Angular momentum equation in continuum mechanics with allowance for internal rotations

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In the isothermal approximation, we develop a theory of the motion of continuous media with allowance for internal angular momentum in which the balance equation of the total mechanical angular momentum, which is the basis of the theory, is formulated in a way that differs from the existing theory. The resulting equations are used to interpret the results of experimental investigation into the behavior of a magnetic liquid in a rotating magnetic field.

1. INTRODUCTION

The creation of a theory of the motion of a continuous medium with allowance for internal angular momentum has been the subject of a number of investigations.[1–5] These studies introduce in addition to the density of the macroscopic angular momentum $\mathbf{L}=\rho[\mathbf{rv}]$ (ρ and $\mathbf{v}$ are, respectively, the density and hydrodynamic velocity of the fluid) the density of internal angular momentum $\mathbf{S}$. In accordance with Refs. 1–5, the balance equation for the total mechanical angular momentum $\mathbf{K}$, which is equal to the sum $\mathbf{L}+\mathbf{S}$, is not a consequence of the momentum balance equation but is an independent equation that occurs in the system of equations together with the balance equations for mass and momentum. In the case of isothermal motion of an incompressible medium, the differential form of the balance equations for mass, momentum, and angular momentum are postulated in the form

$$\frac{\partial v_k}{\partial x_k}=0, \tag{1}$$

$$\frac{\partial p_i}{\partial t}=-\frac{\partial}{\partial x_k}(p_i v_k-p^*_{ik})+F_i \tag{2}$$

$$\frac{\partial K_{ik}}{\partial t}=-\frac{\partial}{\partial x_l}(K_{ik}v_l+j^K_{ikl})+N^K_{ik}, \tag{3}$$

where $p_i=\rho v_i$ is the momentum density, F_i is the volume density of the external forces, $K_{ik}=e_{ikl}K_l$, N^K_{ik} is the density of the source K_{ik}, and j^K_{ikl} is the flux density of K_{ik}, which satisfy

$$N^K_{ik}=(x_iF_k-x_kF_i)+N_{ik},$$

$$j^K_{ikl}=j^L_{ikl}+j^S_{ikl}=-(x_ip^*_{kl}-x_kp^*_{il})+j^S_{ikl}, \tag{4}$$

N_{ik} is the volume density of the external couples, and j^S_{ikl} is the flux density of the tensor of the internal angular momentum $S_{ik}=e_{ikl}S_l$. Equation (2) differs from the momentum balance equation in the absence of internal angular momentum through the fact that the stress tensor p^*_{ik} contains not only a symmetric part p_{ik} but also an antisymmetric part p^a_{ik} ($p^*_{ik}=p_{ik}+p^a_{ik}$). When the balance equation for K_{ik} was postulated, it was assumed that the tensors p^a_{ik} and j^S_{ikl} are independent.

Equation (3) can be transformed to a balance equation for S_{ik}; for this we subtract from (3) the balance equation for the macroscopic angular momentum $L_{ik}=e_{ikl}L_l$; this is a consequence of the momentum balance equation (2). In accordance with Refs. 1–5, the balance equation for L_{ik} can be represented in the form

$$\frac{\partial L_{ik}}{\partial t}=-\frac{\partial}{\partial x_l}(L_{ik}v_l+j^L_{ikl})+N^L_{ik}, \tag{5}$$

where

$$N^L_{ik}=(x_iF_k-x_kF_i)+2p^a_{ik},$$

$$j^L_{ikl}=j^K_{ikl}-j^S_{ikl}=-(x_ip^*_{kl}-x_kp^*_{il}). \tag{6}$$

The flux density j^L_{ikl} is written down in accordance with the expression for j^K_{ikl} in (3). The balance equation for the internal angular momentum can be found in the form

$$\frac{\partial S_{ik}}{\partial t}=-\frac{\partial}{\partial x_l}(S_{ik}v_l+j^S_{ikl})-2p^a_{ik}+N_{ik}. \tag{7}$$

The system of the three balance equations (1), (2), and (7) is augmented by equations that relate the irreversible fluxes and thermodynamic forces. In the domain of applicability of nonequilibrium thermodynamics,[3–5] these equations for fluid media are written in the form

$$p_{ik}=-p\,\delta_{ik}+\eta\left(\frac{\partial v_k}{\partial x_i}+\frac{\partial v_i}{\partial x_k}\right), \tag{8}$$

$$p^a_{ik}=2\tilde{\gamma}(\omega^S_{ik}-\Omega_{ik}),\quad \Omega_{ik}=\frac{1}{2}\left(\frac{\partial v_k}{\partial x_i}-\frac{\partial v_i}{\partial x_k}\right),$$

$$\omega^S_{ik}=\frac{S_{ik}}{I}, \tag{9}$$

$$-j^S_{ik}=\gamma_0\frac{\partial \omega_l}{\partial x_l}\delta_{ik}+\gamma_1\left(\frac{\partial \omega^S_k}{\partial x_i}+\frac{\partial \omega^S_i}{\partial x_k}-\frac{2}{3}\frac{\partial \omega^S_l}{\partial x_l}\right)$$
$$+\gamma_2\left(\frac{\partial \omega^S_k}{\partial x_i}-\frac{\partial \omega^S_i}{\partial x_k}\right), \tag{10}$$

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where p is the hydrostatic pressure, I is the moment of inertia per unit volume, η is the coefficient of dynamic viscosity, $\tilde{\gamma}$, γ_0, γ_1, γ_2 are newly introduced empirical coefficients, and $j^S_{ik}=e_{ilm}j^S_{lmk}/2$.

For $F_i=0$ and $N_{ik}=0$, some specific problems were considered on the basis of the system (1), (2), (7), and (8)–(10). The solutions contain corrections due to allowance for internal rotations, which in some limiting cases can be appreciable. However, experimental confirmation of these solutions is nowhere given. Comparison of the results of experiments made with a ferromagnetic liquid in a rotating homogeneous magnetic field ($F_i=0$, $N_{ik}\neq 0$) with theoretical results obtained by solving this system indicates a qualitative discrepancy between the theoretical and experimental results, as will be discussed in detail in Sec. 4.

It is necessary to point out logical contradictions in the system of equations postulated in Refs. 1–5 found by analyzing this system from the point of view of the flux density and source of the angular momenta.

The balance equation (3) for the total mechanical angular momentum is postulated independently of the momentum balance equation (2). In accordance with (3) and (4), the source of K_{ik} is the sum of the densities of the moment of the body forces $(x_iF_k-x_kF_i)$ and external couples N_{ik}. The source postulate appears indisputable. However, we cannot agree with the expression for j^K_{ikl} in terms of p^*_{ik} and j^S_{ikl} in accordance with (4). The postulate that j^K_{ikl} has the form (4) based on the assumption (which is apparently obvious for the authors of Refs. 1–7) that p^a_{ik} and j^S_{ikl} are independent leads to the need to express the balance equation for L_{ik} in the form (5), in which the flux density j^L_{ikl} is obtained from the expression (4) for j^K_{ikl} in the form

$$j^L_{ikl}=-(x_ip^*_{kl}-x_kp^*_{ik}).$$

We cannot agree with the balance equation for L_{ik} in the form (5), since in accordance with (5) the source of the macroscopic L_{ik} is not only $x_iF_k-x_kF_i$ but also the antisymmetric part of the stress tensor p^a_{ik}. The presence of $2p^a_{ik}$ with minus sign as source simultaneously in Eq. (7) for the internal angular momentum is interpreted as the possibility that the internal angular momentum S_{ik} can be transformed into macroscopic angular momentum even for p^a_{ik}=const. It turns out that for $x_iF_k-x_kF_i=0$ the presence of a homogeneous distribution of p^a_{ik} for fixed bounding surfaces leads to excitation of L_{ik} and, therefore, to macroscopic motion of the fluid, since for $L_{ik}\neq 0$ we find in accordance with the definition of L_{ik} that also the hydrodynamic velocity satifies $v_i\neq 0$. However, this conclusion from the form of Eq. (5) contradicts Eq. (2), a consequence of which is (5). Indeed, Eq. (2) contains only the derivatives $\partial p^a_{ik}/\partial x_k$ and for p^a_{ik}=const reduces to a form that does not contain p^a_{ik} at all. Thus, in accordance with (2) the given homogeneous distribution of p^a_{ik} does not excite macroscopic motion. It is readily understood that the balance equation (5) for the macroscopic angular momentum, which is a consequence of Eq. (2), can also lead under these conditions only to the same result. This is readily seen directly on the basis of Eq. (5), which for a homogeneous distribution of p^a_{ik} reduces to an equation that does not contain p^a_{ik}. Thus, $2p^a_{ik}$ is not a source of macroscopic angular momentum, and the identification in Eq. (5) of $2p^a_{ik}$ as the source of L_{ik} is artificial.

2. CONSISTENT SYSTEM OF EQUATIONS

To construct a consistent theory of the motion of continuous media with allowance for internal angular momentum, it appears expedient to follow the scheme of arguments adopted in the derivation of hydrodynamic equations for media with internal degrees of freedom.[8,9] To find the equations of motion, it is reasonable to proceed from a balance equation for L_{ik} that does not contain p^a_{ik} as source:

$$\frac{\partial L_{ik}}{\partial t}=-\frac{\partial}{\partial x_l}[L_{ik}v_l-(x_ip^*_{ik}-x_kp^*_{il})-j_{ikl}]+(x_iF_k-x_kF_i), \tag{11}$$

where j_{ikl} is a third-rank tensor that has the dimensions of an angular momentum flux density and is related to p^a_{ik} by

$$2p^a_{ik}=\partial j_{ikl}/\partial x_l. \tag{12}$$

It is clear that the balance equation for L_{ik} in the form (11) is equivalent to Eq. (5), but the expressions for the flux density j^L_{ikl} and the source of the macroscopic angular momentum N^L_{ik} in (11) differ appreciably from the corresponding expressions in (5):

$$j^L_{ikl}=-(x_ip^*_{kl}-x_kp^*_{il})-j_{ikl},\quad N^L_{ik}=x_iF_k-x_kF_i. \tag{13}$$

In accordance with (13), the flux density of the total mechanical angular momentum can be found from the definition in the form

$$j^K_{ikl}=j^L_{ikl}+j^S_{ikl}=-(x_ip^*_{kl}-x_kp^*_{il})-j_{ikl}+j^S_{ikl}. \tag{14}$$

The balance equation for K_{ik} can be found from Eq. (3), in which j^K_{ikl} is determined by the expression (14), and not (4), and the source density N^K_{ik} remains, in accordance with what was said above, unchanged:

$$\frac{\partial K_{ik}}{\partial t}=-\frac{\partial}{\partial x_l}[K_{ik}-(x_ip^*_{kl}-x_kp^*_{il})-j_{ikl}+j^S_{ikl}]+(x_iF_k-x_kF_i)+N_{ik}. \tag{15}$$

After subtracting (11) from (15), we find the balance equation for the internal angular momentum:

$$\frac{\partial S_{ik}}{\partial t}=-\frac{\partial}{\partial x_l}(S_{ik}v_l+j^S_{ikl})+N_{ik}. \tag{16}$$

In contrast to (7), Eq. (16) does not contain $-2p^a_{ik}$ as source. In the stationary case for $F_i=0$ and in the presence of a homogeneous distribution of N_{ik} and p^a_{ik}, $v_i=0$ in accordance with what was said above, and Eq. (16) takes the form

$$\partial j^S_{ikl}/\partial x_l=N_{ik}. \tag{17}$$

In addition to the investigations of Refs. 1–7, some studies have been made[10–13] of the stationary case ($dv_i/dt=0$, $dS_{ik}/dt=0$) when a volume density N_{ik} of couples arises through the action of an electric or magnetic field on particles of a medium that possess an electric or magnetic moment, respectively. In the presence of a homogeneous distribution of N_{ik}, we have according to Refs. 10–13 the equation

$$2p^a_{ik}=N_{ik}, \tag{18}$$

which differs from the Cosserat equation, which is the limiting case of Eq. (7) for $dS_{ik}/dt=0$.

In the construction of the theory of the motion of continuous media with allowance for internal angular momentum, it is assumed in the general case that under stationary conditions Eq. (18) holds. With allowance for (18), the relation (17) can be rewritten in the form

$$\partial j^S_{ikl}/\partial x_l=2p^a_{ik}. \tag{19}$$

It follows from comparison of (19) and (12) that

$$j_{ikl}=j^S_{ikl}. \tag{20}$$

Thus, it turns out that the antisymmetric tensor p^a_{ik} and the flux density j^S_{ikl} of the internal angular momentum are not independent but are related by Eq. (19), in accordance with which p^a_{ik} is determined by the divergence of the flux density j^S_{ikl}. When the equations of motion of a continuous medium are postulated with allowance for internal angular momentum, Eq. (20) and the relationship (19) between p^a_{ik} and j^S_{ikl} are assumed to be valid in the general case. In accordance with (13) and (19), the flux density of the macroscopic angular momentum is

$$j^L_{ikl}=-(x_ip^*_{kl}-x_kp^*_{il})-j^S_{ikl} \tag{21}$$

and in accordance with (11) the balance equation for the macroscopic angular momentum must have the form

$$\frac{\partial L_{ik}}{\partial t}=-\frac{\partial}{\partial x_l}[L_{ik}v_l-(x_ip^*_{kl}-x_kp^*_{il})-j^S_{ikl}]+(x_iF_k-x_kF_i). \tag{22}$$

The flux density of the mechanical angular momentum is

$$j^K_{ikl}=j^L_{ikl}+j^S_{ikl}=-(x_ip^*_{kl}-x_kp^*_{il}), \tag{23}$$

and the balance equation for the total mechanical angular momentum is postulated in accordance with (15) in the form

$$\frac{\partial K_{ik}}{\partial t}=-\frac{\partial}{\partial x_l}[K_{ik}v_l-(x_ip^*_{kl}-x_kp^*_{il})]+(x_iF_k-x_kF_i)+N_{ik}. \tag{24}$$

The integral equation for the total mechanical angular momentum corresponding to Eq. (24) takes the form

$$\frac{d}{dt}\int_V \mathbf{K}dV=\int_\sigma[\mathbf{r}\mathbf{p}^*_n]d\sigma+\int_V[\mathbf{r}\mathbf{F}]dV+\int_V\mathbf{N}dV, \tag{25}$$

where σ is the surface that bounds the volume V, $\mathbf{p}^*_n = \mathbf{e}_ip^*_{ik}n_k$ is the force per unit surface, and n_k are the components of the normal to the element of surface. In accordance with (25), the additional torque on the surface of the volume in question due to the internal moment is determined when the relationship (19) holds by the integral of only $[\mathbf{r}\mathbf{p}^a_n]$ over the surface, and we have

$$\int_\sigma[\mathbf{r}\mathbf{p}^a_n]d\sigma=\int_\sigma\mathbf{m}_nd\sigma+\int_V[\mathbf{r}\ \mathrm{curl}\ \mathbf{p}^a]dV, \tag{26}$$

i.e., the integral over the surface of $\mathbf{m}_n$ ($\mathbf{m}_n=-\mathbf{e}_ij^S_{ik}n_k$) is contained in this integral. The second term on the right-hand side of (26) determines the total moment of the "body force" curl $\mathbf{p}^a$ due to the antisymmetric stresses ($p^a_i=e_{ikl}p^a_{kl}/2$).

The system of differential equations with allowance for internal angular momentum can contain the angular momentum equation either in the form of the balance equation for the total mechanical angular momentum (24) or in the form of the balance equation for the internal angular momentum (16). For an incompressible medium, this system of equations, expressed in terms of the axial vectors S_i and p^a_i, has in accordance with the foregoing the form

$$\rho\left(\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right)=\frac{\partial p_{ik}}{\partial x_k}+(\mathrm{curl}\ \mathbf{p}^a)_i+F_i,\quad \frac{\partial v_k}{\partial x_k}=0, \tag{27}$$

$$\frac{\partial S_i}{\partial t}+v_k\frac{\partial S_i}{\partial x_k}=2p^a_i+N_i, \tag{28}$$

$$2p^a_{ik}=\frac{\partial}{\partial x_k}(j^S_{ik}). \tag{29}$$

The postulate for the equation of the internal angular momentum in the form (28) under the condition (29) finds support in the derivation of the equation of the internal angular momentum for a specific medium made in Ref. 14.

3. LINEAR DETERMINING EQUATIONS

The phenomenological equations that close the system (27)–(29) can be found from the requirement that the dissipative function be positive.[8,9] The kinetic energy density u is equal to the sum of the kinetic energy densities of the translational and rotational motions:

$$u=\rho v^2/2+S_i^2/2I.$$

On the basis of Eqs. (27)–(29), we find the balance equation for the kinetic energy:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x_l}\left[uv_l-v_ip^*_{il}-\left(\Omega_i-\frac{S_i}{I}\right)j^S_{il}\right]=-\phi+F_iv_i+\frac{N_lS_l}{I}, \tag{30}$$

where ϕ is the dissipative function:

$$\phi=p_{ik}\left(\frac{\partial v_k}{\partial x_i}+\frac{\partial v_i}{\partial x_k}\right)-j^S_{ik}\frac{\partial}{\partial x_k}\left(\frac{S_i}{I}-\Omega_i\right). \tag{31}$$

The sum $-v_ip^a_{il}-(\Omega_i-S_i/I)j^S_{il}$ under the sign of the divergence on the left-hand side of Eq. (30) determines the energy flux associated with the transport of internal angular momentum. This flux is analogous to the energy flux $-v_ip_{il}$ associated with the transport of momentum. The second term in the expression (31) determines the contribution to the energy dissipation made by internal rotations. From the requirement of positivity of ϕ there follows the well-known expression (8) for p_{ik} and the linear determining equation for j^S_{ik}:

$$j^S_{ik} = -D^S_{iklm} \frac{\partial}{\partial x_l} (S_m - I\Omega_m), \quad (32)$$

where D^S_{iklm} is the tensor of the transport coefficients of the internal angular momentum. The components of this tensor are empirical constants.

It is clear that because of the difference between the equations of the internal angular momentum (7) (Refs. 1–5) and (28) the terms in the expression for the dissipative function that contribute to the energy dissipation introduced by the internal rotations and, accordingly, the linear determining equations (10) (Refs. 1–5) and (32) have different forms.

With allowance for (8) and (32), Eq. (28) can be reduced to the form

$$\frac{\partial S_i}{\partial t} + v_k \frac{\partial S_i}{\partial x_k} + D^S_{iklm} \frac{\partial^2}{\partial x_k \partial x_l} (S_m - I\Omega_m) = N_i. \quad (33)$$

For $\Omega_i = 0$, Eq. (33) is completely analogous to the heat transport equation.

In a number of cases, one can distinguish only two directions along which the transport coefficients of the internal angular momentum can differ appreciably. For example, these may be the directions along and at right angles to the vector **S**. In this case,

$$D^S_{iklm} = D^S_{\perp} \delta_{im} \delta_{kl} + (D^S_{\parallel} - D^S_{\perp}) \delta_{ik} \delta_{im} \delta_{lm}, \quad (34)$$

where in the second term on the right-hand side there is no summation over the repeated indices i and m. By $D^S_{\parallel}$ and $D^S_{\perp}$, we denote the transport coefficients of the internal angular momentum parallel and at right angles to the direction of **S**, respectively. In this case, the equation for the internal angular momentum, Eq. (33), has the form

$$\frac{\partial S_i}{\partial t} + v_k \frac{\partial S_i}{\partial x_k} + D^S_{\perp} \frac{\partial^2}{\partial x_k^2} (S_i - I\Omega_i) + (D^S_{\parallel} - D^S_{\perp}) \frac{\partial^2}{\partial x_i^2} \times (S_i - I\Omega_i) = N_i. \quad (35)$$

In the final term on the left-hand side, there is no summation over the index i.

The volume densities F_i and N_i are determined by the characteristics of the medium and the characteristics of the force fields with which this medium interacts. In the case of magnetizable liquids, F_i and N_i are determined in the form

$$F_i = M_k \frac{\partial B_i}{\partial x_k}, \quad N_i = e_{ikl} \mu_0 M_k H_l. \quad (36)$$

In the majority of problems associated with magnetic liquids, it is sufficient to write down the equations of the electromagnetic field in the quasistationary approximation:

$$\text{curl } \mathbf{H} = 0, \quad \text{div } \mathbf{B} = 0, \quad \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \quad (37)$$

which must be augmented by the equations for the magnetization dynamics. In the case of magnetic liquids, the Fokker–Planck equation for the distribution function with respect to the orientations of the magnetic moments of suspended ferromagnetic particles can serve as an equation for the magnetization dynamics. The magnetization **M** is determined by the mean value of the magnetic moment of the particles.

4. COMPARISON WITH EXPERIMENT

Effects associated with internal angular momentum and its transport in liquid media play an important role in experiments on the interaction of a magnetic field with ferromagnetic liquids. Among the most interesting experiments are the investigations into the behavior of a ferromagnetic liquid in a rotating homogeneous field. Several experimental studies[15–18] have been devoted to investigating this. The corresponding theoretical studies[19–21] were based on the equations of Refs. 1–7. Some results of these theoretical investigations agree with experiment, but the greater part of the results is contradictory and cannot be interpreted. Noncontradictory results in agreement with experiment can be found on the basis of the system of equations (27)–(29) given in the present paper.

In the theoretical treatment, it is assumed that the liquid occupies a cylindrical region of radius r_0. A cylindrical coordinate system (r, φ, z) with z axis along the symmetry axis is used. In accordance with (37), the components of the rotating homogeneous magnetic field are represented in the form

$$H_r = H \cos(\omega t - \varphi), \quad H_\varphi = H \sin(\omega t - \varphi), \quad H_z = 0, \quad (38)$$

where H is the magnitude of the magnetic field **H**, and ω is the angular frequency of its rotation. This field, interacting with the magnetic moments of the particles, sets them into rotation and therefore excites an internal mechanical angular momentum **S**. According to Refs. 19–21, the solution of the equation for the magnetization dynamics can be represented in the form

$$M_r = M \cos(\omega t - \varphi - \varphi_1),$$
$$M_\varphi = M \sin(\omega t - \varphi - \varphi_1), \quad M_z = 0. \quad (39)$$

The magnitude M of the magnetization and the angle between the vectors **H** and **M** can be expressed in terms of the components $M_\parallel$ and $M_\perp$ of the magnetization parallel and perpendicular to the vector **H**:

$$M = (M_\parallel^2 + M_\perp^2)^{1/2}, \quad \varphi_1 = \tan^{-1}(M_\perp / M_\parallel). \quad (40)$$

In accordance with (36), the volume densities **F** and **N** are

$$\mathbf{F} = 0, \quad \mathbf{N} = \mu_0 H M_\perp \mathbf{e}_z. \quad (41)$$

In accordance with the symmetry conditions, it is assumed that

$$\mathbf{v} = v(r) \mathbf{e}_\varphi, \quad \mathbf{\Omega} = \Omega(r) \mathbf{e}_z, \quad \mathbf{S} = S(r) \mathbf{e}_z,$$
$$\boldsymbol{\omega}^s = \omega^s(r) \mathbf{e}_z, \quad \mathbf{p}^a = p^a_z(r) \mathbf{e}_z.$$

In this case, the momentum balance equation (2) or (27) can be reduced to the form

$$\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{v}{r^2} = \frac{1}{\eta} \frac{dp^a_z}{dr}. \quad (42)$$

The solution of this equation must be bounded and vanish at $r = r_0$, since the cylindrical surface is assumed to be fixed. For these boundary conditions, Eq. (42) has a nontrivial solution only in the presence of a region in which the right-hand side of this equation, which plays the role of a driving force, is nonzero. If p_z^a=const holds in the complete interval ($0 \leq r \leq r_0$), then we have $dp_z^a/dr=0$ and Eq. (42) has the trivial solution ($v=0$). The theoretical investigations of Refs. 19–21 were based on the theory developed in Refs. 1–7, and so p_z^a was determined in them by the expression (9). The mean angular velocity ω^S of rotation of the particles must satisfy certain boundary conditions on the bounding surface. In Refs. 19 and 20, a zero-value boundary condition was assumed: $\omega^S(r_0)=0$. When this boundary condition is imposed on ω^S a region of inhomogeneity of ω^S arises near the bounding surface and, therefore, in accordance with (9), a region of inhomogeneity of p_z^a as well. This had the consequence that in Refs. 19 and 20 a nontrivial solution of Eq. (42) determining a very slow rotation of the liquid was found.

In Ref. 21, the boundary condition of free rotation was adopted; in other words, no boundary conditions were imposed on ω^S, since it was assumed that the resistance to rotation of the particles near the wall did not differ appreciably from the resistance to rotation of the particles far from the wall. As a result of this, ω^S and p_z^a were homogeneous in the complete interval $0 \leq r \leq r_0$, and the solution of Eq. (42) was the trivial solution ($v=0$), from which the conclusion was drawn that a magnetic liquid cannot be set into rotation by a rotating homogeneous magnetic field.

In the experimental investigations of Refs. 16–18, in complete agreement with Ref. 21, it was found that a rotating homogeneous field did not generate motion in a magnetic liquid. According to Ref. 22, the rotation observed in the experiment of Ref. 15 did not occur in the volume of the liquid but represented rotation in a thin surface layer and was due to the presence of a meniscus.

The results of the measurement of the velocity in Refs. 16–18 could apparently be regarded as a confirmation of the theory of Refs. 1–7 with boundary conditions of free rotation. However, in these experiments only the velocity and not other physical quantities (for example, ω^S, j_{zr}^L, j_{zr}^S) that characterize the behavior of the magnetic liquid was measured, mainly, it seems, because of the absence of appropriate methods of measurement. The fluxes j_{zr}^L and j_{zr}^S were also not investigated in the theory, since attention was concentrated primarily on quantities that could be measured in the experiment. It is clear that a correct theory must give correct values of all quantities.

In the theory developed in Refs. 1–7, it follows from the expressions (6) and (10) that the flux densities j_{zr}^L and j_{zr}^S are, respectively,

$$j_{zr}^L = r p_z^a \neq 0, \quad j_{zr}^S = 0. \tag{43}$$

The existence of a finite value of the flux density j_{zr}^L of the axial component of the macroscopic angular momentum L_z given $L_z = rv = 0$ is in contradiction with the very concept of a flux density, since, as is entirely clear, the existence of a flux density of some quantity requires presence of the quantity itself.

Vanishing of j_{zr}^S for $S \neq 0$ and $N_z \neq 0$ also cannot be understood from the point of view of the definitions of the participating quantities. Indeed, the presence of a volume density N_z of couples generates an internal angular momentum that under stationary conditions must flow out of the considered volume, i.e., there must exist a nonzero flux density j_{zr}^S.

Correct noncontradictory values of j_{zr}^S and j_{zr}^L can be found on the basis of the theory developed in the present paper. In accordance with Eq. (28) for the internal angular momentum, the component p_z^a can be written with allowance for (41) in the form

$$p_z^a = N_z/2 = \mu_0 H M_\perp /2 = \text{const} \quad \text{for } 0 \leq r \leq r_0. \tag{44}$$

Equation (42) for the determination of v remains unchanged. In accordance with (44), it has, in agreement with the experiment, the trivial solution. The flux density j_{zr}^S of the internal angular momentum in the case p_z^a=const is most readily found from the relation (29):

$$j_{zr}^S = r p_z^a = \mu_0 H M_\perp r/2 \neq 0. \tag{45}$$

The flux density $j_{zr}^L = j_{r\varphi r}^L$ of the macroscopic angular momentum is determined by the expression (21), from which, with allowance for (45), it follows that $j_{zr}^L=0$. The self-consistent values of j_{zr}^S and j_{zr}^L are undoubtedly a confirmation of the correctness of the developed theory.

The presence of a homogeneous distribution of p_z^a now no longer leads to a homogeneous distribution of $S(r)$ or $\omega^S(r)$. To find $S(r)$, we obtain in accordance with (33) and (44) the equation

$$\frac{1}{r}\frac{d}{dr} r \frac{dS}{dr} = -\frac{N_z}{D_\perp^S} = -\frac{\mu_0 H M_\perp}{D_\perp^S}. \tag{46}$$

This equation can be solved under the condition that S and dS/dr are bounded in the region $0 \leq r \leq r_0$. In accordance with what we have said above, no boundary conditions are imposed on S at $r = r_0$. The solution of Eq. (46) can be found in the form

$$S(r) = S(0) - N_z r^2/4D_\perp^S = S(0) - \mu_0 H M_\perp r^2/4D_\perp^S, \tag{47}$$

where $S(0)$ is the value of $S(r)$ at $r=0$. For $\omega\tau_B \ll 1$, where $\tau_B = \beta_0/2kT$ and β_0 is the coefficient of resistance to rotation of suspended particles, the component $M_\perp$ was determined in Refs. 19–21. For arbitrary values of $\omega\tau_B$, the magnetization $M_\perp$ was determined in Refs. 23 and 24 on the basis of a solution of the Fokker–Planck equation. In accordance with this solution,

$$\mu_0 H M_\perp = \beta\omega f(\gamma, \omega\tau_B), \tag{48}$$

where $\gamma = \mu_0 m_0 H/kT$ is the Langevin parameter, and $\beta = n_0\beta_0$. With increasing γ, the function $f(\gamma, \omega\tau_B)$ increases monotonically for all values of $\omega\tau_B$ from $f=0$ at $\gamma=0$ to $f \simeq 1$ ($\mu_0 H M_\perp \simeq \beta\omega$) at $\gamma/2\omega\tau_B \gg 1$ (see Fig. 6 in Ref. 23).

It is assumed that the stationary solution (47) will be stable only when the difference

$$\Delta S = S(0) - S(r_0) = \mu_0 H M_\perp r_0^2/4D_\perp^S \tag{49}$$

is sufficiently small. It can be seen directly from (49) that ΔS depends strongly on the radius r_0 bounding the cylindrical surface. In accordance with (48), the limiting value of ΔS in strong magnetic fields ($\gamma \gg 2\omega\tau_B$) is $\beta\omega r_0/4D_\perp^S$ and, therefore, depends on the particle density, since $\beta = n_0\beta_0$.

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