}{\partial x}=-\frac{2iG(\mu)}{\phi(\mu)}f(\mu),\quad \frac{\partial\mu}{\partial t}=-\frac{2ib(\mu)}{\phi(\mu)(\mu-\rho)}f(\mu),\tag{A3}$$

and the same substitution for $\lambda=\mu^*$ leads to analogous equations for $\mu^*(x,t)$. Both $\mu(x,t)$ and $\mu^*(x,t)$ depend only on the variable $x-Vt$, so that the phase velocity we are looking for is

$$V=\frac{G(\mu)(\mu-\rho)}{b(\mu)}=\frac{H(\mu^*)(\mu^*-\rho)}{c(\mu^*)}.\tag{A4}$$

Cancellation of the singular terms in (10) leads to three relations, two of which can be written in the form

$$\frac{\phi(\rho)(\rho-\mu)}{ib(\rho)}=-\frac{f(\rho)}{a(\rho)},\quad \frac{\tilde{\phi}(\rho)(\rho-\mu^*)}{ic(\rho)}=\frac{f(\rho)}{a(\rho)},\tag{A5}$$

and the third is a consequence of them. From (A.4) and (A.5) we find

$$V^2=\frac{G(\mu)H(\mu^*)b(\rho)c(\rho)}{\phi(\rho)\tilde{\phi}(\rho)b(\mu)c(\mu^*)}\frac{f^2(\rho)}{a^2(\rho)}.\tag{A6}$$

Now we substitute (A.2) into (11) and set $\lambda=\rho$:

$$f^2(\rho)+\phi(\rho)\tilde{\phi}(\rho)(\rho-\mu)(\rho-\mu^*)=P(\rho).$$

This equation, together with (A.5), gives

$$\frac{f^2(\rho)}{a^2(\rho)}=\frac{P(\rho)}{a^2(\rho)+b(\rho)c(\rho)},$$

so that we are led to the required expression for the phase velocity

$$V=Q\sqrt{P(\rho)},\tag{A7}$$

where

$$Q=\sqrt{\frac{G(\mu)H(\mu^*)b(\rho)c(\rho)}{\phi(\rho)\tilde{\phi}(\rho)b(\mu)c(\mu^*)[a^2(\rho)+b(\rho)c(\rho)]}}.\tag{A8}$$

Although the expression for Q appears rather complicated, in practice it is easily calculated for each specific case. In particular, we verify quickly that for the case of SRS, when $\rho=-\Delta/2$, we have $Q=4$, which returns us to Eq. (19). Taking into account that the zeroes λ_i of the polynomial $P(\lambda)$ are Riemann invariants (see, e.g., Ref. 26), we are led by a very simple path back to the Whitham equations.

[1)]In recent papers,[13,14] A. A. Zabolotskiĭ investigated an analogous problem. However, the special periodic solutions he found for the SRS equations were in an ineffective form, which was inadequate to describe the processes of soliton creation within the framework of the SRS modulation theory.

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Translated by Frank J. Crowne