Excitations in small-scale Abrikosov–Josephson vortex structures

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Exact analytical expressions for complete sets of eigenfunctions and eigenvalues describing collective excitations in a Josephson junction are obtained for the first time for stationary small-scale Abrikosov–Josephson vortex structures with a characteristic scale of spatial nonuniformity shorter than the London length. The general relations obtained are applied to the case of collective excitations in an annular junction. The influence of a weak current on a small-scale vortex structure with a mean magnetic field is considered. It is established that a small-scale periodic structure of Abrikosov–Josephson vortices can have significantly greater resistance and similar large-scale structures consisting of Josephson vortices. © *1995 American Institute of Physics.*

1. INTRODUCTION

Since the first paper by Lebwohl and Stephen,[1] collective excitations in a multisoliton chain of vortices in a Josephson junction have continued to attract the attention of investigators. The approach described in Ref. 1 was extended and applied to the case of a Josephson junction of annular geometry as well in the work recently published in Refs. 2 and 3. We note that the interest recently displayed in annular Josephson junctions is no accident. The use of this geometry makes it possible to eliminate the boundary effects appearing on the ends of a Josephson junction and to thereby simplify the comparison of theoretical and experimental results. Here it should also be pointed out that the dynamics of collective excitations were considered in Ref. 2 under conditions such that which the length of the Josephson junction, which corresponds to the period of the states in an annular Josephson structure, is smaller than the Josephson length. In our opinion, under such conditions it becomes possible to use nonlocal Josephson electrodynamics, within which, in particular, exact nonlinear solutions were obtained for a static multifluxon chain of small-scale vortices in Ref. 4, where a detailed discussion of the conditions under which the ordinary description based on the sine–Gordon equation is incorrect was also given.

The material presented below can be regarded as an extension of the approach in Refs. 1–3 under new conditions not previously discussed, such that the characteristic scale of the spatial variation of the structures investigated is not only smaller than the Josephson length λ_j, but also smaller than the London length λ. The appearance of such vortex structures can be associated, for example, with the presence of a magnetic field averaged along the junction $\langle H\rangle$ which approaches the lower critical field in magnitude, but remains smaller than it.[5] In accordance with Refs. 4 and 6, the unusual condition $\lambda>\lambda_j$, which is characteristic of nonlocal Josephson electrodynamics, can be realized at large critical currents, at which the critical current density j_c satisfies the relation

$$j_c>j_0\equiv\frac{\hbar c^2}{16\pi|e|\lambda^3}. \tag{1}$$

As was asserted in Ref. 4, this can occur either for thin junctions of thickness $2d$, for which the relation

$$\kappa>d/l_{fp} \tag{2}$$

(κ is the Ginzburg–Landau parameter, and l_{fp} is the mean free path) holds, or for superconductors with a large value for the Ginzburg–Landau parameter, $\kappa\gg 1$ (for further details, see Refs. 4–7).

Unlike large-scale structures, whose investigation has been traditionally associated with the sine–Gordon equation,[8] the electrodynamics of small-scale Josephson structures is nonlocal[4–7] and is described by the following integrodifferential equation:

$$\sin\varphi+\frac{\beta}{\omega_j^2}\frac{\partial\varphi}{\partial t}+\frac{1}{\omega_j^2}\frac{\partial^2\varphi}{\partial t^2}=\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{\partial\varphi(z',t)}{\partial z'}, \tag{3}$$

where φ is the phase difference between the Cooper pairs on the two sides of the junction, ω_j is the Josephson frequency, β characterizes the dissipative properties of the junction, and the parameter l is expressed in terms of the thickness of the junction $2d$, the Josephson length λ_j, and the London lengths λ_+ and λ_- of the superconductors on opposite sides of the tunnel junction in the following manner:

$$l=\frac{\lambda_j^2(\lambda_++\lambda_-+2d)}{(\lambda_+^2+\lambda_-^2)}.$$

The integral in Eq. (3) has the sense of the Cauchy principal value. A more detailed discussion of the conditions under which the use of Eq. (3) becomes necessary was given in Refs. 4 and 5.

A stationary solution of Eq. (3) having the form of a 2π kink,

$$\varphi_0(z)=\pi+2\ \arctan(z/l), \tag{4}$$

and corresponding to a solitary vortex was obtained in Ref. 6. Other solutions describing periodic chains of vortices were subsequently obtained in Ref. 4. The solution with a nonzero mean magnetic field along such a chain has the form

$$\varphi_0(z)=\pi+2\ \arctan\left[\left(\sqrt{\frac{L^2}{l^2}+1}+\frac{L}{l}\right)\tan\frac{z}{2L}\right]. \tag{5}$$

Here the parameter L is related to the magnetic field averaged along the tunnel junction $\langle H\rangle$ by the expression

$$\langle H\rangle=\frac{\Phi_0}{2\pi L(\lambda_++\lambda_-+2d)}, \tag{6}$$

where $\Phi_0=\pi\hbar c/|e|=2.05\times10^{-7}$ Oe/cm^2 is the magnetic flux quantum. The solution with a zero mean field has the form

$$\varphi_0(z)=\pi+2\ \arctan\left(\sqrt{\frac{L^2}{l^2}-1}\ \sin\frac{z}{L}\right), \tag{7}$$

where $L>l$. The parameter L in Eqs. (5) and (7) characterizes the periodicity of the vortex chain. Vortex structures corresponding to (4), (5), and (7) differ significantly in form from ordinary large-scale Josephson structures described by the sine–Gordon equation. In particular, the vortex magnetic field corresponding to the solutions (4), (5), and (7) is similar in form to Abrikosov vortices and is in no way similar to ordinary Josephson vortices.[4,6] On the other hand, the three qualitatively different types of structures described by (4), (5), and (7) correspond to three fundamentally different types of solutions of the equation of a nonlinear magnet, to which the sine–Gordon equation reduces in the stationary case.

In view of both the qualitative similarity and quantitative differences between the new stationary small-scale vortex structures and ordinary Josephson vortices, it would be natural to call the new structures "Abrikosov–Josephson vortices" (compare Ref. 6). We note that Eqs. (5) and (7), which describe such vortices and utilize trigonometric functions, are considerably more transparent than the traditional formulas that describe ordinary chains of Josephson vortices, which are expressed in terms of elliptic functions.

We now enumerate some known results pertaining to the spectra of weak excitations in the theory of small-scale Josephson structures. First, in Ref. 5 the following asymptotic dependence of the characteristic frequencies of excitations on the mode number $n>0$ and the mean field $\langle H\rangle$ was established for periodic excitations with a period equal to the period of the main structure:

$$\omega_n\propto\sqrt{\langle H\rangle n}. \tag{8}$$

The relation (8) was derived without reference to the stationary states (4), (5), and (7), but under the assumption of a sufficiently strong magnetic field $\langle H\rangle$. This dependence differs qualitatively from the ordinary limit of local Josephson electrodynamics $\omega_n\propto\langle H\rangle n$. The stability of structures (4), (5), and (7) toward weak perturbations was subsequently investigated in Ref. 4, and it was shown that solution (7) is unstable. Finally, the eigenfunctions and corresponding eigenvalues were written out explicitly in Ref. 9, but they correspond only to periodic excitations of stationary structures (4), (5), and (7) which have the same period as the stationary structure itself.

In this paper we present results pertaining to the complete analytical solution of the problem of the spectrum of collective excitations of the stationary small-scale Abrikosov–Josephson vortex structures defined by Eqs. (4), (5), and (7). When such excitations are investigated, the main concern is to solve the following eigenvalue problem:

$$\mathscr{L}\psi\equiv\psi\cos\varphi_0(z)-\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{d\psi(z')}{dz'}=\varepsilon\psi. \tag{9}$$

In Sec. 2 we describe the complete analytical solution of this problem in a class of restricted functions for the three different types of stationary solutions $\varphi_0(z)$ defined by Eqs. (4), (5), and (7). In Sec. 3 we present some consequences which follow from the results of Sec. 2 and pertain to collective excitations in a distributed Josephson junction of annular geometry. Finally, in Sec. 4 we show how the complete set of eigenfunctions and eigenvalues found for the excited states makes it possible to describe the effect of a weak electric current on multisoliton states.

2. EIGENFUNCTIONS AND EIGENVALUES FOR THE PROBLEM OF COLLECTIVE EXCITATIONS IN AN INFINITE JOSEPHSON JUNCTION WITH ABRIKOSOV–JOSEPHSON VORTEX STRUCTURES

We first consider the spectral problem (9) in the case in which the solution of Eq. (3) $\varphi_0(z)$ is given by Eq. (5). In this case Eq. (9) takes the form

$$\left\{\sqrt{1+\frac{l^2}{L^2}}-\frac{(l/L)^2}{\sqrt{1+(l/L)^2}-\cos(z/L)}\right\}\psi(z)$$
$$-\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{d\psi(z')}{dz'}=\varepsilon\psi(z). \tag{10}$$

We present the solution of this spectral problem in a class of restricted functions which exhibits splitting of the spectrum of eigenvalues into two bands.

1) **Lower band.** This band corresponds to the portion of the continuous spectrum

$$0<\varepsilon<\frac{1}{2}\left[\sqrt{1+\left(\frac{l}{L}\right)^2}-1\right],$$

in which the eigenfunctions and the corresponding eigenvalues have the following explicit expressions:

$$\psi_q^-(z)=\left\{-\frac{1/2}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]+i\cos(z/2L)}\right.$$
$$\left.+\frac{lq+\sqrt{l^2q^2+1/4}}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]-i\cos(z/2L)}\right\}$$
$$\times\exp(iqz),$$

$$\varepsilon_q^- = \frac{1}{2}\sqrt{1+\frac{l^2}{L^2}} - \sqrt{l^2q^2+\frac{1}{4}}. \tag{11}$$

Here the variable q is a parameter, which labels the solutions within the band and varies within the band in the range $0 \leqslant q \leqslant 1/2L$.

2) **Upper band.** This band corresponds to the portion of the continuous spectrum

$$\frac{1}{2}\left[\sqrt{1+\left(\frac{l}{L}\right)^2}+1\right] \leqslant \varepsilon < \infty$$

and consists of two subbands a and b. The explicit form of the solutions in this band is specified by the following equations:

a) $$\psi_q^+(z) = \left\{ -\frac{1/2}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]+i\cos(z/2L)} + \frac{lq-\sqrt{l^2q^2+1/4}}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]-i\cos(z/2L)} \right\} \times \exp(iqz),$$

$$\varepsilon_q^+ = \frac{1}{2}\sqrt{1+\frac{l^2}{L^2}} + \sqrt{l^2q^2+\frac{1}{4}}, \quad 0 \leqslant q \leqslant \frac{1}{2L}, \tag{12}$$

b) $$\psi_q(z) = \frac{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]-i\cos(z/2L)}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]+i\cos(z/2L)} \times \exp\left[i\left(q-\frac{1}{2L}\right)z\right],$$

$$\varepsilon_q = \sqrt{1+\frac{l^2}{L^2}} + l\left(q-\frac{1}{2L}\right), \quad q \geqslant \frac{1}{2L}. \tag{13}$$

Thus, within the upper band the parameter q runs through values from zero to infinity, but in the ranges $0 \leqslant 1 < 1/2L$ and $q \geqslant 1/2L$ the eigenfunctions are described by different expressions. At the connecting point $q = 1/2L$, the following relation holds:

$$\psi_{1/2L}^+(z) = \frac{i}{4}\left[\sqrt{1+\frac{l^2}{L^2}} - 1 - \frac{l}{L}\right] \times \left\{\left[1+\sqrt{1+\frac{L^2}{l^2}}\right] \times \psi_{1/2L}(z) + \left[1-\sqrt{1+\frac{L^2}{l^2}}\right]\bar{\psi}_{1/2L}(z)\right\},$$

where $\bar{\psi}$ denotes the complex conjugate of ψ.

It would be convenient in some of the discussion which follows to separate the continuous spectrum into bands so that within each band the parameter labelling the solutions would vary in the range from zero to $1/2L$ in analogy to the lower band. Such a description is customary in Floquet's theorem (see Ref. 10). Under such a division, the lower band remains as a single band (the "minus" band), while the upper band splits into an infinite number of bands, the first of which is subband a (the "plus" band). Subband b now consists of a series of bands, each of which can be assigned a number n, where $n = 1, 2, \ldots$. Introducing the single parameter μ ($0 \leqslant \mu \leqslant 1/2L$), which labels the solutions within each band, we write the final form of the solution in such a representation:

1) **the "minus" band**, $\varepsilon = \varepsilon_\mu^- \in [0; (\sqrt{1+(l/L)^2}-1)/2]$,

$$\psi_\mu^-(z) = \left\{ -\frac{1/2}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]+i\cos(z/2L)} + \frac{l\mu+\sqrt{l^2\mu^2+1/4}}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]-i\cos(z/2L)} \right\} \times \exp(i\mu z),$$

$$\varepsilon_\mu^- = \frac{1}{2}\sqrt{1+\frac{l^2}{L^2}} - \sqrt{l^2\mu^2+\frac{1}{4}}, \quad 0 \leqslant \mu \leqslant \frac{1}{2L}; \tag{14}$$

2) **the "plus" band**, $\varepsilon = \varepsilon_\mu^+ \in [(\sqrt{1+(l/L)^2}+1)/2; \sqrt{1+(l/L)^2}]$,

$$\psi_\mu^+(z) = \left\{ -\frac{1/2}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]+i\cos(z/2L)} + \frac{l\mu+\sqrt{l^2\mu^2+1/4}}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]-i\cos(z/2L)} \right\} \times \exp(i\mu z),$$

$$\varepsilon_\mu^+ = \frac{1}{2}\sqrt{1+\frac{l^2}{L^2}} + \sqrt{l^2\mu^2+\frac{1}{4}}, \quad 0 \leqslant \mu \leqslant \frac{1}{2L}; \tag{15}$$

3) **the n bands**, $n = 1, 2, \ldots$, $\varepsilon = \varepsilon_\mu^n \in [\sqrt{1+(l/L)^2} + (n-1)/2L; \sqrt{1+(l/L)^2}+n/2L)$,

$$\psi_\mu^n(z) = \frac{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]-i\cos(z/2L)}{\sin(z/2L)[\sqrt{(L/l)^2+1}+L/l]+i\cos(z/2L)} \times \exp\left[i\left(\frac{n-1}{2L}+\mu\right)z\right],$$

$$\varepsilon_\mu^n = \sqrt{1+\frac{l^2}{L^2}} + \frac{l(n-1)}{2L} + l\mu,$$

$$0 \leqslant \mu \leqslant \frac{1}{2L}, \quad n = 1, 2, \ldots . \tag{16}$$

We shall use both Eqs. (11)–(13) and Eqs. (14)–(16) to describe the continuous spectrum below. Figures 1 and 2 show the dependence of the eigenvalues ε on the parameter q, which plays the role of the quasimomentum. Figure 1 is similar to Figs. 1, 2, and 3 in Ref. 1 and is an illustration of Eqs. (11)–(13). We note that when $q > 1/2L$, the corresponding dependence is represented by a straight line, which sharply distinguishes the case under consideration from the case of local electrodynamics (see Ref. 4). In Fig. 2 the same dependence is reduced to one Brillouin zone [Eqs. (14)–(16)], in which the quasimomentum varies in the range $0 \leqslant \mu \equiv q \leqslant 1/2L$. Such an interpretation underlines the similarity of the results obtained to the known results of Floquet's theorem (see Refs. 10 and 11).

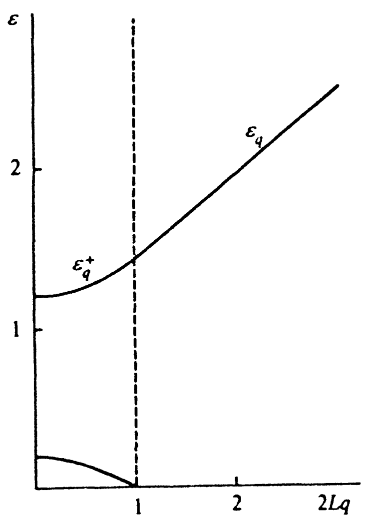

FIG. 1. Dependence of the eigenvalue ε on the quasimomentum q.

To test Eqs. (11)–(13) it is convenient to use the relations

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\left[\frac{e^{i\mu z}}{\sin(\Omega z+i\alpha/2)}\right]$$

$$=\frac{i\, e^{i\mu z}}{\sin(\Omega z+i\alpha/2)},\quad -\Omega<\mu<\Omega,\quad \mathrm{Re}\,\alpha>0, \tag{17}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\left[\frac{\sin(\Omega z-i\alpha/2)}{\sin(\Omega z+i\alpha/2)}\, e^{i\mu z}\right]$$

$$=i\,\frac{\sin(\Omega z-i\alpha/2)}{\sin(\Omega z+i\alpha/2)}\, e^{i\mu z},\quad \mu\geqslant 0,\quad \mathrm{Re}\,\alpha>0, \tag{18}$$

where

$$\Omega=1/(2L),\quad \sinh\alpha=l/L. \tag{19}$$

It would be useful to give the following commentaries on the solutions presented.

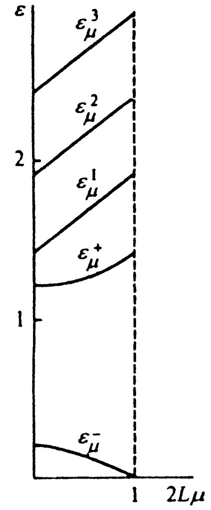

FIG. 2. Dependence of ε on q reduced to one Brillouin zone, $0\leqslant\mu\equiv q\leqslant 1/2L$.

a) Although the original equation is an integrodifferential equation, the band structure of the solutions presented corresponds to the classical results regarding the structure of the restricted solutions of differential equations with periodic coefficients,[10] which are known, in particular, in the theory of parametric resonance[11] and the Bloch theory of electrons in a crystal lattice.[12] Within each band the solutions have the Floquet form (see Ref. 10), and they are labelled by the parameter μ, which runs through all the values within the interval $[0,1/2L)$ and has the same meaning as the analogous parameter in Floquet's theorem.

b) The band structure of the solutions presented is similar to the band structure of Lamé's equation,[1,13] which appears when small perturbations of periodic structures are analyzed in the local electrodynamics of a Josephson junction based on the sine–Gordon equation. There are also two branches in the spectrum of collective excitations separated from one another by a gap in ordinary Josephson electrodynamics, the lower branch corresponding to acoustical modes, the upper branch describing plasma Josephson modes, and the size of the gap being determined by the Josephson frequency. One significant distinguishing feature of the case under consideration is the fact that the formulation of the nonlocal problem includes a class of exponentially increasing eigenfunctions, which are allowed by the Lamé equation. In addition, the eigenfunctions of the Lamé equation cannot be described in such a remarkably simple form as in our case, and they have representations in the form of infinite series[13] or special functions.[1]

c) It is clear that the complex conjugates $\overline{\psi_\mu^+}$, $\overline{\psi_\mu^-}$, and $\overline{\psi_\mu^n}$, $n=1,2,\ldots$ are also solutions of the problem and correspond to the same eigenvalues as ψ_μ^+, ψ_μ^-, and ψ_μ^n, $n=1,2,\ldots$. Thus, each eigenvalue in the spectrum is at least doubly degenerate provided the corresponding eigenfunction and its complex conjugate are linearly independent. The latter is true for all the points in the spectrum except for the three boundaries between bands $\varepsilon=\varepsilon_0^-$, $\varepsilon_{1/2L}^-$, and ε_0^+.

d) The eigenvalues $\varepsilon=\varepsilon_0^-$, $\varepsilon_{1/2L}^-$, ε_0^+, and ε_0^n, $n=1,2,\ldots$, which mark boundaries between bands, correspond to periodic eigenfunctions with a period $4\pi L$, which are equal to the doubled period of the solution $\varphi_0(z)$. According to Ref. 9, they exhaust all the solutions of the problem in this class.

Next, the following expressions, which relate the system of eigenfunctions

$$\{\psi_\mu^+,\overline{\psi_\mu^+},\psi_\mu^-,\overline{\psi_\mu^-};\psi_\mu^n,\overline{\psi_\mu^n},n=1,2,\ldots\} \tag{20}$$

and the complete system of functions $\{e^{i\eta z},\eta\in\mathbf{R}\}$, are valid:

$$e^{i[(2p+1)\Omega+\mu]z}=e^{-\alpha}\psi_\mu^{2p+2}-2\sinh\alpha\sum_{k=1}^{p}e^{-k\alpha}\psi_\mu^{2p-2k+2}$$

$$-i(e^{-p\alpha}+e^{-(p+1)\alpha})\left\{\left(1-\frac{l\mu}{\sqrt{l^2\mu^2+1/4}}\right)\right.$$

$$\left.\times\psi_\mu^-+\left(1+\frac{l\mu}{\sqrt{l^2\mu^2+1/4}}\right)\psi_\mu^+\right\},$$

$$p=0,1,2,\ldots,\quad 0\leqslant\mu<\Omega; \tag{21}$$

$$e^{i(2p\Omega+\mu)z}=e^{-\alpha}\psi_\mu^{2p+1}-2\sinh\alpha\sum_{k=1}^{p}e^{-k\alpha}\psi_\mu^{2p-2k+1}$$

$$+\frac{i(e^{-p\alpha}+e^{-(p+1)\alpha})}{2\sqrt{l^2(\Omega-\mu)^2+1/4}}\{\overline{\psi_{\Omega-\mu}^-}-\overline{\psi_{\Omega-\mu}^+}\},$$

$$p=0,1,2,\ldots,\quad 0<\mu<\Omega; \tag{22}$$

$$e^{2ip\Omega z}=e^{-\alpha}\psi_0^{2p+1}-2\sinh\alpha\sum_{k=1}^{p}e^{-k\alpha}\psi_0^{2p-2k+1}$$

$$+\frac{i(e^{-p\alpha}+e^{-(p+1)\alpha})}{\cosh\alpha}\overline{\psi_\Omega^-}-\frac{e^{-(p+2)\alpha}}{2\cosh\alpha}\psi_0^1$$

$$+\frac{e^{-p\alpha}}{2\cosh\alpha}\overline{\psi_0^1},\quad p=0,1,2,\ldots . \tag{23}$$

Here Ω and α are defined by Eqs. (19), and the sums are assumed to be equal to zero, if the lower boundary of the summation surpasses the upper. It follows from the relations (21)–(23) that the system of eigenfunctions (20) is also complete. This, in turn, provides some indication that Eqs. (14)–(16) describe all the solutions of the problem (9) with $\varphi_0(z)$ defined by (5). We note that the explicit form of the solution presented leads to the marginal stability of the stationary periodic structure of the Abrikosov–Josephson vortices described by Eq. (5).

We now present the results pertaining to the problem of the spectrum of weak perturbations of a vortex chain described by Eq. (7). In this case the eigenvalue problem (9) takes the form

$$\left[1-\frac{2l^2}{l^2+(L^2-l^2)\sin^2(z/L)}\right]\psi(z)$$

$$-\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{d\psi(z')}{dz'}=\varepsilon\psi(z). \tag{24}$$

Using (17) and (18), we can show that the eigenvalue spectrum of the problem (24) has the following band structure, which is qualitatively similar to the band structure of the problem (10).

1) The **lower band**, which corresponds to the portion of the continuous spectrum

$$-l^2/L^2\leqslant\varepsilon\leqslant 0,$$

is represented by the following eigenfunctions and eigenvalues:

$$\psi_q^-(z)=\left\{-\frac{L^2-l^2}{L\sin(z/L)+il\cos(z/L)}\right.$$

$$\left.+\frac{2L^2lq+\sqrt{4L^4l^2q^2+(L^2-l^2)^2}}{L\sin(z/L)-il\cos(z/L)}\right\}\exp(iqz),$$

$$\varepsilon_q^-=\frac{1}{2}\left(1-\frac{l^2}{L^2}\right)-\sqrt{l^2q^2+\frac{1}{4}\left(1-\frac{l^2}{L^2}\right)^2}. \tag{25}$$

Here the parameter q varies within the band in the range $0\leqslant q\leqslant 1/L$.

2) The **upper band** corresponds to the portion of the continuous spectrum

$$1-l^2/L^2\leqslant\varepsilon\leqslant\infty.$$

In analogy to the problem (10) this band consists of two subbands:

a) $$\psi_q^+(z)=\left\{-\frac{L^2-l^2}{L\sin(z/L)+il\cos(z/L)}\right.$$

$$\left.+\frac{2L^2lq-\sqrt{4L^4l^2q^2+(L^2-l^2)^2}}{L\sin(z/L)-il\cos(z/L)}\right\}\exp(iqz),$$

$$\varepsilon_q^+=\frac{1}{2}\left(1-\frac{l^2}{L^2}\right)+\sqrt{l^2q^2+\frac{1}{4}\left(1-\frac{l^2}{L^2}\right)},\quad 0\leqslant q<\frac{1}{L}, \tag{26}$$

b) $$\psi_q(z)=\frac{L\sin(z/L)-il\cos(z/L)}{L\sin(z/L)+il\cos(z/L)}\exp\left[i\left(q-\frac{1}{L}\right)z\right];$$

$$\varepsilon_q=1+l(q-1)/L),\quad q\geqslant 1/L. \tag{27}$$

It is clear that the complex conjugates $\overline{\psi_q^-}$, $\overline{\psi_q^+}$, and $\overline{\psi_q}$ are also solutions of the problem and correspond to the same eigenvalues as ψ_q^-, ψ_q^+, and ψ_q. Thus, the band structure of the spectrum of weak excitations for the state (7) is qualitatively similar to the structure of the spectrum for the state (5). At the same time, the presence of negative eigenvalues in the spectrum (25)–(27) points out the instability of the state (7). The completeness of the system of eigenfunctions (25)–(27) is proved in complete analogy to the case of the system (11)–(13).

To conclude this section we point out that to obtain the set of eigenfunctions and eigenvalues of Eq. (9) when $\varphi_0(z)$ corresponds to the 2π kink (4), it is sufficient to note that the expression (5) transforms into (4) upon the limiting transition $L\to\infty$. Passing to the limit $L\to\infty$ in Eqs. (11)–(13), we have the set of eigenfunctions and eigenvalues of Eq. (9) for the case of a 2π kink:

$$\psi(z)=\frac{1}{z^2+l^2},\quad \varepsilon=0, \tag{28}$$

$$\psi_q(z)=\frac{z-il}{z+il}e^{iqz},\quad \varepsilon_q=1+ql,\quad q\geqslant 0. \tag{29}$$

It follows from the equations obtained that the solution of the 2π-kink type is marginally stable. We note that the solutions (28) and (29) were previously presented in Ref. 9.

3. COLLECTIVE EXCITATIONS IN AN ANNULAR DISTRIBUTED JOSEPHSON JUNCTION

Let us consider a distributed Josephson junction having the form of a thin ring of radius R. We assume that R is quite large (compare Refs. 2 and 3), that $\lambda_\pm>\lambda_j$, and that the approximation (3) can be used to describe the distributions of the phase jump of $\varphi(z)$ in the junction. It is clear that in order for either of the periodic structures (5) and (7) to be realized in such a junction, the period of the structure must be an integral number m times smaller than the length of the junction $2\pi R$. We focus on the case of the stable chain (5), which has the smallest period $2\pi L$. In an annular junction this formation can appear, if $R=mL$, and at large values of L such a structure has the form of m single Josephson vortices evenly distributed over the entire circumference of the junction. It follows from (6) that such a state can appear only for selected values of the mean field equal to

$$\langle H_m\rangle=\frac{m\Phi_0}{2\pi R(\lambda_++\lambda_-+2d)}. \tag{30}$$

This does not mean that such stationary formations cannot correspond to other values of $\langle H\rangle \neq \langle H_m\rangle$, since in the present case we have been dealing only with structures defined by (5).

To investigate the perturbations of the structures indicated, the solutions which "fit" into the total circumference of the junction an integral number of times must be selected from the complete set of all the restricted solutions of the problem (10). In other words, these are the solutions whose smallest period equals $2\pi R/m$, where m is an integer. This problem was solved for $m=1,2$ in Ref. 9. This can easily be done for arbitrary m using the explicit expressions (11)–(13). Here the results will differ, depending on whether m is even or odd. A single description for the cases of even and odd m can be proposed by the traditional method using the solutions of the problem (10) in the doubled period $4\pi R$. To obtain a complete system of such solutions, L must be set equal to R/m in Eqs. (11)–(13), and the values of q must be assumed to be equal to $q=k/2R$, $k=0,1,\dots,m$ in Eqs. (11) and (12) and $q=(k+m)/2R$, $k=0,1,\dots$ in Eq. (13), respectively. Now the explicit forms of the eigenfunctions and eigenvalues are given by the following expressions:

$$1)\ \psi_k^-(z,m)=\left\{-\frac{1/2}{\sin(mz/2R)[\sqrt{(R/lm)^2+1}+R/lm]+i\cos(mz/2R)}+\frac{(lk/2R)+\sqrt{(lk/2R)^2+1/4}}{\sin(mz/2R)[\sqrt{(R/lm)^2+1}+R/lm]-i\cos(mz/2R)}\right\}\exp\left(\frac{ikz}{2R}\right),$$

$$\varepsilon_k^-(m)=\frac{1}{2}\left[\sqrt{1+\left(\frac{lm}{R}\right)^2}-\sqrt{1+\left(\frac{lk}{R}\right)^2}\right],\quad k=m,m-1,\dots,0; \tag{31}$$

$$2)\ \psi_k^+(z,m)=\left\{-\frac{1/2}{\sin(mz/2R)[\sqrt{(R/lm)^2+1}+R/lm]+i\cos(mz/2R)}+\frac{(lk/2R)-\sqrt{(lk/2R)^2+1/4}}{\sin(mz/2R)[\sqrt{(R/lm)^2+1}+R/lm]-i\cos(mz/2R)}\right\}\exp\left(\frac{ikz}{2R}\right),$$

$$\varepsilon_k^+(m)=\frac{1}{2}\left[\sqrt{1+\left(\frac{lm}{R}\right)^2}+\sqrt{1+\left(\frac{lk}{R}\right)^2}\right],\quad k=0,1,\dots,m-1; \tag{32}$$

$$3)\ \psi_k(z,m)=\frac{\sin(mz/2R)[\sqrt{(R/lm)^2+1}+R/lm]-i\cos(mz/2R)}{\sin(mz/2R)[\sqrt{(R/lm)^2+1}+R/lm]+i\cos(mz/2R)}\exp\left(\frac{ikz}{2R}\right),$$

$$\varepsilon_k(m)=\sqrt{1+\left(\frac{lm}{R}\right)^2}+\frac{lk}{2R},\quad k=0,1,\dots \tag{33}$$

It follows from (21)–(23) that the system (31)–(33) obtained is a complete system of eigenfunctions in the doubled period $4\pi R$. In order to isolate the eigenfunctions which are periodic with a period $2\pi R$ from the system (31)–(33), it must be assumed in Eqs. (31) and (32) that k and m have the same parity and only even values of k must be taken in Eq. (33). Thus, the complete system of eigenfunctions which are periodic with a period $2\pi R$ for $m=2p$ (even m) will be

$$\{\psi_{2s}^-(z,m),\ \overline{\psi_{2s}^-(z,m)},\quad s=p,p-1,\dots,0;$$
$$\psi_{2s}^+(z,m),\ \overline{\psi_{2s}^+(z,m)},\quad s=0,1,\dots,p-1;$$
$$\psi_{2s}(z,m),\ \overline{\psi_{2s}(z,m)},\quad s=0,1,\dots\}, \tag{34}$$

while the analogous system for $m=2p+1$ (odd m) is

$$\{\psi_{2s+1}^-(z,m),\ \overline{\psi_{2s+1}^-(z,m)},\quad s=p,p-1,\dots,0;$$
$$\psi_{2s+1}^+(z,m),\ \overline{\psi_{2s+1}^+(z,m)},\quad s=0,1,\dots,p-1;$$
$$\psi_{2s}(z,m),\ \overline{\psi_{2s}(z,m)},\quad s=0,1,\dots\}. \tag{35}$$

The system of functions (31)–(33) makes it possible to solve the problem of describing the weakly excited states $\varphi(z)$ near the stable equilibrium state (5) in an annular Josephson junction:

$$\varphi(z)=\varphi_0(z)+\delta\psi(z)e^{-i\omega z}. \tag{36}$$

Here δ is some small amplitude, and $\psi(z)$ is represented by any of the functions $\psi_k^-(z,m)$, $\psi_k^+(z,m)$, and $\psi_k(z,m)$ which are periodic with a period $2\pi R$. The characteristic frequency ω of such collective excitations is expressed in terms of the corresponding eigenvalue $\varepsilon=\varepsilon_k^-(m)$, $\varepsilon_k^+(m)$, or $\varepsilon_k(m)$ by means of the formula

$$\omega=\omega_j\sqrt{\varepsilon-(\beta/2\omega_j)^2}-i(\beta/2). \tag{37}$$

We note that in the asymptotic limit of large eigenvalues ε the dependence (37) coincides with the dependence (8) previously obtained.

4. INFLUENCE OF A WEAK CURRENT

In this section we consider the influence of a weak current j on the vortex structure (5). An analysis of the corresponding problem within traditional Josephson electrodynamics can be found in Refs. 3, 14, and 15, where it was shown, in particular, that a motionless chain of Josephson fluxons begins to move slowly as a single unit under the action of a weak current and that the current–voltage characteristic is then linear. The influence of a current on the nonlocal structure (5) of Abrikosov–Josephson vortices can be described by the following equation:[16,17]

$$\sin\varphi+\frac{\beta}{\omega_j^2}\frac{\partial\varphi}{\partial t}+\frac{1}{\omega_j^2}\frac{\partial^2\varphi}{\partial t^2}=\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{dz'}{z'-z}\frac{\partial\varphi(z',t)}{\partial z'}+I(t). \tag{38}$$

We take a dimensionless current density $I(t)=j(t)/j_c$, which depends only on time. Setting $I(t)$ small, we seek a solution of Eq. (38) in the form of an asymptotic expansion in this variable. In such an expansion the secular terms must be eliminated. We eliminate these secular terms by assuming, in analogy to the local case, that the perturbed state moves as a single unit according to a certain law $z_0(t)$, which is determined by the current $I(t)$. In a first approximation with respect to the small current, the solution of Eq. (38) can be written in the form

$$\varphi(\xi,t)=\varphi_0(\xi)+\varphi_1(\xi,t)+\ldots,\quad \xi=z-z_0(t).$$

Here φ_0 is described by Eq. (5), and φ_1 obeys the equation

$$\varphi_1\cos\varphi_0(\xi)+\frac{\beta}{\omega_j^2}\frac{\partial\varphi_1}{\partial t}+\frac{1}{\omega_j^2}\frac{\partial^2\varphi_1}{\partial t^2}-\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\times\frac{\partial\varphi_1(\xi',t)}{\partial\xi'}=\frac{1}{\omega_j^2}\frac{d\varphi_0(\xi)}{d\xi}\left(\beta\frac{dz_0}{dt}+\frac{d^2z_0}{dt^2}\right)+I(t). \tag{39}$$

Considering the structure (5) in an annular Josephson junction, we use an expansion of Eq. (39) in the complete system of eigenfunctions considered in the previous section which are periodic with a period $2\pi R$. The subsequent arguments are simplified when the following notations are introduced:

$$\chi_m^-(\xi)=\frac{2R^2}{ml(ml-R+\sqrt{R^2+l^2m^2})}\overline{\psi_m^-(\xi,m)}=\frac{R}{\sqrt{R^2+m^2l^2}-R\cos(m\xi/R)}, \tag{40}$$

$$\chi_0(\xi)=\frac{1}{2}[\psi_0(\xi,m)+\overline{\psi_0(\xi,m)}]=\frac{R-\sqrt{R^2+m^2l^2}\cos(m\xi/R)}{\sqrt{R^2+m^2l^2}-R\cos(m\xi/R)}. \tag{41}$$

These two functions are eigenfunctions of the spectral problem (10) and correspond to the eigenvalues $\varepsilon_m^-(m)=0$ and $\varepsilon_0(m)=\sqrt{1+(ml/R)^2}$. The use of the functions (40) and (41) is beneficial, since the following relations hold:

$$1=\frac{l^2m^2\chi_m^-(\xi)+R^2\chi_0(\xi)}{R\sqrt{R^2+l^2m^2}},\quad \frac{d\varphi_0}{d\xi}=\frac{m^2l}{R^2}\chi_m^-(\xi).$$

This makes it possible to represent the solution of Eq. (39) in the form

$$\varphi_1(\xi,t)=A_m^-(t)\chi_m^-(\xi)+A_0(t)\chi_0(\xi). \tag{42}$$

For the amplitudes A_m^- and A_0 we now obtain the equations

$$\frac{d^2}{dt^2}A_m^-(t)+\beta\frac{d}{dt}A_m^-(t)=0, \tag{43}$$

$$\frac{d^2}{dt^2}A_0(t)+\beta\frac{d}{dt}A_0(t)+\omega_j^2\sqrt{1+\frac{m^2l^2}{R^2}}A_0(t)=\frac{R\omega_j^2I(t)}{\sqrt{R^2+l^2m^2}}, \tag{44}$$

and the condition for elimination of the secular terms has the form

$$\frac{d^2z_0}{dt^2}+\beta\frac{dz_0}{dt}=-\frac{Rl\omega_j^2}{\sqrt{R^2+m^2l^2}}I(t). \tag{45}$$

We first discuss the consequences of Eq. (45) under the assumption that the initial problem is solved. Then the solution of Eq. (45) can be represented in the form

$$z_0(t)=z_0(0)+\frac{1-\exp(-\beta t)}{\beta}v(0)-\frac{Rl\omega_j^2}{\sqrt{R^2+m^2l^2}}\times\int_0^t dt'\int_0^{t'}dt''\exp[\beta(t''-t')]I(t''), \tag{46}$$

where $z_0(0)$ and $v(0)$ are integration constants. If $I=\text{const}$ holds for $t>0$, Eq. (46) has the following form:

$$z_0(t)=z_0(0)+\frac{1-\exp(-\beta t)}{\beta}v(0)-\frac{Rl\omega_j^2I}{\beta^2\sqrt{R^2+m^2l^2}}\times[\beta t-1+\exp(-\beta t)]. \tag{47}$$

In the limit $\beta t\ll 1$ this relation corresponds to uniformly accelerated motion:

$$z_0(t)=z_0(0)+v(0)t-\frac{Rl\omega_j^2It^2}{2\sqrt{R^2+m^2l^2}}. \tag{48}$$

In the resistive case, in which $\beta\neq 0$, in the limit $t\to\infty$ Eq. (47) corresponds to uniform steady motion:

$$z_0(t)=z_0(0)+\frac{v(0)}{\beta}+\frac{Rl\omega_j^2 I}{\beta^2\sqrt{R^2+m^2l^2}}-vt, \tag{49}$$

where

$$v=\frac{Rl\omega_j^2 I}{\beta\sqrt{R^2+m^2l^2}}. \tag{50}$$

In the other characteristic case, in which we have $I=I_0\cos\omega t$ at $t>0$, Eq. (46) gives

$$z_0(t)=z_0(0)+\frac{1-\exp(-\beta t)}{\beta}v(0)-\frac{Rl\omega_j^2 I_0}{(\omega^2+\beta^2)\sqrt{R^2+m^2l^2}}\times\left[\exp(-\beta t)-\cos\omega t+\frac{\beta}{\omega}\sin\omega t\right]. \tag{51}$$

Hence, in particular, in the nondissipative limit, where $\beta\ll\omega$ and $\beta t\ll 1$, we have

$$z_0(t)=z_0(0)+v(0)t-\frac{2Rl\omega_j^2 I_0\sin^2(\omega t/2)}{\omega^2\sqrt{R^2+m^2l^2}}. \tag{52}$$

The perturbed motion emerging here together with the appearance of a constant velocity is also characterized by oscillations. In the opposite resistive limit, where $\beta\neq 0$ in the limit $t\to\infty$, Eq. (51) gives

$$z_0(t)=z_0(0)+\frac{v(0)}{\beta}-\frac{Rl\omega_j^2 I_0}{(\omega^2+\beta^2)\sqrt{R^2+m^2l^2}}\times\left(1-\cos\omega t+\frac{\beta}{\omega}\sin\omega t\right). \tag{53}$$

Equation (46) and the ensuing equations up to (53) describe the motion of the vortex structure (5) as a whole. The new feature not observed in ordinary Josephson electrodynamics is the dependence of the properties of such motion on the parameters characterizing the Josephson junction in our nonlocal electrodynamics. It should be noted that the uniform velocity (50) corresponds to the motion obtained in the small-current limit from the general relation (7) in Ref. 17, in which an analytical description of a nonlinear periodic structure of small-scale vortices traveling with a constant velocity in the strong-dissipation limit of nonlocal Josephson electrodynamics was given.

Apart from the appearance of the motion of the structure (5) as a whole, the perturbation (42) should be borne in mind. According to Eq. (43), the amplitude A_m^- undergoes only relaxational damping. The consequences of Eq. (44), which has steady-state solutions, are more interesting. For example, in the case of $I=\text{const}$, such a steady-state solution has the form

$$\varphi_1(\xi)=\frac{R^2I[R-\sqrt{R^2+m^2l^2}\cos(m\xi/R)]}{(R^2+m^2l^2)[\sqrt{R^2+m^2l^2}-R\cos(m\xi/R)]}. \tag{54}$$

In another case of a sinusoidally varying current, the steady-state solution of Eq. (44) has the form

$$A_0(t)=\frac{I_0R^2\omega_j^2}{\sqrt{R^2+m^2l^2}}\times\frac{(\sqrt{R^2+m^2l^2}\omega_j^2-R\omega^2)\cos\omega t+R\omega\beta\sin\omega t}{(\sqrt{R^2+m^2l^2}\omega_j^2-R\omega^2)^2+\beta^2\omega^2R^2}. \tag{55}$$

At small values of β this equation describes the resonant excitation of a spatially nonuniform vortex state with the spatial structure (41), which occurs under the action of a spatially uniform current flowing through the periodic vortex structure (5).

The relations obtained make it possible, in particular, to make some definite statements regarding the current–voltage characteristic of an annular junction containing the static vortex structure (5), at least at small currents. Taking into account the equation

$$\left\langle\frac{1}{\sqrt{R^2+m^2l^2}-R\cos(m\xi/R)}\right\rangle=\frac{1}{ml} \tag{56}$$

for averaging around a ring, we can write the following relation:

$$V=\frac{\hbar}{2|e|R}\left[m\frac{dz_0}{dt}-(\sqrt{R^2+m^2l^2}-ml)\frac{dA_0}{dt}\right]. \tag{57}$$

In the special case of $I=\text{const}$, for steady motion we therefore have

$$V=-\frac{\hbar ml\omega_j^2 I}{2|e|\beta\sqrt{R^2+m^2l^2}}=-\frac{ml}{\sqrt{R^2+m^2l^2}}R_sj, \tag{58}$$

where R_s is the ohmic resistance of a unit of area of the tunnel junction. The expression (58) corresponds to the weak-current limit of Eq. (12) in Ref. 17, which was derived in the theory of a resistive small-scale nonlinear traveling structure of Josephson vortices of the Abrikosov type.

The current–voltage characteristic corresponding to a small-scale vortex structure with an oscillating current has not previously been considered. In accordance with (53) and (55), here Eq. (57) gives the following expression for steady motion in the resistive limit $\beta\neq 0$

$$V=\frac{\hbar\omega_j^2 I_0}{2|e|\sqrt{R^2+m^2l^2}}\Bigg[-ml\frac{\omega\sin\omega t+\beta\cos\omega t}{\omega^2+\beta^2}+R\omega(\sqrt{R^2+m^2l^2}-ml)\times\frac{\omega_j^2\sqrt{R^2+m^2l^2}\sin\omega t-R\omega\beta\cos\omega t}{(\omega_j^2\sqrt{R^2+m^2l^2}-\omega^2R)^2+\omega^2\beta^2R^2}\Bigg]. \tag{59}$$

According to Eqs. (52) and (55), in the opposite nondissipative limit $\beta\ll\omega$ under the assumption that $v(0)=0$ we have

$$\frac{dz_0}{dt}=-\frac{Rl\omega_j^2 I_0\sin\omega t}{\omega\sqrt{R^2+m^2l^2}},$$

$$\frac{dA_0}{dt}=-\frac{R^2\omega_j^2\omega I_0\sin\omega t}{\sqrt{R^2+m^2l^2}[\omega_j^2\sqrt{R^2+m^2l^2}-\omega^2R]}.$$

These equations and (57) give

$$V=-\frac{\hbar\omega_j^2 I_0(ml\omega_j^2-R\omega^2)}{2|e|\omega(\omega_j^2\sqrt{R^2+m^2l^2}-\omega^2 R)}\sin\omega t. \qquad (60)$$

The latter expression corresponds to resonance at a frequency $\omega_j[1+(ml/R)^2]^{1/4}$.

Equation (58) can be compared with the result in Refs. 3 and 15, which was obtained in the classical theory of Josephson junctions, in which

$$V=-R_s j\frac{\pi^2}{4K(k)E(k)}, \qquad (61)$$

where k is determined from the equation

$$\pi R=\lambda_j kK(k)m. \qquad (62)$$

Here $K(k)$ and $E(k)$ are complete elliptic integrals, λ_j is the Josephson length, and m is the number of fluxons in a period equal to $2\pi R$.

In the limit $m \gg \pi R/\lambda_j$ the classical theory, like Eq. (58), gives the ordinary Ohm's law. In the opposite limit of a small number of fluxons, it follows from the customary theory that

$$V=-R_s jm\lambda_j/R. \qquad (63)$$

Conversely, Eq. (58) gives the following current–voltage characteristic:

$$V=-R_s jml/R. \qquad (64)$$

Bearing in mind the smallness of λ_j in comparison with l (see Refs. 4 and 6), we can argue that small-scale Josephson structures can have a significantly larger resistance than traditional large-scale vortex structures.

5. CONCLUSIONS

Summing up all the material presented above, we can state that exact (and very simple) analytical expressions have been obtained for the complete systems of eigenfunctions and eigenvalues of the collective excitations in an infinite Josephson junction containing stationary small-scale Abrikosov–Josephson vortices. For example, in the absence of dissipation ($\beta=0$), the frequency of the collective excitations in a junction with one Abrikosov–Josephson fluxon has the form $\omega=\omega_j\sqrt{1+ql}$, $q\geqslant 0$. In the case of a junction filled by a chain of Abrikosov–Josephson fluxons with a mean magnetic field [see Eq. (5)] there are two branches of characteristic frequencies for collective excitations. This is similar to traditional (local) Josephson electrodynamics. The acoustical (lower) branch of collective excitations is characterized by the frequency

$$\omega=\omega_j\sqrt{\frac{1}{2}\sqrt{1+\frac{l^2}{L^2}}-\sqrt{l^2q^2+\frac{1}{4}}},\quad 0\leqslant q\leqslant\frac{1}{2L}. \qquad (65)$$

The frequency of the collective excitations on the plasma (upper) branch of the spectrum have the form

$$\omega=\omega_j\sqrt{\frac{1}{2}\sqrt{1+\frac{l^2}{L^2}}+\sqrt{l^2q^2+\frac{1}{4}}},\quad 0\leqslant q\leqslant\frac{1}{2L}, \qquad (66)$$

$$\omega=\omega_j\sqrt{\sqrt{1+\frac{l^2}{L^2}}+l\left(q-\frac{1}{2L}\right)},\quad q\geqslant\frac{1}{2L}. \qquad (67)$$

In addition, the spectrum of collective excitations corresponding to the multifluxon filling of a tunnel junction with a zero mean magnetic field [see Eq. (7)] was obtained. Such a state is unstable with a growth rate

$$\gamma=\omega_j\sqrt{\sqrt{l^2q^2+\frac{1}{4}\left(1-\frac{l^2}{L^2}\right)^2}-\frac{1}{2}\left(1-\frac{l^2}{L^2}\right)},$$

$$0\leqslant q\leqslant\frac{1}{L}. \qquad (68)$$

The general results obtained were used to find the spectrum of excitations in an annular junction and to describe the influence of a weak electric current on an annular junction containing Abrikosov–Josephson vortices. To solve the former problem, the values of q which correspond to the annular geometry of the structure were identified. The investigation of the latter problem showed that the action of a weak current on a periodic vortex structure results in movement of the entire vortex structure as a whole and in the appearance of spatially nonuniform perturbations in it. The current–voltage characteristics of an annular Josephson junction containing Abrikosov–Josephson fluxons were obtained, and their dependence both on the parameters characterizing the junction and on the number of fluxons in it was investigated. Here we stress the new resonant dependence of the current–voltage characteristic of a Josephson junction described by Eq. (60), which appears at the frequency of a collective excitation in an annular junction containing m Abrikosov–Josephson vortices. In addition, a simple comparison of Eqs. (63) and (64) demonstrates the significant difference between our proposed current–voltage characteristic of a junction with m Abrikosov–Josephson fluxons and the characteristic of a similar junction with m Josephson vortices.

The new predicted spectra of collective excitations confront experimentalists with a challenge no less complicated than the one offered by the publication of Ref. 1 in 1967. We note that there is still no exhaustive experimental confirmation of the results in Ref. 1.

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