Nature of the universal properties of amorphous solids

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It is shown that multicenter excitations determine the universal low-energy spectral properties of an arbitrary ensemble of defect centers with an internal degree of freedom. This universality is manifested by a quasiuniform distribution of the tunneling amplitudes with respect to the energies and logarithms together with disappearance of the dependence on the parameters of the original defect centers. © *1996 American Institute of Physics.* [S1063-7761(96)02101-5]

1. INTRODUCTION

The anomalous low-temperature properties of amorphous systems that are totally different in nature demonstrate the pronounced universal character of their behavior (or, at least, its very nearly universal character). These properties have been described using the widely known tunneling model, which is based on a representation of an ensemble of randomly distributed double-well systems.[1,2] A key assumption here was that the distributions of the two-well systems with respect to the magnitude of the relative displacement of the levels in the wells Δ and the logarithm of the interwell tunneling amplitude Δ_0 are uniform. The first part of this assumption seems indisputable. However, the assumption that the distribution with respect to $\ln(\Delta_0)$ is uniform together with the simultaneous requirement that it hold over a very broad range of Δ_0 (to account for the logarithmic distribution of the relaxation times τ over many orders of magnitude) in the general case does not have sufficient foundation.

It seems to us that the universality of the low-energy spectral properties of amorphous systems appears as a result of a many-particle interaction between defect centers having a dynamic degree of freedom. The unavoidable appearance of such centers is attributed to (besides some other possible causes) the removal of the "spatial degeneracy" characteristic of crystal structures by local stresses. A vivid example of this is provided by a crystalline chain (or ribbon) experiencing compression in the longitudinal direction. Instead of one orientation in space, there are now at least two orientations with similar energies and with a degree of freedom corresponding to transitions between the configurations. In this sense we obtain a variant of a double-well center. The amorphous structure developed will unavoidably have an appreciable concentration of such dynamic defect centers. If a double-well model is used to describe them, the distribution with respect to the tunneling amplitudes Δ_0 can be comparatively arbitrary. Of course, such a primary system of double-well centers itself will not have universal spectral properties.

The main purpose of the present work is to investigate the shaping of the low-energy spectral properties of amorphous systems appearing as a result of the interaction of the primary centers and multicenter clusters. We see that as the energy decreases, an increasingly large number of clusters become involved in this process. At the same time, the relative role of the primary structure of the defect centers diminishes. As a result, we arrive at a universal structure of spectral properties that is essentially insensitive to the distribution of the parameters of the primary defect centers. The amplitudes of the many-particle transitions then have a nearly uniform logarithmic distribution. This predetermines the logarithmic distribution of the relaxation times, and the appearance of enormous values of τ is attributed to displacement of the center of gravity to the excitation of large clusters.

The entire treatment relies heavily on the assumption that the absolute value of the interaction between the defect centers decreases with the distance according to a $1/R^3$ law (or as $1/R^d$, where d is the dimensionality of the space, in the general case). It is generally known that such a law always holds in a dielectric medium owing to the indirect interaction between the defects through the deformation field. In a metal the indirect interaction mediated by the conduction electrons leads to the same law. In all cases the distances are large compared with the interatomic distances, but we are interested in just such distances.

It should be noted that several years ago Yu and Leggett[3,4] hypothesized that a $1/R^3$ law for the interaction between defects might be responsible for the universality of the anomalous properties of amorphous systems. However, the question of the nature of the excitations in this case remained open. The results obtained in the present work provide evidence supporting that hypothesis.

In the case of the $1/R^3$ law the logarithmic increase in the effective interaction region with the size of the system is a significant point. It suggests that a renormalization group procedure can be used. Such a procedure was first demonstrated by Levitov[5] in the problem of the localization and delocalization of elementary excitations in a system with a dipole–dipole interaction. The use of a similar approach allows us to obtain a general renormalization group equation, whose solution leads to the main results.

When low-energy excitations are found in a system of interacting centers with $T \to 0$, correct determination of the ground state is of fundamental importance. Its stability requires positive values for the energies of any many-particle excitations, thereby introducing a hierarchy of constraints. The influence of the latter on the density of states is similar to the familiar Coulomb gap in semiconductors,[6–8] although it appears in a different form. In the problem under consideration the constraints introduced by the stability requirements are very significant and must be taken into account from the very outset.

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To make the problem clear from a physical standpoint, we begin with a treatment of spectral properties in the limit of weak interaction between the centers, restricting ourselves to two-particle and three-particle interactions.

2. MANY-PARTICLE EXCITATIONS. WEAK INTERACTIONS

Let us consider an amorphous medium with randomly distributed, double-well defect centers with an arbitrary distribution of the parameters. The interaction Hamiltonian of such a system in the standard pseudospin representation has the form

$$H=H_0+U+V, \tag{2.1}$$

$$H_0=\sum_i \omega_i S_i^z, \quad U=\frac{1}{2}\sum_{ij} U_{ij}\, S_i^z S_j^z \quad V=\sum_i \Delta_{0i} S_i^x. \tag{2.2}$$

The magnitude of the interaction between centers can be represented in the form

$$U_{ij}=u_{ij}/R_{ij}^3. \tag{2.3}$$

We assume that the interaction constants u_{ij} for different pairs of centers are uncorrelated and have random signs. In that case we can write

$$\langle |u_{ij}| \rangle = U_0, \tag{2.4}$$

We assume that the energy spread of the centers W is large in comparison to the interaction of the centers at the mean distance:

$$W \gg nU_0, \tag{2.5}$$

where n is the density of the centers. In this case the distribution of the energies of the one-particle excitations (with $\Delta_{0i}=0$)

$$\varepsilon_i=\omega_i+\sum_j U_{ij}S_j^z \tag{2.6}$$

is essentially given by the original spread of the ω_i.

The distribution function of the parameters of the original centers in the absence of a correlation between ε_i and Δ_{0i} can be represented in the form

$$P(\Delta,\Delta_0)=P_0(\Delta)P_0'(\Delta_0), \quad \int P_0'(\Delta_0)d\Delta_0=1,$$

$$P_0(\Delta)=\frac{1}{\Omega}\left\langle \sum_i \delta(\Delta-\varepsilon_i)\right\rangle \tag{2.7}$$

(Ω is the volume of the system). Since the distribution function is calculated per unit volume,

$$\int P_0(\Delta)d\Delta=n. \tag{2.8}$$

We adopt the natural assumption that the transition amplitudes Δ_{0i} lie in a restricted energy range with the characteristic value

$$\Delta_{0*} \ll W. \tag{2.9}$$

We begin by considering the density of states at low energies

$$\Delta_{0*} < \varepsilon \ll W. \tag{2.10}$$

This allows us to omit the term V in the Hamiltonian (2.1).

The stability of the ground state requires that $\varepsilon_i>0$. This also applies to many-particle excitations. Considering the case of a weak interaction between centers in this section, we restrict ourselves to an analysis of the role of two-particle and three-particle excitations, whose stability conditions can be written in the form

$$\varepsilon_{ij}=\varepsilon_i+\varepsilon_j-U_{ij}>0,$$

$$\varepsilon_{ijk}=\varepsilon_i+\varepsilon_j+\varepsilon_k-U_{ij}-U_{ik}-U_{jk}>0. \tag{2.11}$$

To first order in the interaction, the contribution of the pair excitations to the density of states can be found directly:

$$P_2(\Delta)=\frac{1}{2\Omega}\left\langle \sum_{ij} \delta(\Delta-\varepsilon_{ij})\right\rangle$$

$$=\frac{1}{2}P_0^2(0)\int d\mathbf{R}_{ij}\int d\varepsilon_i \int d\varepsilon_j$$

$$\times\left\langle \delta\left(\Delta-\varepsilon_i-\varepsilon_j+\frac{u_{ij}}{R_{ij}^3}\right)\right\rangle_u . \tag{2.12}$$

Since the small values ε_i, $\varepsilon_j \ll W$ contribute to the integral, the functions $P_0(\Delta)$ are replaced by the limiting values $P_0(0)$. The notation $\langle \ldots \rangle_u$ signifies averaging over the u_{ij}. Taking into account that $\varepsilon_i>0$, we find

$$P_2(\Delta)=\frac{1}{2}P_0^2(0)\int d\mathbf{R}_{ij}\left\langle \left(\Delta+\frac{u_{ij}}{R_{ij}^3}\right)\theta\left(\Delta+\frac{u_{ij}}{R_{ij}^3}\right)\right\rangle_u .$$

Here $\theta(x)$ is a standard unit step function. Let the interaction region be restricted to the radius $R_{\max}$, which corresponds to

$$U_0/R_{\max}^3 \gg \Delta_{0*}. \tag{2.13}$$

In the range of energies $\Delta<U_0/R_{\max}^3$ only the positive u_{ij} make a contribution. Assuming that the distribution of the u_{ij} is symmetric about zero, we have

$$P_2(\Delta)=P_0(0)\chi\left(\xi+\frac{1}{3}\frac{\Delta}{U_0R_{\max}^{-3}}\right). \tag{2.14}$$

Here

$$\chi=\pi P_0(0)\langle |u| \rangle=\pi P_0(0)U_0, \quad \xi=\ln(R_{\max}/R_{\min}). \tag{2.15}$$

The value of $R_{\min}$ can be specified to logarithmic accuracy by the relation

$$U_0/R_{\min}^3=W, \tag{2.16}$$

and the contribution of the incidental nearby pairs, which cause a shift of the level exceeding W, can be ignored.

The second term in parentheses on the right-hand side of Eq. (2.14) is small compared with $\xi \gg 1$, and we ultimately find

$$P_2(\Delta)\approx P_0(0)\chi\xi. \tag{2.17}$$

The dimensionless parameter $\chi \approx \pi n U_0/W$ is small compared with unity. The perturbation theory considered in this

section foresees that the product $\chi\xi$ is small. In a first approximation with respect to this product, the density of states of the one-particle excitations $P_1(\Delta)$ is also renormalized owing to the condition of stability with respect to all the two-particle excitations ε_{ij} [see (2.11)] involving the subject center i. The general expression for $P_1(\Delta)$ in this case can be written in the form

$$P_1(\Delta)=\frac{1}{\Omega}\sum_i\left\langle\delta(\Delta-\varepsilon_i)\prod_j\theta(\varepsilon_{ij})\right\rangle. \tag{2.18}$$

If $\theta(\varepsilon_{ij})$ is rewritten as

$$\theta(\varepsilon_{ij})=1-\theta(u_{ij}/R_{ij}^3-\varepsilon_i-\varepsilon_j)$$

and it is taken into account that this expression differs from unity only in a small part of the phase space, we have

$$\begin{aligned}P_1(\Delta)&\approx\frac{1}{\Omega}\sum_i\left\langle\delta(\Delta-\varepsilon_i)\left[1-\sum_j\theta\left(\frac{u_{ij}}{R_{ij}^3}-\varepsilon_i-\varepsilon_j\right)\right]\right\rangle\\&=P_0(0)\left[1-P_0(0)\int d\mathbf{R}_{ij}\int d\varepsilon\right.\\&\left.\times\left\langle\theta\left(\frac{u_{ij}}{R_{ij}^3}-\Delta-\varepsilon\right)\right\rangle_u\right].\end{aligned} \tag{2.19}$$

Under the same assumptions as in the derivation of (2.12), we ultimately obtain

$$P_1(\Delta)\approx P_0(0)(1-2\chi\xi). \tag{2.20}$$

This expression demonstrates the tendency for a decrease in the density of states of the one-particle excitations (a "dipole quasigap").

Three-particle excitations contribute to the density of states of the low-frequency excitations in the second order with respect to $\chi\xi$. Again, taking into account that centers with energy $\varepsilon\ll W$ are involved in three-particle excitations with a total energy $\Delta<U_0/R_{\max}^3$, we have

$$\begin{aligned}P_3(\Delta)=\frac{1}{6}P_0^3(0)&\int d\mathbf{R}_{ik}\int d\mathbf{R}_{jk}\int d\varepsilon_i\int d\varepsilon_j\int d\varepsilon_k\\&\times\left\langle\delta\left(\Delta-\varepsilon_i-\varepsilon_j-\varepsilon_k+\frac{u_{ij}}{R_{ij}^3}+\frac{u_{ik}}{R_{ik}^3}+\frac{u_{jk}}{R_{jk}^3}\right)\right.\\&\times\theta\left(\varepsilon_i+\varepsilon_j-\frac{u_{ij}}{R_{ij}^3}\right)\theta\left(\varepsilon_i+\varepsilon_k-\frac{u_{ik}}{R_{ik}^3}\right)\\&\left.\times\theta\left(\varepsilon_k+\varepsilon_j-\frac{u_{jk}}{R_{jk}^3}\right)\right\rangle_u.\end{aligned} \tag{2.21}$$

(for simplicity, we chose a concrete pair of vectors in the triangle ijk, and the corresponding symmetrization will be restored at the end.)

After removal of the δ function due to the integration with respect to ε_k, the following four inequalities demarcate the integration phase volume ($\varepsilon_i>0$, $\Delta\approx0$):

$$\begin{aligned}&\varepsilon_i+\varepsilon_j>\frac{u_{ij}}{R_{ij}^3},\quad\varepsilon_i+\varepsilon_j<\frac{u_{ij}}{R_{ij}^3}+\frac{u_{ik}}{R_{ik}^3}+\frac{u_{jk}}{R_{jk}^3},\\&\varepsilon_i<\frac{u_{ij}}{R_{ij}^3}+\frac{u_{ik}}{R_{ik}^3},\quad\varepsilon_j<\frac{u_{ij}}{R_{ij}^3}+\frac{u_{jk}}{R_{jk}^3}.\end{aligned} \tag{2.22}$$

The largest contribution (to logarithmic accuracy) to the integral is made by configurations in which one side of the triangle (let it be $\mathbf{R}_{ij}$ in the present case) is small compared with the other two. Such a choice introduces the additional factor 3 into (2.21), since any of the sides of the triangle could be selected as the smallest side. In this case $R_{ij}\sim R_{jk}$. We introduce the variable $y=\varepsilon_i+\varepsilon_j$. It follows from the first two inequalities in (2.22) that

$$U_{ij}<y<U_{ij}+U_{ik}+U_{jk}. \tag{2.23}$$

The inequalities (2.23) presume that $u_{ij}>0$, and when $R_{ik}\sim R_{jk}$, if follows from (2.22) that $u_{ik}+u_{jk}>0$.

Then

$$\int d\varepsilon_i\int d\varepsilon_j\ldots=\int d\varepsilon_i\int_{\varepsilon_i}dy\ldots\approx U_{ij}(U_{ik}+U_{jk})\ldots$$

In accordance with the remark already made, here we omitted terms quadratic in U_{ik} and U_{jk}.

When the remaining integrals of $U_{ij}U_{ik}$ in (2.21) are taken, we switch from integration with respect to $\mathbf{R}_{ik}$ to integration with respect to $\mathbf{R}_{ij}$, setting $R_{ij}<R_{ik}$:

$$\int d\mathbf{R}_{ij}\int d\mathbf{R}_{ik}\theta(R_{ik}-R_{ij})U_{ij}U_{ik}=(4\pi)^2u_{ij}u_{ik}\xi^2/2,$$

where $\xi=\ln(R_{\max}/R_{\min})$ [see (2.15)]. The integral of $U_{ij}U_{jk}$ leads to a similar result.

The last step is the averaging of the product $u_{ij}(u_{ik}+u_{jk})$, each of whose cofactors is positive. The assumption regarding the symmetry of the distribution function for u_{ij} is transferred directly to the sum $u_{ik}+u_{jk}$:

$$\langle u_{ij}(u_{ik}+u_{jk})\rangle+\frac{1}{4}U_0^2\frac{\langle|u_{ik}+u_{jk}|\rangle}{U_0}.$$

We finally arrive at the following expression for the density of states (2.21) [see (2.15)]:

$$P_3\approx P_0(0)(\chi\xi)^2\langle|u_1+u_2|\rangle/U_0. \tag{2.24}$$

Two contributions to P_1 appear in the same order with respect to $\chi\xi$. The first is associated with the stability of the three-center excitations containing the center whose excitation is being studied. The second is attributed to the fact that in passing from (2.18) to (2.19) we took into account only the contribution which is linear with respect to the instability condition. The bilinear contribution must be taken into account already in the order under consideration. It is easy to understand that the corresponding correction to P_1 has a positive sign. The calculations lead to the following result:

$$P_1\approx P_0(0)\{1-2\chi\xi-(\chi\xi)^2\langle|u_1+u_2|\rangle/U_0+4(\chi\xi)^2\}. \tag{2.25}$$

In the approximation under consideration the renormalization of P_2 is associated with the condition of stability of all the pairs sharing a center with the pair under analysis, as well

as the groups of three which include that pair. Again, omitting the calculations, we present the final result:

$$P_2 \approx P_0(0)\chi\xi\{1-\chi\xi\langle|u_1+u_2|\rangle/U_0\}. \tag{2.26}$$

An analysis of (2.24), (2.25), and (2.26) makes it possible to draw several conclusions. The low-energy density of states $P(\Delta)$ depends on the single dimensionless parameter $\chi\xi$, which is a correlation parameter by nature. At a sufficiently low value of the energy Δ the function $P(\Delta)$ does not depend at all on the energy. As $\chi\xi$ increases, the role of the many-particle excitations increases, while the role of the one-particle excitations decreases. This also applies to larger configurations: $P_n \propto (\chi\xi)^{n-1}$. It can be concluded from the form of the expressions obtained that many-particle excitations begin to dominate when $\chi\xi \geqslant 1$.

3. DYNAMICAL PROPERTIES WITH CONSIDERATION OF THE PAIRWISE INTERACTION BETWEEN DEFECT CENTERS

Let us now include intracenter transitions in the discussion, i.e., let us turn to the complete Hamiltonian (2.1) with a nonzero V (2.2). In order to highlight the special role of excitations involving more than one particle in this case, we consider the limit $\chi\xi \ll 1$ and analyze the contribution of pair (two-center) excitations to the spectral characteristics of the system. As will be shown, already in this case the distribution function of the effective parameters exhibits a uniform distribution with respect to the logarithm of the transition amplitude (and thus with respect to the logarithm of the relaxation time) under an arbitrary $P_0'(\Delta_0)$ in (2.7).

Let us identify the Hamiltonian of an individual pair. When (2.6) is taken into account, we have

$$H_p = \varepsilon_1 S_1^z + \varepsilon_2 S_2^z + \Delta_{01} S_1^x + \Delta_{02} S_x^x - U_{12}(S_1^z + S_2^z)/2 + U_{12} S_1^z S_2^z. \tag{3.1}$$

We transform to a representation in which the one-center Hamiltonians are diagonal. In spin space this transformation corresponds to rotation of the quantization axes:

$$S_i^z = \frac{\varepsilon_i}{E_i}\tilde{S}_i^z - \frac{\Delta_{0i}}{E_i}\tilde{S}_i^x, \quad S_i^x = \frac{\Delta_{0i}}{E_i}\tilde{S}_i^z + \frac{\varepsilon_i}{E_i}\tilde{S}_i^x. \tag{3.2}$$

Here $E_i = \sqrt{\varepsilon_i^2 + \Delta_{0i}^2}$ is the excitation energy of a two-level center.

Using (3.2), we find the Hamiltonian of a pair H_p in the new representation:

$$H_p = h_0 + h', \tag{3.3}$$

$$h_0 = E_1\tilde{S}_1^z + E_2\tilde{S}_2^z - \frac{\varepsilon_1}{2E_1}U_{12}\tilde{S}_1^z - \frac{\varepsilon_2}{2E_2}U_{12}\tilde{S}_2^z + U_{12}\frac{\varepsilon_1\varepsilon_2}{E_1E_2}\tilde{S}_1^z\tilde{S}_2^z + U_{12}\frac{\Delta_{01}\Delta_{02}}{E_1E_2}\tilde{S}_1^x\tilde{S}_2^x, \tag{3.4}$$

$$h' = -U_{12}\frac{\varepsilon_1\Delta_{02}}{E_1E_2}\tilde{S}_1^z\tilde{S}_2^x - U_{12}\frac{\Delta_{01}\varepsilon_2}{E_1E_2}\tilde{S}_1^x\tilde{S}_2^z + \frac{U_{12}}{2} \times \left(\frac{\Delta_{02}}{E_1}\tilde{S}_1^x + \frac{\Delta_{02}}{E_2}\tilde{S}_2^x\right). \tag{3.5}$$

We assume below that

$$\Delta_{0i} < \varepsilon_i. \tag{3.6}$$

This allows us to set $\varepsilon_i/E_i \approx 1$ in (3.4) and (3.5). Then for the energy of an *up–up* pair excitation, from (3.4) we have

$$\Delta = E_1 + E_2 - U_{12}. \tag{3.7}$$

The transition amplitude between the ground and excited states of a pair is determined by the last term in (3.4) in second order with respect to h'. However, the contribution of h' vanishes up to terms quadratic in Δ_{0i}. As a result, the transition amplitude is

$$\Delta_0 = \frac{1}{2}\frac{\Delta_{01}\Delta_{02}}{E_1E_2}U_{12}. \tag{3.8}$$

Let us now determine the distribution function of the parameters Δ and Δ_0 characterizing pair excitations:

$$P_2(\Delta,\Delta_0) = \frac{1}{2}P_0^2(0)\int d\Delta_{01}P_0'(\Delta_{01}) \times \int d\Delta_{02}P_0'(\Delta_{02})\int d\varepsilon_1\int d\varepsilon_2\int d\mathbf{R}_{12} \times \left\langle \delta(\Delta-\varepsilon_1-\varepsilon_2+U_{12}) \times \delta\left(\Delta_0 - \frac{1}{2}\frac{\Delta_{01}\Delta_{02}}{\varepsilon_1\varepsilon_2}\frac{|u|}{R_{12}^3}\right)\right\rangle_u. \tag{3.9}$$

Taking (3.6) into account, in this expression we replaced E_i by ε_i in the arguments of the δ functions. Bearing in mind that $\varepsilon_i \ll W$, as in the preceding section, we took P_0 out of the integral sign in the zeroth argument. The appearance of the absolute value $|u|$ in the second δ function is due to the fact that the distribution function was defined for the absolute values of the transition amplitudes, i.e., $\Delta_0 > 0$ in (3.9).

To consider the low-energy spectral properties, we assume that

$$\Delta < U_0/R_{\max}^3. \tag{3.10}$$

In this case the first δ function automatically eliminates only pairs with $u > 0$. Then removing the second δ function by integrating over $\mathbf{R}_{12}$, at fixed Δ_{0i} we have

$$J = \frac{\pi}{3}\frac{\Delta_{01}\Delta_{02}\langle|u|\rangle}{\Delta_0^2}\int d\varepsilon_1\int\frac{d\varepsilon_2}{\varepsilon_1\varepsilon_2}\delta \times\left(\Delta-\varepsilon_1-\varepsilon_2+\frac{2\Delta_0\varepsilon_1\varepsilon_2}{\Delta_{01}\Delta_{02}}\right) \times\frac{\pi\Delta_{01}\Delta_{02}U_0}{3\Delta_0^2}\int\frac{d\varepsilon_1}{\varepsilon_1|\varepsilon_1-\Delta|}\theta \times\left(\frac{\varepsilon_1-\Delta}{2\Delta_0\varepsilon_1/\Delta_{01}\Delta_{02}-1}\right). \tag{3.11}$$

The range of permissible values of Δ_0 is dictated by the boundaries of the region where R is defined. Here it is significant that at each value of R the characteristic values of the energies of the one-particle excitations in (3.9) be

$$\varepsilon_1, \varepsilon_2 \sim U_0/2R^3. \tag{3.12}$$

Taking this into account, we find approximately

$$2\Delta_{01}\Delta_{02}R_{\min}^3/U_0 < \Delta_0 < 2\Delta_{01}\Delta_{02}R_{\max}^3/U_0. \tag{3.13}$$

In (3.11) the θ function permits two energy regions. In the first case we have the requirements

$$\varepsilon_1 > \Delta, \quad \varepsilon_1 > \Delta_{01}\Delta_{02}/2\Delta_0. \tag{3.14}$$

Since we are interested in the limit of small Δ and Δ_0, it is natural to assume that the second inequality is stronger, i.e.,

$$\Delta_{01}\Delta_{02} > 2\Delta_0\Delta. \tag{3.15}$$

In this case the lower limit of the integral in (3.11) is given by

$$\varepsilon_{\min} = \Delta_{01}\Delta_{02}/2\Delta_0. \tag{3.16}$$

The inequality (3.6), which we assumed from the start, now requires

$$\Delta_{02} > 2\Delta_0 \tag{3.17}$$

(the subscripts 1 and 2 appear symmetrically in the original expression (3.9); therefore, the inequalities (3.17) and (3.14) naturally hold after the replacement $1 \leftrightarrow 2$).

The integral in Eq. (3.11) is determined by the lower limit and is simply equal to $1/\varepsilon_{\min}$.

The second region corresponds to

$$\varepsilon_1 < \Delta, \quad \varepsilon_1 < \Delta_{01}\Delta_{02}/2\Delta_0.$$

It can, however, be shown that the second of these inequalities is at variance with (3.8) and (3.12); therefore, this region does not make a contribution to J (3.11).

Using (3.16) for the final calculation of (3.11) and recalling (3.9), we have

$$P_2(\Delta,\Delta_0) = \frac{1}{3}P_0^2(0)\frac{\pi U_0}{\Delta_0}\int d\Delta_{01}P_0'(\Delta_{01}) \times \int d\Delta_{02}P_0'(\Delta_{02})\,\theta(\Delta_{01}-2\Delta_0)\,\theta(\Delta_{02} - 2\Delta_0)\,\theta(\Delta_{01}\Delta_{02}-2\Delta_0\Delta). \tag{3.18}$$

Here the θ function in the integral appeared as a consequence of the inequalities (3.15) and (3.17).

We use Δ_0 to denote the characteristic scale corresponding to the distribution P_0'. Assuming that

$$\Delta_0, \quad \Delta \ll \Delta_{0*} \tag{3.19}$$

and that P_0' does not have singularities at zero, we ultimately find [see the normalization in (2.7)]

$$P_2(\Delta,\Delta_0) = \frac{1}{3}\frac{P_0(0)\chi}{\Delta_0}, \tag{3.20}$$

where χ is defined by (2.15). The inequality (3.19) for Δ is easily satisfied, since there is no lower bound on Δ. The lower bound on Δ_0 introduced by the inequality (3.13),

$$\Delta_0 > 2\Delta_{01}\Delta_{02}R_{\min}^3/U_0 \sim \Delta_{0*}^2/W, \tag{3.21}$$

is easily combined with (3.19), opening up a broad range of variation for Δ_0 in which (3.20) is valid.

Thus, pair excitations of interacting double-well centers with a totally arbitrary distribution $P'(\Delta_{01})$ that is nonsingular at zero have a distribution of the transition amplitude that is characteristic of the universal properties of amorphous systems in the general case.

We note that if $U_0/R_{\max}^3 \geqslant \Delta_{0*}$ [see (3.13)], we have

$$\int_{\Delta_{0\min}}^{\Delta_{0\max}} P_2(\Delta,\Delta_0)d\Delta_0 \approx P_0(0)\chi\xi.$$

This result coincides with (2.17).

4. DERIVATION OF RENORMALIZATION GROUP EQUATIONS

The results obtained in Sec. 2 demonstrate that the effective interaction constant $\chi\xi$, which is responsible for the formation of multicenter clusters with collective excitations having an energy $\varepsilon \approx 0$, increases logarithmically with the volume of the region under consideration. In a big enough system, a situation corresponding to a strong interaction is realized nearly always, at least when $T \to 0$. All this is a direct consequence of the interaction of the centers according to a $1/R^d$ law in d-dimensional space.

The appearance of multicenter clusters with a probability of order unity requires $\chi\xi \approx 1$. Since $\chi \ll 1$, the corresponding size of the region is exponentially large:

$$R_*/R_{\min} \simeq \exp(1/\chi). \tag{4.1}$$

Here R_* is the distance scale on which a particular arbitrary cluster finds a partner with which it forms a more complex cluster.

We begin, as in Sec. 2, with $V=0$ in the Hamiltonian (2.1). Let the radius of the interaction region be restricted at first to $R_1 \gg R_{\min}$ and let the distribution function of the collective excitations involving n centers with an energy $\varepsilon < U_0/R_1^3$ be denoted by $P_n(\varepsilon,R_1)$. We determine the change in the distribution function when the interaction radius increases to R_2:

$$1 \ll R_2/R_1 \ll \exp(1/\chi). \tag{4.2}$$

Here

$$\chi \ln(R_2/R_1) \ll 1,$$

and the probability of the formation of new clusters with an excitation energy $\varepsilon < U_0/R_2^3$ is low according to this parameter (see Sec. 2). At the same time, the formation of new clusters occurs statistically at distances $R \gg R_1$. This allows us to treat the clusters as point formations. On the other hand, the displacement of the energy into the region $\varepsilon \sim U_0/R_2^3$ suggests that the energy of the collective excitations of both clusters should be of the same order. Since other intracluster excitations will have an energy $\varepsilon > U_0/R_1^3$, this means that clusters enter larger formations as single objects.

It is clear from the structure of the results in Sec. 2 for P_n that the probability of a ternary union of clusters will be down by an additional factor of $\chi \ln(R_2/R_1)$, and that the probability of an m-fold union will be down by

$[\chi \ln(R_2/R_1)]^{m-2}$. Therefore, when the difference $P_n(\varepsilon,R_2)-P_n(\varepsilon,R_1)$ is calculated, we can restrict ourselves to consideration of only the pairwise unions.

As the interaction radius increases, the value of P_n decreases due to the "dipole-gap" effect, since the number of configurations that must satisfy the stability condition increases. On the other hand, there is an increase in P_n due to the pairwise interaction of small clusters. The general equation then has the form

$$P_n(\varepsilon,R_2)=\left\langle P_n(\varepsilon,R_1)\prod_{k,j}\theta\left(\varepsilon+\varepsilon^{(k)}-\frac{u_{kn}}{R_j^3}\right)\right\rangle + \frac{1}{2\Omega}\sum_{k=1}^{n-1}\sum_j\left\langle\delta\left(\varepsilon-\varepsilon^{(k)}-\varepsilon^{(n-k)}+\frac{u_{k,n-k}}{R_j^3}\right)\right\rangle. \tag{4.3}$$

Here $\varepsilon^{(k)}$ is the excitation energy of a k cluster; u_{km} is the interaction constant of k and m clusters. The same arguments that were used to go from (2.18) to (2.19) allow us to switch from the product in the first term in (4.3) to the sum, which, conversely, determines the instability condition.

By considering an excitation energy $\varepsilon<U_0/R_2^3$, we can calculate the right-hand side of (4.3) in the limit $\varepsilon\to 0$. In this case the contribution to (4.3) after the transformation just noted is made only by the configurations with $u_{km}>0$. As before, we assume that a distribution of the u_{km} that is symmetric about zero is realized. Bearing in mind (4.2), in calculating the right-hand side of (4.3) we can assume that the distribution functions do not differ from $P_k(\varepsilon,R_1)$. Taking into account all that has been stated, we have

$$P_n(0,R_2)=P_n(0,R_1) -\frac{1}{2}P_n(0,R_1)\sum_{k=1}^{\infty}\int_{R_1}^{R_2}d\mathbf{R}\int d\varepsilon_1 P_k(0,R_1) \times\left\langle\theta\left(\frac{|u_{kn}|}{R^3}-\varepsilon_1\right)\right\rangle_u + \frac{1}{4}\sum_{k=1}^{n-1}\int_{R_1}^{R_2}d\mathbf{R}\int d\varepsilon_1\int d\varepsilon_2 P_k(0,R_1) \times P_{n-k}(0,R_1)\left\langle\delta\left(\frac{|u_{k,n-k}|}{R^3}-\varepsilon_1-\varepsilon_2\right)\right\rangle_u. \tag{4.4}$$

This expression can be calculated directly. As a result we obtain

$$P_n(0,R_2)-P_n(0,R_1)=-2\pi P_n(0,R_1) \times\sum_{k=1}^{\infty}\langle|u_{kn}|\rangle P_k(0,R_1)\ln\left(\frac{R_2}{R_1}\right) + \pi\sum_{k=1}^{n-1}\langle|u_{k,n-k}|\rangle P_k(0,R_1)P_{n-k}(0,R_1)\ln\left(\frac{R_2}{R_1}\right). \tag{4.5}$$

Noting that P_n varies little in going from R_1 to R_2, we arrive at the general renormalization group equation

$$\frac{\partial\tilde P_n(t)}{\partial t}=-2\tilde P_n(t)\sum_{k=1}^{\infty}b_{kn}\tilde P_k(t) + \sum_{k=1}^{n-1}b_{k,n-k}\tilde P_k(t)\tilde P_{n-k}(t). \tag{4.6}$$

Here we have adopted the notation [see (2.15)]

$$t=\chi\xi(R),\quad \xi(R)=\ln\frac{R}{R_{\min}},$$
$$\tilde P_n(t)=\frac{P_n(0,\xi(R))}{P_0},\quad b_{km}=\frac{\langle|u_{km}|\rangle}{U_0}. \tag{4.7}$$

It should be noted that the solution of Eq. (4.6) corresponding to a definite t [or $\xi(R)$] gives the distribution function corresponding to the energy range $\varepsilon<U_0/R^3$.

We seek a solution for $R=R_{\max}$. The limiting radius $R_{\max}$ yields the largest spatial region in which the cluster under consideration can combine with some other cluster to form a larger object. At finite temperatures $R_{\max}=R_T$ and $U_0/R_T^3\approx T$ (at large distances the interaction between the clusters is smaller than T, and their excitations are independent). If the cluster excitation energy ε is less than T, the $\tilde P_n$ do not depend on ε. Otherwise, $R_{\max}$ reduces effectively to R_ε, which is defined by the relation $U_0/R_\varepsilon^3\approx\varepsilon$. Thus, to logarithmic accuracy the final solution takes the form $\tilde P_n(t_m)$:

$$t_m=\chi\ln(R_{\max}/R_{\min}),\quad R_{\max}=\min(r_T,R_\varepsilon).$$

Note that if all clusters with $n>1$ are omitted from Eq. (4.6), the renormalized equation takes the form

$$\partial\tilde P_1/\partial t=-2\tilde P_1^2,$$

which has the simple solution

$$P_1(R_{\max})=P_0/[1+2\pi U_0P_0\ln(R_{\max}/R_{\min})].$$

At $T=0$ this solution coincides with the result previously obtained for a one-particle density of states (see Refs. 7 and 8). We shall see below that consideration of clusters with $n>1$ results in significant alteration of this result.

If it is assumed that the coefficients b_{km} are multiplicative, i.e.,

$$b_{km}=b_kb_m \tag{4.8}$$

[b_{km}=const is a special case of (4.8)], it turns out that Eq. (4.6) has an integral of the motion

$$I_1=\sum_{k=1}^{\infty}k\tilde P_n(t). \tag{4.9}$$

To prove this, we multiply the left- and right-hand sides of (4.6) by n and sum from $n=1$ to ∞. We transform the second term on the right in (4.6):

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}(n-k+k)(b_k\tilde P_k)(b_{n-k}\tilde P_{n-k}) =\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}[k(b_k\tilde P_k)(b_{n-k}\tilde P_{n-k})+(n-k) \times(b_{n-k}\tilde P_{n-k})(b_k\tilde P_k)]. \tag{4.10}$$

After the replacement of variables $m=n-k$, this expression cancels identically with the first term on the right in (4.6). When $t\to 0$, we obtain $\tilde{P}_n\sim\delta_{n0}$ and, consequently,

$$I_1=1, \tag{4.11}$$

Let us now find the generalized renormalization group equation that takes into account the dynamical character of the problem, i.e., let us turn to the complete Hamiltonian (2.1) with a nonzero term V (2.2).

Each cluster is characterized not only by an excitation energy Δ (when $\Delta_{0i}=0$), but also by a coherent amplitude Δ_0, which couples the excited state to the ground state. We introduce the distribution function of the parameters for an m cluster, $P_m(\Delta,L,R)$. For the ensuing treatment, instead of Δ_0 it is convenient to use the variable

$$L=\ln(\Delta_0/W). \tag{4.12}$$

Returning to Eqs. (4.3) and (4.4), we should introduce the new distribution functions and additional integration with respect to L_1 in the "outgoing" term. As a result, for this term in (4.5) we have ($\Delta\to 0$)

$$-2\pi P_n(0,L,R_1)\theta(|L|-3\xi_1)\sum_{k=1}^{\infty}\langle|u_{kn}|\rangle\int dL'\,\theta \times(|L'|-3\xi_1)P_k(0,L',R_1)\ln(R_2/R_1),$$

$$\xi_1=\ln(R_1/R_{\min}). \tag{4.13}$$

The appearance of the θ function (4.13) is a result of the requirement that the cluster energy $\varepsilon=\sqrt{\Delta^2+\Delta_0^2}$ be small compared with the interaction at the current distance R. In (4.13) this leads to an upper bound on the tunneling amplitudes of the excitations, for which the stability requirements can be significant.

The "incoming" term is a sum of terms that are structurally similar to (3.9) and correspond to the pairing of k and $n-k$ clusters. The transition amplitude of an n cluster formed from k and $n-k$ clusters can be represented in a form analogous to (3.8):

$$\Delta_0^{(n)}=\frac{1}{2}\frac{\Delta_0^{(k)}\Delta_0^{(n-k)}}{E_kE_{n-k}}\frac{u_{k,n-k}}{R^3}.$$

Using (3.12), we can rewrite this result in an approximate form:

$$\Delta_0^{(n)}\simeq\Delta_0^{(k)}\Delta_0^{(n-k)}R^3/u_{k,n-k}.$$

Taking into account that the dependence of the coupling constants $u_{k,n-k}$ on the number of particles in the clusters is not stronger than a power function (see Sec. 6), while the transition amplitudes depend exponentially on the number of particles, we can neglect the difference between the coupling constant and U_0. Then the δ function appearing upon the transition to integration over the expanded parameter space can be written with consideration of the preceding definitions in the form $\delta(L-L_1-L_2-3\xi_1)$. Using this expression for the "incoming" term, we find

$$\pi\sum_{k=1}^{n-1}\langle|u_{k,n-k}|\rangle\int dL_1\int dL_2P_k(0,L_1,R_1)P_{n-k} \times(0,L_2,R_1)\delta(L-L_1-L_2-3\xi_1) \times\theta(|L_1|-3\xi_1)\theta(|L_2|-3\xi_1)\ln(R_2/R_1). \tag{4.14}$$

The same arguments that were used to go from (4.1) to (4.5) and then to (4.6) make it possible to arrive at the generalized renormalization group equation:

$$\frac{\partial\tilde{P}_n(L,t)}{\partial t}=-2\tilde{P}_n(L,t)\theta\left(|L|-\frac{3t}{\chi}\right) \times\sum_{k=1}^{\infty}b_{kn}\int dL'\,\theta\left(|L'|-\frac{3t}{\chi}\right)\tilde{P}_k(L',t) +\sum_{k=1}^{n-1}b_{k,n-k}\int dL_1\int dL_2\tilde{P}_k(L_1,t) \times\tilde{P}_{n-k}(L_2,t)\delta\left(L-L_1-L_2-\frac{3t}{\chi}\right) \times\theta\left(|L_1|-\frac{3t}{\chi}\right)\theta\left(|L_2|-\frac{3t}{\chi}\right). \tag{4.15}$$

Here

$$\tilde{P}_n(L,t)=P_n(0,L,T)/P_0(0). \tag{4.16}$$

5. SOLUTION OF THE RENORMALIZATION GROUP EQUATIONS WHEN b_{km}=const

The solution of Eq. (4.6) depends on the form of the coefficients b_{km}. We begin by considering the simplest model $\langle|u_{km}|\rangle$=const, which maintains the main qualitative features of the problem. In this case

$$b_{km}=1. \tag{5.1}$$

We sum both sides of Eq. (4.6) over n and introduce the total density of states

$$\tilde{P}(t)=\sum_{k=1}^{\infty}\tilde{P}_k(t). \tag{5.2}$$

After transforming the second term on the right-hand side of (4.6) in analogy to (4.10), we find

$$\partial\tilde{P}(t)/\partial t=-\tilde{P}^2(t).$$

With the initial condition $\tilde{P}(0)=1$, the solution of this equation has the form

$$\tilde{P}(t)=1/(1+t). \tag{5.3}$$

When this expression is used, the equation for $n=1$ takes the form

$$\partial\tilde{P}_1(t)/\partial t=-2\tilde{P}_1(t)/(1+t)$$

and has the solution

$$\tilde{P}_1(t)=1/(1+t)^2. \tag{5.4}$$

Substituting (5.4) and (5.3) into (4.6), we easily find

$$\tilde{P}_2(t)=t/(1+t)^3. \tag{5.5}$$

In the general case,

$$\tilde{P}_n(t)=(1+t)^{-2}\int_0^t dt_1 \times(1+t_1)^2\sum_{k=1}^{n-1}\tilde{P}_k(t_1)\tilde{P}_{n-k}(t_1),\quad n>1. \tag{5.6}$$

The structure of this expression and (5.5) determine the general form of the solution,

$$\tilde{P}_n(t)=\left(\frac{t}{1+t}\right)^{n-1}\frac{1}{(1+t)^2}, \tag{5.7}$$

whose validity is easily verified directly. The asymptote of this expression at $t\gg 1$ is

$$\tilde{P}_n(t)=t^{-2}\exp[-(n-1)/t]. \tag{5.8}$$

Two significant conclusions follow from a comparison of (5.8) and (5.3). When $t\gg 1$, the low-energy density of states is uniformly comprised (in the model under consideration, but see the next section) of all m-particle excitations with $m\leqslant t$. In this case the total density of states (5.3) [with the notation of (4.7) and (2.15)]

$$P(0,\xi_R)\approx 1/\pi U_0\xi_R \tag{5.9}$$

ceases to depend at all on the initial distribution of the defect centers.

Let us consider the solution of the generalized equation (4.15) in the adopted model (5.1). First let $3\xi<|L_*|$, where $|L_*|=\ln(\Delta_{0*}/W)$. We are interested in the solution corresponding to typical values $\Delta_0\ll\Delta_{0*}$. In this case none of the θ functions in the integrand in (4.15) imposes any constraints, and they can all be omitted. We sum the left- and right-hand sides of (4.15) over n and introduce the total distribution function $\tilde{P}(L,t)$. As a result, we obtain

$$\frac{\partial\tilde{P}(L,t)}{\partial t}=-2\tilde{P}(L,t)\tilde{P}(t)+\int dL_1\int dL_2\tilde{P}(L_1,t)\tilde{P}(L_2,t)\times\delta\left(L-L_1-L_2-\frac{3t}{\chi}\right). \tag{5.10}$$

Here $\tilde{P}(t)$ is defined by (5.2). We introduce the Fourier transform

$$\tilde{P}(L,T)=\frac{1}{2\pi}\int dQ\tilde{P}(Q,t)\exp(iQL). \tag{5.11}$$

For the Fourier components we obtain

$$\frac{\partial\tilde{P}(Q,t)}{\partial t}=-2\tilde{P}(Q,t)\tilde{P}(t)+\exp\left(-\frac{3iQt}{\chi}\right)\tilde{P}^2(Q,t).$$

This equation can be solved using the substitution $\tilde{P}=1/z$. Taking into account the initial condition

$$\tilde{P}(Q,t=0)=P_0'(Q) \tag{5.12}$$

and the explicit form of (5.3), we find

$$\tilde{P}(Q,t)=\left[(1+t)^2\left(\frac{1}{P_0'(Q)}-\int_0^t\frac{dy}{(1+y)^2}\exp\left(-\frac{3iQy}{\chi}\right)\right)\right]^{-1}. \tag{5.13}$$

As previously noted, we are interested in the behavior of the distribution function when $|L|\gg|L_*|$. In (5.11) the corresponding contribution is made by Fourier components with small Q, for which we necessarily have

$$QL_*\ll 1. \tag{5.14}$$

We expand the denominator of (5.13) in small Q, retaining linear terms. We take into account that $P_0'(Q)\approx 1-iQL_*$ (from a rigorous standpoint, this is the definition of L_*):

$$\tilde{P}(Q,t)\approx(1+t)^{-2}\frac{i}{|\tilde{L}|}\left(Q+\frac{i}{(1+t)|\tilde{L}|}\right)^{-1}. \tag{5.15}$$

Here

$$|\tilde{L}|=|L_*|-\frac{3}{\chi}\left[\ln(1+t)-\frac{t}{t+1}\right]. \tag{5.16}$$

Substituting (5.15) into (5.11), we find

$$\tilde{P}(L,t)\approx(1+t)^{-2}|\tilde{L}|^{-1}\exp\left\{-\frac{|L|}{(1+t)|\tilde{L}|}\right\}. \tag{5.17}$$

The self-consistent nature of the approximation made during the derivation of (5.15) and, consequently, of (5.17) suggests that $t\gg 1$. As was shown above, under just this condition the center of gravity of the distribution of the low-energy excitations shifts toward multicenter clusters. The latter, in particular, have values of $|L|$ that are large compared with $|L_*|$.

It follows from (5.17) that the distribution function actually does not depend on L over the range

$$(1+t)|\tilde{L}|>|L|>|L_*| \tag{5.18}$$

when $t\gg 1$. This means that the distribution function has a universal form that exhibits a logarithmically uniform distribution of tunneling amplitudes

$$P(\Delta_0,t)\simeq P_0(0)/\Delta_0 t^2|\tilde{L}|. \tag{5.19}$$

This expression is valid over the broad range of Δ_0 specified by (5.18):

$$\Delta_{0*}>\Delta_0>\Delta_{0*}(\Delta_{0*}/W)^t f(t),\qquad f(t)=\exp\{3[(1+t)\ln(1+t)-t]/\chi\}. \tag{5.20}$$

The ratio $\Delta_{0*}/W\ll 1$ is actually a small parameter. In (5.20) the progressive displacement of the lower boundary of permissible values of Δ_0 with increasing t is typical. The occurrence of the factor $(\Delta_{0*}/W)^t$ has a clear physical meaning. As has been established [see (5.8)], when $t\gg 1$ the low-energy density of states is governed by many-particle excitations of clusters with $n\leqslant t$. Indeed, it is clear from the treatment of two-particle excitations that the range of permissible values of Δ_0 is shifted downward to Δ_{0*}^2/W [see (3.21)]. This value is dictated by the inequality (3.13). If we refer to

the general equation (4.15), we can easily see that a similar inequality holds. However, it takes the following form for each term in the sum over k at fixed n:

$$\Delta_0 > (\Delta_0^{(k)})_{\min} (\Delta_0^{(n-k)})_{\min} R_{\min}^3 / U_0,$$

where $(\Delta_0^{(k)})_{\min}$ is the minimum coherent amplitude of a k cluster. The inequality is identical for all terms in the sum:

$$\Delta_0 > \Delta_{0*} (\Delta_{0*}/W)^{n-1} \tag{5.21}$$

[in fact, it follows from (5.8) that $n_{\max} - 1 \approx t$]. Nevertheless, the estimate (5.21) does not take into account the decrease in the size of the phase volume due to the fact that an increasingly narrower range of R near $R_{\min}$ is responsible for the range of the lowest values of Δ_0. Just this circumstance resulted in the appearance of the factor $f(t) \gg 1$ in (5.20), which, however, does not make any qualitative changes in the picture described.

We have thus far analyzed the solution of Eq. (5.10) under the condition $3\xi < |L_*|$. The results enable us to understand how this solution can be extended to $3\xi > |L_*|$, where the integrands in (5.10) must contain $\theta(|L_i| - 3\xi)$ [see (4.15)]. As ξ increases, the ensuing constraints require the displacement of $\Delta_0^{(k)}$ toward smaller values. This means that as ξ increases, the two-particle clusters are excluded, then the three-particle clusters are excluded, and so on up to $n_{\max} \approx |t|$. Since, on the other hand,

$$\Delta_0 < U_0 / R_{\max}^3,$$

the limiting value of $\xi_{\max}$ must be found from the relation

$$\Delta_{0*} (\Delta_{0*}/W)^t f(t) = U_0 R_{\max}^3, \tag{5.22}$$

which can be rewritten in the form

$$(1+t)|L_*| - 3[(1+t)\ln(1+t) - t]/\chi = 3\xi_{\max}.$$

In these expressions $t = \chi \xi_{\max}$, and (5.22) is a self-consistent equation for $\xi_{\max}$:

$$t_m = \chi \xi_{\max} \approx \exp(\chi |L_*|/3). \tag{5.23}$$

It was assumed during the analysis performed above for $t \gg 1$ that $\chi |L_*|/3 \gg 1$. The result (5.23) indicates a sharp displacement of the lower boundary of Δ_0 toward smaller values [in (5.20) $t \to t_m$].

Substituting (5.23) into (5.17), we find the final expression for the distribution function:

$$P(L) \approx \exp\left(-\frac{L}{3\xi_{\max}} \right) \frac{1}{3\chi \xi_{\max}^2}. \tag{5.24}$$

The integral of this expression with respect to L coincides with (5.3) when $t \gg 1$.

The distribution function has the universal form

$$P(\Delta_0) \approx \frac{P_0(0)}{3\chi \xi_{\max}^2} \frac{1}{\Delta_0} \tag{5.25}$$

over the range of Δ_0 specified by the inequalities (5.20), but now with the replacement $t \to t_m$. We stress that this result does not depend on the character of the distribution of Δ_{0i} for the original defect centers.

6. SOLUTION OF THE EQUATIONS WHEN $b_{km} = \sqrt{km}$

In the preceding section we considered the solution of the renormalization group equations under the assumption that the interaction between clusters does not depend on the number of particles in a cluster. In real systems such a dependence exists in the general case, and it can be important in shaping the universal properties of amorphous systems.

The interaction between clusters is determined by the sum of the pairwise interactions of centers in different clusters. Accordingly, the interaction constant is determined by the sum of the pairwise interaction constants

$$u_{km} = \sum_{\substack{i \in k \\ j \in m}} u_{ij}.$$

If each center can be effectively assigned a certain scalar charge, which has a symmetric distribution with respect to the sign after averaging over the configurations of the cluster or due to certain physical factors, the coefficients in Eqs. (4.6) and (4.15) take the form

$$\langle |u_{km}| \rangle = \sqrt{km} U_0. \tag{6.1}$$

Such a situation arises, in particular, in amorphous metallic systems, where the indirect interaction between the centers mediated by the conduction electrons is decisive. This interaction has the form (2.3) with

$$u_{ij} = u \cos(2k_F R_{ij}). \tag{6.2}$$

As we have seen, the interaction between clusters is effective at distances that are large compared with the size of the clusters. In this case

$$R_{ij} = R + \mathbf{n}(\mathbf{r}_i - \mathbf{r}_j).$$

Here R is the distance between the centers of the clusters, $\mathbf{n} = \mathbf{R}/R$, and $\mathbf{r}_i$ and $\mathbf{r}_j$ are the positions of centers i and j in different clusters.

The interaction of k and m clusters is characterized by the quantity

$$u_{km} = \mathrm{Re}\left\{ \exp(2ik_F R) \left[\sum_{i=1}^{k} \exp(ik_F \mathbf{n} \mathbf{r}_i) \right] \times \left[\sum_{j=1}^{m} \exp(-ik_F \mathbf{n} \mathbf{r}_j) \right] \right\}.$$

Since $k_F R_k \gg 1$ (R_k is the size of the k cluster; $k \geqslant 2$), averaging over the cluster configurations with selection of a definite sign for the sum in the square brackets gives a value $\sim \sqrt{k}(\sqrt{m})$. (If $\cos(2k_F R) > 0$, sums with identical signs are selected; otherwise, sums with opposite signs are selected.)

In amorphous insulators the feasibility of using Eq. (6.1) calls for special analysis. It can be roughly based on the fact that in a deformation interaction there are at least six independent components of the stress tensor of an individual center, which determine the coupling between centers.

We consider Eq. (4.6), introducing coefficients of the form (6.1):

$$\frac{\partial \tilde{P}_n(t)}{\partial t} = -2\sqrt{n}\tilde{P}_n(t)\sum_{k=1}^{\infty}\sqrt{k}\tilde{P}_k(t) + \sum_{k=1}^{n-1}\sqrt{k}\sqrt{n-k}\tilde{P}_k(t)\tilde{P}_{n-k}(t). \quad (6.3)$$

Since the multiplicative condition (4.8) holds, the equation has the integral of the motion (4.9). When $t \gg 1$, the contribution of large n becomes significant in (4.9), and the sum can be replaced by the integral

$$I_1 = \int_1^{\infty} dn\, n\tilde{P}_n(t) = 1. \quad (6.4)$$

The dependence of the coefficients in Eq. (6.3) on the number of particles in the clusters renders the structure of the solution complex and nonobvious. This necessitates a preliminary analysis of Eq. (6.3). We begin with some exact assertions, which follow from the structure of Eq. (6.3). We sum the left- and right-hand sides of (6.3) with respect to n. With consideration of the definition (5.2), we have

$$\frac{\partial \tilde{P}(t)}{\partial t} = -[I_{1/2}(t)]^2, \quad I_m(t) = \sum_{k=1}^{\infty}\tilde{P}_k k^m. \quad (6.5)$$

On the other hand,

$$\frac{\partial \tilde{P}_1}{\partial t} = -2\tilde{P}_1(t)I_{1/2}(t),$$

or

$$\tilde{P}_1(t) = \exp\left\{-2\int_0^t dt' I_{1/2}(t')\right\}. \quad (6.6)$$

It would seem that the asymptotic behavior of $\tilde{P}_1$ when $t \to \infty$ can lead to one of two results: a) $\tilde{P}_1 \to 0$, b) $\tilde{P}_1 \to \text{const} \neq 0$. In the former case the function $I_{1/2}(t)$ must have the asymptote

$$I_{1/2}(t) \sim t^{-\gamma}, \quad \gamma \leqslant 1. \quad (6.7)$$

It turns out that case (b) cannot occur at all. In fact, in that case

$$I_{1/2}(t) \underset{t\to\infty}{\to} \text{const},$$

and $\tilde{P}_1$ tends exponentially to zero as $t \to \infty$, in contradiction to (b). Substituting (6.7) into (6.5) and noting that $\tilde{P}(t)$ is a positive definite quantity, we quickly obtain

$$\gamma > 1/2. \quad (6.8)$$

Thus, $I_{1/2}(t)$ and the zeroth moment of the distribution function $\tilde{P}(t)$ asymptotically tend to zero at large t.

The first moment (6.4) is preserved by virtue of the multiplicative relationship, and does not decay with increasing t.

Let us now find the equation for the second moment. For this purpose we multiply both sides of (6.3) by n^2 and sum over n. In the "incoming" term in (6.3) we utilize the relation

$$n^2 = k^2 + (n-k)^2 + 2k(n-k).$$

The first two terms compensate the "outgoing" term exactly. Consideration of the latter term leads to the equation

$$\partial I_2(t)/\partial t = 2[I_{3/2}(t)]^2. \quad (6.9)$$

The asymptotic decay of $I_{1/2}(t)$ and $\tilde{P}(t)$ with increasing t and the absence of such decay in the case of I_1 indicate that the contribution of the large n, which compensates the decay, becomes significant for the first and higher moments. The second moment already increases with increasing t, as follows from (6.9). This obviously occurs for all the moments $I_m(t)$ with $m > 1$ [in particular, for $I_{3/2}(t)$] owing to the dominant contribution from large n.

When t is small, the functions $\tilde{P}_n(t)$ with $n > 1$ increase with increasing t, since $\tilde{P}_n(0) = 0$. On the other hand, when t is sufficiently large, $\tilde{P}_n(t)$ should decrease and cause the overall decay of $\tilde{P}(t)$. Thus, $\tilde{P}_n(t)$ for a given n passes through a maximum. As t increases, the value of n corresponding to the function $\tilde{P}_n(t)$ passing through a maximum at the respective value of t increases. In fact, it is the position of the front $n_{\max}(t)$, which propagates toward larger n as t increases. The comparatively rapid motion on the front prevents all the moments $I_m(t)$ with $m \geqslant 1$ from decreasing as t increases. We turn to Eq. (6.3) and rewrite it in the form

$$\frac{\partial \tilde{P}_n(t)}{\partial t} = \sum_{k=1}^{n-1}[\sqrt{k}\tilde{P}_k(t) - \sqrt{n}\tilde{P}_n(t)][\sqrt{n-k}\tilde{P}_{n-k}(t) - \sqrt{n}\tilde{P}_n(t)] - n(n-1)\tilde{P}_n^2(t) - 2\sqrt{n}\tilde{P}_n(t)\sum_{k=n}^{\infty}\sqrt{k}\tilde{P}_k(t). \quad (6.10)$$

Bearing in mind that an asymptotic solution has been found for $t \gg 1$ and, therefore, for $n_{\max}(t) \gg 1$, let us consider the range of n corresponding to the inequality

$$1 \ll n \ll n_{\max}(t). \quad (6.11)$$

It stands to reason, and will be proved below, that unlike I_1 and I_2, $I_{1/2}(t)$ is determined by the lower limit. In this case the last term on the right in (6.10) behaves approximately as

$$n^2\tilde{P}_n^2. \quad (6.12)$$

It is not difficult to show that the first term on the right in (6.10) is determined by $k \sim n/2$ and that the role of the fringes of the integration range $k \sim 1$ and $n-k \sim 1$ is negligible. As a result, this term also acquires the structure (6.12), and it thus turns out that the entire right-hand side of Eq. (6.10) has this structure. Here the overall sign must be negative to reflect the decrease in $\tilde{P}_n$ with increasing t when $n < n_{\max}(t)$. The resulting approximate equation is easily solved:

$$\tilde{P}_n(t) \approx \eta/n^2 t, \quad \eta \sim 1. \quad (6.13)$$

Since the sums corresponding to the first and higher moments converge at the upper limit, in (6.9) [as before in (6.4)] we can switch from summation to integration. The negligible role of the fringes of the summation range allows us to switch from summation to integration in Eq. (6.3) or (6.10), which now takes the form

$$\frac{\partial \tilde{P}_n}{\partial t}=\int_0^n(\sqrt{x}\tilde{P}_n-\sqrt{n}\tilde{P}_n)(\sqrt{n-x}\tilde{P}_{n-x}-x\sqrt{n}\tilde{P}_n)dx$$

$$-n^2\tilde{P}_n^2-2\sqrt{n}\tilde{P}_n\int_0^\infty\sqrt{x}\tilde{P}_x dx. \tag{6.14}$$

We use the equations for the first and second moments to establish the relation between $n_{\max}$ and t. Substituting (6.13) into (6.4), we find

$$I_1=\eta\ \ln(n_{\max})/t.$$

We see that $n_{\max}(t)$ increases exponentially with t. The exponential increase in $n_{\max}$ is also maintained in the more refined form of the distribution

$$\tilde{P}_n(t)=\eta\ \ln^\alpha n/n^2t. \tag{6.15}$$

In this case

$$I_1=\eta\ \ln^{\alpha-1} n_{\max}/t(\alpha+1)=1. \tag{6.16}$$

If $I_{3/2}(t)$ and $I_2(t)$ are calculated with the distribution (6.15) and plugged into Eq. (6.9), under the assumption $\alpha \ll n_{\max}$ and with consideration of (6.16) we find

$$\frac{dn_{\max}}{dt}=8(\alpha+1)\frac{n_{\max}}{\ln n_{\max}}. \tag{6.17}$$

For the compatibility of (6.16) and (6.17) we must set $\alpha=1$. Note that we have intentionally maintained the inverse linear dependence of $\tilde{P}_n$ on t. Such a dependence is dictated by the bilinear structure of the right-hand side of Eq. (6.14) with respect to $\tilde{P}_k$.

The exponential increase in $n_{\max}$ is a significant feature of the problem under consideration. It is dictated by the dependence of the coefficients in the original equation (6.3) on the number of particles in a cluster. When we considered the case in which the coefficients do not depend on the number of the cluster in the preceding section, we found that $n_{\max}$ increased only linearly with increasing t.

We now show that the solution (6.15) with $\alpha=1$ satisfies the general equation (6.14). Substituting (6.15) into the left- and right-hand sides of Eq. (6.14), we find

$$-\frac{\eta\ \ln n}{n^2t^2}=\left(\frac{\eta}{t}\right)^2\left[A\frac{\ln^2 n}{n^2}-B\frac{\ln n}{n^2}+C\frac{1}{n^2}\right], \tag{6.18}$$

where the dimensionless coefficients have the form

$$A=\int_0^1 dx\left(\frac{1}{x^{3/2}}-1\right)\left[\frac{1}{(1-x)^{3/2}}-1\right]-1-2\int_1^\infty\frac{dx}{x^{3/2}},$$

$$B=\int_0^\infty dx\frac{\ln x}{x^{3/2}}-2\int_0^1 dx\frac{\ln x}{x^{3/2}}\left[\frac{1}{(1-x)^{3/2}}-1\right], \tag{6.19}$$

$$C=\int_0^1 dx\frac{\ln x\ \ln(1-x)}{x^{3/2}(1-x)^{3/2}}.$$

It is easy to verify that the coefficient A is identically equal to zero. For the coefficients B and C we found $B\approx 25$ and $C\approx 10$. Assuming that $\ln n\gg 1$, we can neglect the term C/n^2. It then follows from (6.18) that the solution in the form (6.15) with $\alpha=1$ does, in fact, satisfy Eq. (6.14). Determining the value of η, we ultimately have

$$\tilde{P}_n\approx 4\times 10^{-2}\ \ln n/n^2t. \tag{6.20}$$

We note that we can, in fact, find the solution exactly by retaining the $1/n^2$ term. This solution has the form

$$\tilde{P}_n\approx 4\times 10^{-2}\ \ln(nt^a)/n^2t,\quad a=C/B\approx 0.4. \tag{6.21}$$

If we plug the value found of η into (6.16), we have

$$n_{\max}\approx\exp(\sqrt{50t}). \tag{6.22}$$

Attention should be focused on the large numerical coefficient in the exponent. This result indicates that excitations of large clusters occur by $t\sim 1$. The solution (6.20) or (6.21) corresponds to $n<n_{\max}(t)$.

A numerical investigation of Eqs. (6.3) confirmed both the onset of the regime (6.13) and the exponential increase in the limiting scale of the active clusters $n_{\max}$.

We now include intracenter transitions in the treatment. In this case the problem reduces to solving Eq. (4.15) with the coefficients b_{km} determined according to Eq. (6.1). We, however, will not perform a general analysis of the solution of that equation, but we shall derive an approximate solution on the basis of the results of the preceding section.

As the analysis of (5.20) and (5.21) made clear, the characteristic amplitude of coherent transitions for an n cluster $\Delta_0^{(n)}$ can be represented in the form

$$\Delta_0^{(n)}\simeq\Delta_{0*}(\Delta_{0*}/W)^{n-1}.$$

Using (6.21) for the general distribution function, we have

$$\tilde{P}(\Delta_0,t)=4\times 10^{-2}\frac{1}{t}\int_1^{\varphi(t)}\frac{dn}{n^2}\ln(nt^a)$$

$$\times\delta\left(\Delta_0-\Delta_{0*}\left(\frac{\Delta_{0*}}{W}\right)^{n-1}\right),$$

$$\varphi(t)=n_{\max}(t).$$

After the trivial integration, returning to the old notation of (4.7) and (4.16), we find

$$P(\Delta_0)\approx 4\times 10^{-2}\frac{P_0(0)}{t}\frac{|L_*|}{\ln^2(W/\Delta_0)}\ln(n_{\text{eff}}t^a)\frac{1}{\Delta_0}$$

$$=\frac{4\times 10^{-2}}{\pi U_0\xi}\frac{|L_*|}{\ln^2(W/\Delta_0)}\ln(n_{\text{eff}}(\chi\xi)^a)\frac{1}{\Delta_0}. \tag{6.23}$$

Here

$$n_{\text{eff}}=\frac{\ln(W/\Delta_0)}{\ln(W/\Delta_{0*})} \tag{6.24}$$

is the effective number of particles in the clusters responsible for excitation with a transition amplitude Δ_0. The smaller the value of Δ_0, the greater the corresponding value of n_{eff}. Also, the amplitude Δ_0 in (6.23) is restricted to the following values:

$$\Delta_{0*}>\Delta_0>W(\Delta_{0*}/W)^{\varphi(t)-1}. \tag{6.25}$$

The exponential increase in $\varphi(t)$ according to (6.22) causes the upper bound on Δ_0 to be extremely weak. The expression (6.23) holds over essentially the entire interesting range of small Δ_0 and, accordingly, long relaxation times.

The derived distribution differs comparatively little from a uniform distribution in $\ln \Delta_0$. It is interesting that this occurs despite the appreciable decay of P_n with increasing n. As opposed to (5.25), (6.23) contains the factor $\ln^2(W/\Delta_0)$ in its denominator and the factor $\ln(n_{\text{eff}}\, t^a)$, which is even more weakly dependent on Δ_0, in its numerator. Although their roles are unquestionably minor, they should cause some deviation from the logarithmic distribution of the relaxation times when $\Delta_0 \to 0$.

The distribution (6.23) is logarithmically weakly dependent on the parameters of the original centers, as well as their density. This is one more important aspect of universality. It is interesting that the naturally arising numerical coefficient in (6.23) also predetermines a purely quantitative scale that can be compared with experiment.

In obtaining these results we did not take into account the contraction of the phase volume discussed in the preceding section, which produced the factor f in (5.20). Since the limiting scale of the active clusters is now determined by $\varphi(t)$, rather than t, the factor $(\Delta_{0*}/W)^{\varphi(t)}$ should appear in (5.22) instead of $(\Delta_{0*}/W)^{(t)}$. The exponential increase in $\varphi(t)$ (6.22) is responsible for the fact that there are no upper bounds on t. The quantitative role of this factor in (6.24) is not of great significance, since $\varphi(t) \gg t$.

7. DISCUSSION

The results of the preceding sections were obtained under the assumption that $T=0$. Specific constraints that must be taken into account in a general treatment appear at finite temperatures.

This applies above all to the size of the coherence region, which is actually determined by the condition

$$U_0/R_{\max}^3 \approx T. \tag{7.1}$$

In fact, at larger distances the interaction between clusters is smaller than the temperature, and their excitation can be considered independent. When $R < R_{\max}$, clusters with an energy determined by (3.12) are involved in the formation of new clusters. This energy is large compared with T, and it can, therefore, be assumed that they are in the ground state. All the features dictated by the stability condition are maintained here.

We offer one comment. Condition (7.1) in its general form is, in fact, weaker: $u_{kn}/R^3 \sim T$. This condition can lead to the replacement $U_0 \to U_0\sqrt{k}\sqrt{n}$ in (7.1). This would introduce weaker logarithmic corrections in the ensuing relations, which we ignore.

When excitations with energies and tunneling amplitudes that are small compared with the temperature are considered, the maximum size of the coherence region is given by

$$\xi_{\max} = \xi_T = \frac{1}{3}\ln\left(\frac{W}{T}\right). \tag{7.2}$$

This value of ξ should appear in the distributions (6.23) and (5.25), if the dimension of the entire system is greater than the corresponding value of R_T. For excitations at a higher energy $\varepsilon = \sqrt{\Delta^2+\Delta_0^2}$, the cut-off radius is given by

$$U_0/R_\varepsilon^3 \approx \varepsilon. \tag{7.3}$$

Accordingly, $\xi = \xi_\varepsilon$ should be substituted into (6.23):

$$\xi_\varepsilon = \frac{1}{3}\ln\left(\frac{W}{\varepsilon}\right). \tag{7.4}$$

Many-particle excitations will be significant when

$$\chi\xi_{\max} \gg 1. \tag{7.5}$$

In many cases (see Sec. 6) the following relation is more than adequate:

$$\chi\xi_{\max} \geqslant 1. \tag{7.6}$$

The most significant result of the present work is the proof that collective excitations of multicenter clusters specify a distribution function of the parameters $P(\Delta,\Delta_0)$ having a uniform [see (5.25)] or quasiuniform [see (6.23)] distribution in $\ln \Delta_0$ (apart from the uniform distribution in Δ) under these conditions. At the same time, it was shown that the inequality (7.5) [or the weaker inequality (7.6)] predetermines the appearance of a broad scale of small Δ_0 [see (5.20) and (6.25)], where this uniform distribution is valid. This means that there is a logarithmic distribution of relaxation times over a broad range of times. It is noteworthy that these universal properties of amorphous systems actually appear for an arbitrary distribution of the parameters of the isolated defect centers. Moreover, under the natural assumption regarding the behavior of the coupling constant of an n cluster $b_n \sim \sqrt{n}$ adopted in Sec. 6, the derived distribution function (6.23) scarcely depends on the density of the original defect centers. This significant result makes it possible to understand the reason for the quantitative similarity of different amorphous systems.

In conjunction with the increase in $\xi_{\max}$ (7.2) as the temperature is lowered, the theory developed might make it possible to understand the extremely nontrivial results obtained in Pobell's laboratory in Bayreuth.[9,10] It was found that at ultralow temperatures, samples in which the presence of defects is due to their polycrystallinity exhibit properties characteristic of amorphous systems. The scales of the defects were comparable to those observed in ordinary amorphous systems.

From the estimate of (2.15)

$$\chi \approx \pi n U_0/W$$

it follows that the closer the spread W of the levels of the primary centers to the interaction between centers at a mean distance $U_0 n$, the greater the value of χ. Therefore, the greater the degree to which the spread of the levels is associated with the interaction of dynamically active centers (which have an internal degree of freedom) and the lower the value of T, the more easily the condition (7.5) together with the universality of the spectral properties is realized.

The solution of the nonlinear equation (6.3) led to the appearance of a definite numerical factor in the solution (6.21) and, as a consequence, in the general distribution function (6.23). For this reason a quantitative comparison of

(6.23) with experimental results is of special interest. The internal friction in glasses can serve as a representative example here.

Let us consider the so-called "plateau region,"[11] where the internal friction scarcely depends on T and the frequency ω. In this region (see, for example, Ref. 11)

$$Q^{-1} \approx P \Delta_0 U, \tag{7.7}$$

where U is the deformation interaction constant of the centers responsible for the absorption of sound. This region corresponds to the condition

$$\omega \tau \approx 1, \tag{7.8}$$

where, considering that $\varepsilon \sim T$, the relaxation time τ equals (see Ref. 12)

$$\tau^{-1} = A \Delta_0^2 T, \quad A \sim 10^8 \ \mathrm{s}^{-1} \mathrm{K}^{-3}.$$

When ω and T are fixed, from (7.8) we find Δ_0, as well as n_{eff} (6.24). Clusters with $n \simeq n_{\mathrm{eff}}$ are responsible for the absorption of sound. According to the results of the preceding section,

$$U \approx U_0 n_{\mathrm{eff}}. \tag{7.9}$$

As a result, substituting (6.23) into (7.7) and taking into account (7.9), (6.24), and (7.2), we find

$$Q^{-1} \approx 4 \times 10^{-2} \frac{1}{\ln(W/T)} \frac{1}{\ln(W/\Delta_0)}$$

[here we omitted a factor $\ln(n_{\mathrm{eff}} t^a)$, which is close to unity].

At typical values $\omega \simeq 10^3$ Hz and $W \simeq 10^2$ K, we obtain

$$Q^{-1} \sim 10^{-3},$$

which is very close to the experimentally determined values of the internal friction (see, for example, Refs. 9 and 10). The purely logarithmic dependence on the parameters of the medium, which results in relatively weak variation of Q^{-1} upon passage from one glass to another, is noteworthy. The latter is usually specially stressed when the absorption of sound in amorphous systems is analyzed.

Thus, the results obtained enable us to account for the quasiuniversality of the observed properties of amorphous systems on both a qualitative and a quantitatively level.

We offer one comment in conclusion. The smallness of the coupling constant $\chi \ll 1$ was utilized extensively throughout the preceding treatment. However, an increase in the coefficients in Eq. (4.6) results in enhancement of the effective interaction. We can evaluate the emergent value of χ_{eff} from the first term on the right-hand side of Eq. (6.3):

$$\chi_{\mathrm{eff}} \simeq \chi \sqrt{n} \sum_{k=1}^{\infty} \sqrt{k} \tilde{P}_k(t).$$

The sum in the definition of the effective coupling constant is determined by the lower limit $n \simeq 1$, and P_1 decreases as $1/t$. Therefore the condition $\chi_{\mathrm{eff}} \ll 1$ is violated when

$$n \sim (t/\chi)^2 \gg 1.$$

This additional constraint on the largest size of a cluster for very large n imposes an essentially negligible lower bound on Δ_0 in the distribution (6.23).

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