The resistance of a one-dimensional chain of periodically spaced random scatterers

D. M. Sedrakyan, D. A. Badalyan, V. M. Gasparyan, and A. Zh. Khachatryan

Erevan State University, Erevan, Republic of Armenia
(Submitted 1 March 1995; resubmitted 30 May 1995)
Zh. Éksp. Teor. Fiz. **109**, 243–255 (January 1996)

We develop a method that makes it possible to reduce the problem of finding the ensemble average of the resistance of a one-dimensional chain of periodically spaced random scatterers to solving a finite-difference (recurrence) relation. This equation is solved, in particular, for a potential of the Kronig–Penney type. We solve the problem for an incident-electron energy corresponding to the center of the band and study the solution for two limiting cases. Finally, we show that the dependence of the Landauer resistance ρ on the chain's length is a general exponential function, which in the weak scattering limit becomes purely exponential.
 [S1063-7761(96)01701-1]

1. INTRODUCTION

The problem of an electron traveling through a one-dimensional disordered system has been studied both analytically[1] and numerically.[2,3] The interest in it rests on the fact that studying the behavior of resistance as a function of the characteristic parameters of the system and external factors makes it possible to monitor the changes in the nature of one-electron states. To calculate the resistance of a one-dimensional system, a section with a random static potential is placed between two semi-infinite "perfectly conducting" electrodes, in which an electron moves freely. It is the fraction of electrons that have passed through the "nonideal region" from one electrode to the other that determines the "traffic" capacity of the system. If the nonideal region is a one-dimensional metal with stationary scatterers, where all electronic states are localized, the dimensionless Landauer resistance averaged over various realizations of the random scattering impurities is expressed in terms of the chain length L at absolute zero by the following formula:[4–8]

$$\langle\rho\rangle=\frac{1}{2}(e^{L/\xi}-1),$$

where we have put $\hbar=e^2=1$, and ξ is the electronic-state localization radius, which depends on the shape of the potentials inside the chain and not on the chain length or the way that averaging over the ensemble was achieved.

Finding the energy dependence of the localization radius, $\xi(E)$, constitutes a complex problem, so that solutions for only simple models have been found. For instance, in weak scattering, i.e., scattering on a potential of the white-noise type, the localization radius has been calculated both without an external electric field and with a uniform electric field.[9,10] For strong scattering the localization radius has been calculated in Ref. 11, with a new method employed, which made it possible to trace the changes in electronic states when the interaction between electrons and scattering centers increases. This method of finding analytic solutions works only in some specific cases: for potentials of the Kronig–Penney type, for strong scattering, and for resonant passage of an electron. In the general case, i.e., for an arbitrary number of irregular amplitudes, one is forced to resort to computer calculations when dealing with a one-dimensional chain.

In Ref. 12 it was found that the desired determinant can be represented by finite series. These series simplify the procedure of averaging the Landauer resistance over the random values of the amplitude, i.e., finding $\langle\rho_N\rangle$.

In this paper we develop a new method for calculating the ensemble average of the resistance of a one-dimensional chain, $\langle\rho_N\rangle$. The method reduces finding the desired quantity $\langle\rho_N\rangle$ to solving a recurrence relation of finite degree N, where N is the number of sites in the one-dimensional chain. This makes it possible to find the dependence of the localization radius on the electron's energy.

We also solve the recurrence relation for a potential of the Kronig–Penney type and find the Landauer resistance of a one-dimensional chain for an electron energy corresponding to the band center.

In Sec. 2 we formulate the problem and obtain an important determinant D_N in the form of a series in the interaction potentials. Averaging it over the random values of the scattering-potential amplitudes, we arrive at an expression for the average chain resistance $\langle\rho_N\rangle$. We derive the recurrence (difference) relation for $\langle\rho_N\rangle$ in Sec. 3 and suggest a method for its solution. To illustrate the new method of calculating D_N and $\langle\rho_N\rangle$, in Sec. 4 we find the solution to the equation for any finite N when the interaction-potential amplitudes over the entire length of the chain are the same, i.e., for the Kronig–Penney potential. Finally, in Sec. 5 we solve the recurrence (difference) relation for a one-dimensional chain of length $L=Na$, where a is the site separation (the distance between two neighboring sites), for an electron energy corresponding to the center of the band. As the solution for strong scattering shows, the average resistance $\langle\rho_N\rangle$ increases with L not according to a purely exponential law but by a general exponential law.

2. STATEMENT OF THE PROBLEM

We select a model in which delta-like potentials with arbitrary amplitudes V_n are located at the sites of a one-dimensional chain with coordinates $x=na$, where a is the period of the one-dimensional structure:

 1063-7761/96/010129-06$10.00

$$V(x)=\sum_{n=1}^{N} V_n\delta(x-na). \tag{1}$$

The solution of the Schrödinger equation for an electron that is outside this structure, i.e., for $x\leqslant x_1$ and $x\geqslant x_N$, and has the energy $E=k^2$ ($\hbar=2m_0=1$, and m_0 is the mass of a free electron) can be written in the usual form

$$\begin{aligned}&\psi(k,x)=e^{ikx}+r(k)e^{-ikx}, \quad x\leqslant x_1,\\ &\psi(k,x)=t(k)e^{ikx}, \quad x\geqslant x_N,\end{aligned} \tag{2}$$

where $t(k)$ and $r(k)$ are, respectively, the transmission and reflection amplitudes. As shown in Refs. 1–3 and 13–15, the transmission coefficient can be written as

$$T=|t(k)|^2=|D_N|^{-2}, \tag{3}$$

so that the expression for the Landauer resistance acquires the form

$$\rho_N=T^{-1}-1=|D_N|^2-1, \tag{4}$$

where D_N is the determinant of the matrix

$$(D_N)_{lp}=\delta_{lp}+\frac{iV_N}{2k}\exp[ik(x_l-x_p)]. \tag{5}$$

The determinant of (5) satisfies the following recurrence relation:[11]

$$D_N=A_ND_{N-1}-B_ND_{N-2}, \tag{6}$$

where

$$A_N=1+B_N+\frac{iV_N}{2k}[1-e^{2ika}], \quad N>1,$$

$$A_1=1+\frac{iV_1}{2k}, \quad B_N=\frac{V_N}{V_{N-1}}e^{2ika},$$

$$D_0=1, \quad D_{-1}=0,$$

and the determinant D_{N-1} (or D_{N-2}) is obtained by deleting N (or $N-1$) rows and columns from D_N.

After painstaking but straightforward transformations that use the recurrence relation (6), we arrive at the following expression for the determinant of (5):

$$D_N=1+\sum_{p=1}^{N}\sum_{1=j_1<\cdots<j_p}^{N} f_p(x_{j_1},\ldots,x_{j_p})\prod_{l=1}^{p}\frac{iV_{jl}}{2k}. \tag{7}$$

Here the phase factor f_p is the determinant of a matrix of order p whose elements are

$$(f_p)_{nl}=\exp(ik|x_n-x_l|), \quad f_1=1,$$

and hence

$$f_p=\prod_{l=1}^{p-1}\{1-\exp[2ik(x_{j_{l+1}}-x_{j_l})]\}. \tag{8}$$

Substituting this into Eq. (7) for the determinant D_N, we arrive at the following expression:

$$D_N=1+\sum_{p=1}^{N}\sum_{1=j_1<\cdots<j_p}^{N}\frac{iV_{j_1}}{2k}\cdots\frac{iV_{j_p}}{2k}\times\prod_{l=1}^{p-1}\{1-\exp[2ik(x_{j_{l+1}}-x_{j_l})]\}. \tag{9}$$

Now we attempt to calculate the chain resistance $\langle\rho_N\rangle$, averaging it over the random values of the interaction amplitude. Specifically, we assume that the amplitudes V_n of the delta-like potentials are independent random quantities, each of which can take on any value with the interval from $-\frac{1}{2}W$ to $\frac{1}{2}W$ with a probability density $P(V_n)$, where $P(V_n)$ is an arbitrary even function.

Thus, for the ensemble-average $\langle\rho_N\rangle$ we can write

$$\langle\rho_N\rangle=W^{-N}\int_{-W/2}^{W/2}\cdots\int_{-W/2}^{W/2}P(V_1)\times P(V_2)\cdots P(V_n)D_ND_N^{*}dV_1\cdots dV_N. \tag{10}$$

Substituting (9) into Eq. (10) and integrating, we obtain

$$\langle\rho_N\rangle=\sum_{p=1}^{N}2^{p-1}\alpha^{p}\sum_{1=j_1<\cdots<j_p}^{N}\times\prod_{l=1}^{p-1}\{1-\exp[2ik(x_{j_{l+1}}-x_{j_l})]\}, \tag{11}$$

where

$$\alpha=W^{-1}\int_{-W/2}^{W/2}P(V_j)\frac{V_j^2}{4k^2}dV_j.$$

In particular, at $P(V_j)=1$ we have $\alpha=W^2/48k^2$.

Summation in (11) is possible only when the interaction of the electron with the lattice sites is weak ($\alpha\ll 1$) or strong ($\alpha\gg 1$) (see Refs. 4–11). In the general case, direct summation runs into insurmountable difficulties. These difficulties, however, can be overcome, as we show below, by reducing the calculation of $\langle\rho_N\rangle$ to solving a recurrence (difference) relation for the desired function.

3. THE RECURRENCE RELATION FOR ρ_N

As noted earlier, although the expression obtained is exact, it does not make it possible to calculate directly, i.e., without a computer, the dependence of $\langle\rho_N\rangle$ on the chain length and the incident-electron energy for arbitrary α's, except for several special cases, say at $ka=\pi m$ ($m=1,2,\ldots$), which corresponds to resonant passage of the electron. Therefore, the need arises to find for $\langle\rho_N\rangle$ a new representation or an equation whose solution would enable finding the functional dependence of $\langle\rho_N\rangle$ on α, M, and ka.

Separating the first term in the sum in (11) and writing the sum over the inner indices explicitly, we can write the following expression for $\langle\rho_N\rangle$

$$\langle\rho_N\rangle=\alpha\Bigg\{\sum_{j_1=1}^{N}1+\sum_{p=2}^{N}(2\alpha)^{p-1}\sum_{j_p=p}^{N}\sum_{j_{p-1}=p-1}^{j_p-1}\cdots\sum_{j_1=1}^{j_2-1}\times[1-\cos(2ka(j_p-j_{p-1}))]\cdots$$

$$[1-\cos(2ka(j_2-j_1))]\Big\}. \tag{12}$$

To lift the restrictions in summing over the indices j_p in (12), we introduce the functions

$$V_{j_{p+1},j_p}=2\alpha\{1-\cos[2ka(j_{p+1}-j_p)]\}\theta(j_{p+1}-j_p), \tag{13}$$

where $\theta(x)$ is the Heaviside step function:

$$\theta(x)=\begin{cases}1, & x>0,\\ 0, & x<0.\end{cases}$$

We also note that at $x=0$, i.e., at $j_{p+1}=j_p$, the introduced function vanishes, $V_{j_p,j_p}=0$. Now, if we write the sum over p explicitly, we can write Eq. (12) as

$$\langle\rho_N\rangle=\alpha\Big\{N+\sum_{j_2,j_1=1}^{N}V_{j_2j_1}+\sum_{j_3,j_2,j_1=1}^{N}V_{j_3j_2}V_{j_2j_1}+\cdots+\sum_{j_p,\cdots,j_1=1}^{N}V_{j_pj_{p-1}}\cdots V_{j_2,j_1}+\sum_{j_N,\cdots,j_1}^{N}V_{j_N,j_{N-1}}\cdots V_{j_2,j_1}\Big\}. \tag{14}$$

It is convenient to do the summation over the indices j_p by introducing a triangular matrix $\hat{V}$ whose elements are the V_{j_{p+1},j_p}:

$$\hat{V}=\begin{vmatrix}0 & 0 & \dots & 0\\ V_{21} & 0 & \dots & 0\\ V_{31} & V_{32} & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ V_{N1} & V_{N2} & \dots & 0\end{vmatrix}. \tag{15}$$

Using the properties of triangular matrices and summing over the inner indices j_p, we can write the expression (14) in the following form:

$$\langle\rho_N\rangle=\alpha\Big\{\sum_{j_2,j_1=1}^{N}(\hat{I})_{j_2j_1}+\sum_{j_2,j_1=1}^{N}(\hat{V})_{j_2j_1}+\sum_{j_3,j_1=1}^{N}(\hat{V}^2)_{j_3j_1}+\cdots\sum_{j_N,j_1=1}^{N}(\hat{V}^{N-1})_{j_Nj_1}\Big\}, \tag{16}$$

where $\hat{I}$ is the identity matrix. Here we have allowed for the fact that

$$(\hat{V}^p)_{j_lj_l}=0,\quad (\hat{V}^N)_{j_lj_{l'}}=0. \tag{17}$$

This property follows directly from the definition of matrix (15). Now (16) can be interpreted as a geometric series for the matrix $\hat{V}$. Allowing for (17), we obtain

$$\langle\rho_N\rangle=\alpha\sum_{i,j=1}^{N}\left(\frac{\hat{I}}{\hat{I}-\hat{V}}\right)_{ij}. \tag{18}$$

This implies that the calculation of $\langle\rho_N\rangle$ reduces to finding the matrix elements of $(\hat{I}-\hat{V})^{-1}$. If we introduce the notation

$$R_{N-1}=\sum_{i,j}\left(\frac{\hat{I}}{\hat{I}-\hat{V}}\right)_{ij}, \tag{19}$$

then

$$\langle\rho_N\rangle=\alpha R_{N-1}. \tag{20}$$

Now we wish to calculate R_{N-1}. The direct matrix can easily be shown to have the following form:

$$\hat{I}-\hat{V}=\begin{vmatrix}1 & 0 & 0 & \dots & 0 & 0\\ -c_1 & 1 & 0 & \dots & 0 & 0\\ -c_2 & -c_1 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -c_{N-1} & -c_{N-2} & -c_{N-3} & \dots & -c_1 & 1\end{vmatrix}, \tag{21}$$

where

$$c_j=2\alpha[1-\cos 2kaj],\quad j=1,2,\dots,N-1.$$

The triangular matrix (21) with unit diagonal elements can be represented in the form of a product of elementary triangular matrices $\hat{L}_p$,

$$\hat{I}-\hat{V}=\sum_{p=1}^{N-1}\hat{L}_p, \tag{22}$$

in which the lower part of each matrix contains zeros except in the pth row, where up to the diagonal element equal to unity the c_j form the following pattern:

$$\hat{L}_p=\begin{vmatrix}1 & 0 & 0 & \dots & \dots & \dots & 0\\ 0 & 1 & 0 & \dots & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ -c_p & -c_{p-1} & -c_{p-2} & \dots & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & 1\\ 0 & 0 & 0 & \dots & \dots & \dots & 1\end{vmatrix}.$$

The inverse matrix $(\hat{I}-\hat{V})^{-1}$ can be expressed in terms of a product of the inverse matrices $\hat{L}_p^{-1}$:

$$(\hat{I}-\hat{V})^{-1}=\prod_{p=N-1}^{1}\hat{L}_p^{-1}, \tag{23}$$

where the inverse matrices $\hat{L}_p^{-1}$ differ from the $\hat{L}_p$ only in the sign of the subdiagonal elements. Using the rule of multiplying triangular matrices with ones in the diagonal and carrying out the multiplication of the matrices $\hat{L}_p^{-1}$ in (23), we arrive at the final result:

$$(\hat{I}-\hat{V})^{-1}=\begin{vmatrix}A_0 & 0 & 0 & \dots & 0 & 0\\ A_1 & A_0 & 0 & \dots & 0 & 0\\ A_2 & A_1 & A_0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ A_{N-1} & A_{N-2} & A_{N-3} & \dots & A_1 & A_0\end{vmatrix}, \tag{24}$$

where

$$A_p=\sum_{i=1}^{p} c_i A_{p-i}, \quad A_0=1, \quad p=1,2,\ldots,N-1. \tag{25}$$

We note once more that the sum of the matrix elements in (24) yields the solution of the problem. Indeed, according to (19) we have

$$R_{N-1}=\sum_{i,j} (\hat{I}-\hat{V})^{-1}_{ij}=\sum_{p=0}^{N-1} S_p, \tag{26}$$

where S_{N-1} stands for the sum of elements of the first column in matrix (24), S_{N-2} of the second column, etc. Applying (25) to S_{N-1}, S_{N-2}, etc., we arrive at the following recurrence relation:

$$S_p=1+\sum_{i=1}^{p} c_i S_{p-i}. \tag{27}$$

Substituting (27) into (26), we finally get

$$\begin{aligned} R_{N-1}&=\sum_{p=0}^{N-1} 1+\sum_{i=1}^{N-1} c_i \sum_{p=i}^{N-1} S_{p-i} \\ &=N+\sum_{i=1}^{N-1} c_i R_{N-1-i}, \end{aligned} \tag{28}$$

where $R_0=1$. The recurrence relation (28) is an inhomogeneous finite difference equation for the desired function R_{N-1}.

Knowing R_{N-1}, we can use (20) to find $\langle \rho_N \rangle$. The recurrence relation (28) can be interpreted as a system of N linear equations in the unknown quantities $R_0,\ldots,R_{N-1}$. We write (28) in a form convenient for further calculations:

$$\begin{aligned} &1\cdot R_0+0\cdot R_1+\ldots+0\cdot R_{N-1}=1, \\ &-c_1\cdot R_0+1\cdot R_1+0\cdot R_2+\ldots+0\cdot R_{N-1}=2, \\ &-c_2\cdot R_0-c_1\cdot R_1+1\cdot R_2+\ldots+0\cdot R_{N-1}=3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &-c_{N-1}R_0-c_{N-2}R_1-c_{N-3}R_2-\ldots+1\cdot R_{N-1}=N. \end{aligned} \tag{29}$$

The determinant made up of the coefficients of the unknown quantities $R_0,\ldots,R_{N-1}$ is equal to unity. Hence the quantity R_{N-1} that we are interested in is given by the following expression:

$$R_{N-1}=\begin{Vmatrix} 1 & 0 & 0 & \ldots & 1 \\ -c_1 & 1 & 0 & \ldots & 2 \\ -c_2 & -c_1 & 1 & \ldots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{N-1} & -c_{N-2} & -c_{N-3} & \ldots & N \end{Vmatrix}. \tag{30}$$

Note that for the special case $ka=\pi$, Eq. (30) immediately yields the well-known result for the resonant case,

$$\langle \rho_N \rangle = \alpha N,$$

since all the c_j vanish when $ka=\pi$.

For arbitrary values of the electron energy but small values of N, the determinant (30) can also be easily calculated. But for $N \gg 1$, when calculating (30) is difficult, Eq. (30) must be solved directly by interpreting it as a finite difference equation for the desired function R_{N-1}.

As a rule, the solution of Eq. (28), which yields the resistance averaged over the different realizations of the random scattering impurities, has the form of a sum of general exponential functions of N. This important property of the solution is illustrated below by specific examples, which show that only in the weak-scattering limit does the dependence of $\langle \rho_N \rangle$ on the chain length become purely exponential.

4. SOLUTION OF THE RECURRENCE RELATION FOR THE KRONIG–PENNEY POTENTIAL

As Eqs. (3) and (4) show, the transmission coefficient T and the Landauer resistance $\langle \rho_N \rangle$ of a one-dimensional chain can be expressed in terms of the determinant D_N. The problem of finding D_N for the Kronig–Penney potential, i.e., solving the problem with the amplitude of the interaction of the electrons with the chain sites being the same, can easily be reduced to solving an equation of type (28). Indeed, comparing (9) with (11), we see that at $V_1=V_2=\cdots=V$ the expression for D_N-1 coincides with that for $\langle \rho_N \rangle$ if in the latter we put $\alpha=iV/4k$ and replace the cosine with an exponential. Since at $j_p=j_{p-1}$ both functions are equal to unity, the summation for $\langle \rho_N \rangle$ carried out in Sec. 3 can be repeated here for D_N-1, which yields

$$D_N-1=\alpha R_{N-1}, \tag{31}$$

where R_{N-1} satisfies Eq. (28) in which $\alpha=iV/4k$ and

$$c_l=\frac{iV}{2k}(1-e^{2ikal}). \tag{32}$$

Now let us solve Eq. (28) with a function c_l specified by (32). We introduce a new summation index $n=N-1-l$ and write Eq. (28) in the following form:

$$R_{N-1}=N+\sum_{n=0}^{N-2} c_{N-1-n} R_n. \tag{33}$$

We look for a solution of Eq. (33) in the form

$$R_{N-1}=\sum_{j=1}^{p} A_j x_j^{N-1}+A_0. \tag{34}$$

Here x_j are the roots of the characteristic equation, p is the number of roots of this equation, and A_0 and A_j are as-yet unknown coefficients that are independent of N.

Substituting (34) into Eq. (33) and requiring that the new equation be satisfied for any value of N, we arrive at the following equations for p, x_j, A_0, and A_j:

$$1+\frac{2\alpha}{1-x_j}-\frac{2\alpha}{1-x_j e^{-2ika}}=0, \tag{35}$$

$$2\alpha A_0+1=0, \tag{36}$$

$$\sum_{j=1}^{p} \frac{2\alpha A_j}{e^{2ika}-x_j} + \frac{2\alpha A_j}{e^{2ika}-1} = 0. \tag{37}$$

Equation (35) for the x_j is quadratic and in general has two complex-valued roots. This implies, in particular, that $p=2$. By eliminating the denominators we can write Eq. (35) in the form

$$x^2 - x[(1+2\alpha)+(1-2\alpha)e^{2ika}] + e^{2ika} = 0.$$

We look for a solution of this equation in the form $x = ze^{ika}$. Then Eq. (35) becomes

$$z^2 - z[(1+2\alpha)e^{-ika}+(1-2\alpha)e^{ika}] + 1 = 0. \tag{38}$$

The roots of this equation are

$$z_j = e^{\pm i\beta}, \quad j=1,2, \tag{39}$$

where

$$\cos\beta = \cos(ka) + \frac{V}{2k}\sin(ka).$$

Hence according to (39) we finally get $x_j = e^{i(ka\pm\beta)}$. The constant A_0 can be found immediately from Eq. (36): $A_0 = -1/2\alpha$. Since the characteristic equation has two roots, only the complex-valued constants A_1 and A_2 need to be found. To do this we have one equation (37), which means we need another equation. For this we take the condition $R_0 = 1$. Substituting $N=1$ into (34), we obtain

$$\sum_{j=1}^{2} A_j + A_0 = 1. \tag{40}$$

Below we show that instead of A_j it is convenient to introduce a new constant $B_j = 2\alpha A_j/x_j$. Then to determine B_j, instead of (37) and (40) we have

$$B_1 e^{ika+i\beta} + B_2 e^{ika-i\beta} = 1 + \frac{iV}{2k},$$

$$\frac{B_1(e^{i\beta}-e^{ika})}{1-\cos(ka-\beta)} + \frac{B_2(e^{-i\beta}-e^{ika})}{1-\cos(ka+\beta)} = -\frac{i}{\sin ka}. \tag{41}$$

The solution of these equations shows that B_1 and B_2 are real and are given by the following expression:

$$B_{1,2} = \frac{1}{2} \pm \frac{(V/2k)\cos(ka) - \sin(ka)}{2\sin\beta} \tag{42}$$

(the upper sign corresponds to B_1 and the lower to B_2). Substituting these constants into (34) and employing (31), we finally get

$$D_N = \sum_{j=1}^{2} B_j x_j^N$$

$$= e^{ikaN}\left\{\left[\frac{1}{2} + \frac{(V/2k)\cos(ka)-\sin(ka)}{2\sin\beta}\right]e^{i\beta N}\right.$$

$$\left. + \left[\frac{1}{2} - \frac{(V/2k)\cos(ka)-\sin(ka)}{2\sin\beta}\right]e^{-i\beta N}\right\}, \tag{43}$$

where β is specified by Eq. (39).

5. THE RESISTANCE ρ_N FOR AN ELECTRON ENERGY CORRESPONDING TO THE CENTER OF THE BAND

Now we wish to solve Eq. (28) for a fixed energy corresponding to the center of the band, i.e., we examine the case $ka = \pi/2 + \pi m$ $(m = 1,2,\ldots)$. In this case,

$$c_n = \begin{cases} 0, & \text{for } n \text{ odd}, \\ 4\alpha, & \text{for } n \text{ even}, \end{cases}$$

and Eq. (28) assumes the following simple form:

$$R_{N-1} = N + \sum_{n=0}^{N-2} 4\alpha R_n. \tag{44}$$

We again seek a solution in the form (34). By requiring that this solution satisfy Eq. (44) we get

$$1 - \frac{4\alpha x_j}{x_j^2 - 1} = 0, \tag{45}$$

$$2\alpha A_0 = -1, \tag{46}$$

$$\sum_{j=1}^{p} \frac{4\alpha A_j}{x_j^2 - 1} + A_0 = 0. \tag{47}$$

The solution of the first equation can be written immediately:

$$x_j = 2\alpha \pm \sqrt{1+4\alpha^2}. \tag{48}$$

Since the characteristic equation for x_j has two roots, we need only determine the two constants A_1 and A_2. Employing (45) and (48) and introducing new constants $B_j = 2\alpha A_j/x_j$ instead of A_j, we can write Eq. (47) as

$$B_1 + B_2 = 1. \tag{49}$$

We obtain the missing equation for the constants B_1 and B_2 from the requirement that $R_0 = 1$. Substituting $N=1$ into (34) and replacing A_j with B_j, we get

$$B_1 x_1 + B_2 x_2 = 1 + 2\alpha. \tag{50}$$

The solution of Eqs. (49) and (50) has the form

$$B_1 = \frac{1}{2}\left(1 + \frac{1}{b}\right), \quad B_2 = \frac{1}{2}\left(1 - \frac{1}{b}\right), \tag{51}$$

where $b = \sqrt{1+4\alpha^2}$. If we introduce the notation $a = 2\alpha$ and substitute the derived constants into the solution (34), we finally obtain

$$\langle\rho_N\rangle = \alpha R_{N-1} = \frac{1}{2}\left[\frac{(a+b)^N + (a-b)^N}{2}\right.$$

$$\left. + \frac{(a+b)^N - (a-b)^N}{2b}\right]. \tag{52}$$

We can write the above formula in a more convenient form by introducing the notation $a = \sinh x$ and $b = \cosh x$. Then we have

$$\langle\rho_N\rangle = \frac{1}{2}\left[\cosh(Nx) + \frac{\sinh(Nx)}{\cosh x} - 1\right] \tag{53}$$

for N even, and

$$\langle \rho_N \rangle = \frac{1}{2}\left[\sinh(Nx) + \frac{\cosh(Nx)}{\cosh x} - 1 \right] \tag{54}$$

for N odd.

We also note that this solution can be obtained from the solution (43) if in the latter we put $ka = \pi/2$ and $\alpha = iV/4k$. The two results coincide at $ka = \pi/2$ because the c_n do.

The solution (52)–(54) is remarkable because it was obtained without resorting to perturbation theory, and is therefore valid for any amplitudes of the interaction of electrons with sites for an arbitrary number N of scattering centers. It shows that when the electron energy corresponds to the center of the band, the average electrical resistance for a one-dimensional chain of random scatterers is a general exponential function of the chain length.

In the asymptotic case of $\alpha \ll 1$, the solution (52) yields

$$\langle \rho_N \rangle \approx \tfrac{1}{2}(e^{2\alpha N} - 1), \tag{55}$$

with the same result obtained in Refs. 4–8. In the other limiting case, $\alpha \gg 1$, the solution (52) yields

$$\langle \rho_N \rangle \simeq 2^{2(N-1)} \alpha^N. \tag{56}$$

The condition $\alpha \gg 1$ means that each scattering event can be interpreted as occurring independently of the others; hence the result (56) is obvious.[9]

Using the solutions (52)–(54), we can show that when $N \to \infty$, the electron localization radius is

$$a\xi^{-1} = N^{-1}\left[\sinh(Nx) + \frac{\cosh(Nx)}{\cosh x} \right]. \tag{57}$$

In particular, in the case of weak coupling ($\alpha \ll 1$) this yields

$$a\xi^{-1} \simeq 2\alpha - \frac{2\alpha^2}{N}. \tag{58}$$

The first term on the right-hand side of (58), which is twice the localization radius, was obtained in Ref. 1. The reason is that in the weak-disorder limit the following relationship is valid:[16]

$$\ln\langle \rho \rangle = 2\langle \ln \rho \rangle.$$

In the case of strong coupling ($\alpha \gg 1$) from (57) we obtain the following asymptotic expression for ξ:

$$a\xi^{-1} \approx \ln 4\alpha - \frac{1}{N}\left(\ln 2 - \frac{1}{2\alpha} \right). \tag{59}$$

Note that the first term on the right-hand side of (59) was obtained in Ref. 13. Strictly speaking, the result (59) is formal. Indeed, we have found that $\xi \ll a$ when an electron moves freely between the impurity centers. Hence the result must be interpreted as the penetrability of a single impurity barrier being low rather than ξ being small. This agrees with the result (56).

In conclusion we note that the method developed here for determining $\langle \rho \rangle$, which we applied for energies corresponding to the edges and center of the energy band, can also be used to solve the problem with an arbitrary energy.

One of the authors (D. M. S.) is grateful to the Department of Theoretical Physics of the P. N. Lebedev Physics Institute of the Russian Academy of Sciences for the hospitality shown and for making it possible to work in the Department in the final stages of the present investigation.

The results listed in this paper have been obtained thanks in part to the financial support of the International Science Foundation (Grant No. MVL000).

[1] V. I. Perel' and D. G. Polyakov, Zh. Éksp. Teor. Fiz. **86**, 352 (1984) [Sov. Phys. JETP **59**, 204 (1984)].
[2] G. M. Soukoulis, I. V. Jose, E. N. Economou, and Ping Sheng, Phys. Lett. **50**, 764 (1983).
[3] E. Cota, I. V. Jose, and M. Ya. Azbel, Phys. Rev. B **32**, 6157 (1985).
[4] R. Landauer, Philos. Mag. **21**, 863 (1970).
[5] P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980).
[6] V. I. Mel'nikov, Fiz. Tverd. Tela (Leningrad) **23**, 782 (1981) [Sov. Phys. Solid State **23**, 444 (1981)].
[7] A. A. Abrikosov, Solid State Commun. **37**, 997 (1981).
[8] N. Kumur, Phys. Rev. B **31**, 5513 (1985).
[9] I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems*, Wiley, New York (1988).
[10] V. N. Prigogin, Zh. Éksp. Teor. Fiz. **79**, 2338 (1980) [Sov. Phys. JETP **52**, 1185 (1980)].
[11] V. M. Gasparian, Sov. Phys. Solid State **36**, 1061 (1989).
[12] V. M. Gasparian and Ash. Gh. Khachatryan (Khachatrian), Solid State Commun. **12**, 1061 (1993).
[13] V. M. Gasparian, B. L. Altshuler, A. G. Aronov, and Z. A. Kasamanian, Phys. Lett. A **132**, 201 (1988).
[14] F. Bentosela, A. Grecchi, and E. Zironi, Phys. Rev. B **31**, 6909 (1985).
[15] V. M. Gasparyan, Fiz. Tverd. Tela (Leningrad) **31**, 162 (1989). [Sov. Phys. Solid State **31**, 266 or 1005 (1989)].
[16] B. Kramer, *Localization and Transport in One-Dimensional Disordered Systems*, Preprint PTB (1988).

Translated by Eugene Yankovsky