Interaction of ultrashort light pulses with thin-film resonant structures

S. M. Zakharov

Institute of High-Performance Computing Systems, Russian Academy of Sciences, 117334 Moscow, Russia
(Submitted 1 June 1995)
Zh. Eksp. Teor. Fiz. **108**, 829–841 (September 1995)

We develop the theory of interaction of coherent light pulses with thin-film planar Fabry–Perot resonator structures filled with media containing resonant atoms. Anomalous propagation effects have been detected for ultrashort pulses. We prove the McCall–Hahn "areas theorem," which plays the role of one of the integrals of the motion of the system. The occurrence of bistable and soliton solutions is discussed. © *1995 American Institute of Physics.*

1. INTRODUCTION

In recent times, the nonlinear optics of thin-film planar resonant structures (PRS) (Fabry–Perot resonators) has attracted the attention of many investigators.[1–7] On the one hand, the study of such structures is of purely practical interest in view of the frequent necessity for efficient control of laser radiation, and this type of setup could provide a modular basis for optoelectronics.[8–12] On the other hand, substantial progress has been achieved recently in understanding the physics of nonlinear dynamics, both in open dissipative systems and exactly integrable Hamiltonian systems.

Planar resonant structures constitute an example of the simplest physical model that admits the possibility to consistently take into account the boundaries when utilizing a resonant medium of finite thickness under the conditions for formation of standing waves. This model also subsumes the problem of nonlinear surface wave propagation. In addition, the nonlinear coupling between the field of the transmitted wave and the optical properties of the resonant medium can lead to such important physical phenomena as optical bistability and multistability, self-pulsation under quasicontinuous light interaction, creation of ultrashort solitons, etc.[13,14] Under certain conditions, the nonlinear dynamics of such systems can display the properties of dynamical chaos.[15,16]

The main difficulty in the theoretical description of the dynamical behavior of planar resonant systems, and with it the feature that distinguishes these systems from, for example, ring resonators, is the need to take into account the interference of counterpropagating waves, and the emergence of high-frequency spatial structures. The latter, generally speaking, makes it impossible to study the problem in the framework of slowly changing amplitudes of the field and polarization of the medium.[1,6,17] Progress in this field was achieved mainly due to the substantial simplifications afforded by the single-mode approximation and uniform field approximation, known also as the mean field approximation.[18] These simplifications are essential to an analytic description of quasistationary solutions, while in the general case, the problem must be solved numerically.[17–19] For that reason, the main interest in the cited literature was in the study of optical bistability and other related phenomena.

On the other hand, the interaction of ultrashort optical pulses (USP) with a PRS that contains nonlinear media of finite extent has received far less attention. As we shall see below, the interaction of the USP with a PRS turns out to be closely related to the transmission of light through a surface layer of resonant atoms whose thickness is less than the wavelength of light.[20–25]

In this paper, we study the nonlinear interaction of USP of light with thin-film Fabry–Perot PRS filled with resonant media of "two-level" atoms, under conditions conducive to standing wave formation. Our main interest to search for possible analytic solutions and appropriate approximations. The theoretical model is a general one that enables one to study the dynamical behavior of resonant systems interacting with external optical fields. For USP of light in which the excitation pulses are much narrower than typical polarization and population-inversion relaxation times of the resonant medium, as well as under conditions with no phase modulation, it is possible to formulate the analog of the McCall–Hahn "areas theorem."[26] The given integral of the motion permits one to establish a correspondence between the "areas" of the incident, transmitted, and reflected light. For certain parameters of the nonlinear interaction, the transmission of light through the PRS under conditions of USP is bistable in nature. In addition, we study regions of stable solutions.

2. FUNDAMENTAL EQUATIONS AND APPROXIMATIONS

We analyze USP in PRS on the basis of the semiclassical approach, which is commonly used to describe the propagation of laser pulses in a resonant medium, both in the coherent and the incoherent cases. As is well known, this approach is justified provided one can ignore the quantum fluctuations of the electromagnetic field.

We represent the field inside the resonator as a superposition of two counterpropagating traveling waves,

$$E(z,t)=\{E_+(z,t)\exp(i\omega t-ikz)+E_-(z,t)\exp(i\omega t+ikz)+\text{c.c.}\}\tag{1}$$

with amplitudes $E_\pm(z,t)$ that vary slowly in space and time. For the field to be represented in this form, we require that

$$\tau\omega\gg 1,\quad |\mathbf{d}_{12}|E_\pm\ll\hbar\omega,\quad \alpha\ll k,\tag{2}$$

where τ is the characteristic width of the exciting light pulses, ω is a frequency close to the resonance transition frequency ω_{21}; $\mathbf{d}_{12}$ is the reduced dipole moment of the resonance transition; α is the absorption coefficient, and k is the wave number. In addition, it follows from the "slowness" of the amplitudes $E_\pm(z,t)$ that over times of order $2\pi/\omega$ and distances of order $2\pi/k$, variations in $E_\pm(z,t)$ can be ne-

 1063-7761/95/090452-07$10.00

glected. This last circumstance permits one to view the field E as a function of "quasi-independent" variables z, t, η (where $\eta=kz$):[2]

$$E(z,t)\equiv E(z,t,\eta)=\{E_+(z,t)\exp(-i\eta)+E_-(z,t)\exp(i\eta)\}\exp(i\omega t)+\text{c.c.} \tag{3}$$

Substitution of Eq. (3) into the Maxwell equation

$$\frac{\partial^2 E}{\partial z^2}-\frac{n_0^2}{c^2}\frac{\partial^2 E}{\partial t^2}=\frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2}$$

yields

$$\begin{aligned} &i\left(\frac{n_0\omega}{c}-k\right)\{E_+(z,t)\exp(-i\eta)+E_-(z,t)\exp(i\eta)\}\\ &+\left\{\frac{\partial E_+(z,t)}{\partial z}+\frac{n_0}{c}\frac{\partial E_+(z,t)}{\partial t}\right\}\exp(-i\eta)\\ &+\left\{-\frac{\partial E_-(z,t)}{\partial z}+\frac{n_0}{c}\frac{\partial E_-(z,t)}{\partial t}\right\}\exp(i\eta)\\ &=-i\frac{2\pi\omega}{cn_0}\tilde{P}(z,t,\eta), \end{aligned} \tag{4}$$

where $\tilde{P}(z,t,\eta)$ is the slowly-varying polarization amplitude as a function of time but not space.

Multiplying Eq. (4) by $\exp(\pm i\eta)$ and noting that

$$\int_{-\pi}^{+\pi} d\eta\,\exp(im\eta)=2\pi\delta_{m0},\quad m=0,\pm1,\pm2,\dots, \tag{5}$$

where δ_{m0} is the Kronecker symbol, we obtain the equations for the slowly varying amplitudes of the counterpropagating waves in the resonator:

$$\begin{aligned} &i\left(\frac{n_0\omega}{c}-k\right)E_++\left\{\frac{\partial E_+(z,t)}{\partial z}+\frac{n_0}{c}\frac{\partial E_+(z,t)}{\partial t}\right\}\\ &=-\frac{i}{2\pi}\frac{2\pi\omega}{cn_0}\int_{-\pi}^{+\pi} d\eta\,\exp(i\eta)\tilde{P}(z,t,\eta),\\ &i\left(\frac{n_0\omega}{c}-k\right)E_-+\left\{-\frac{\partial E_+(z,t)}{\partial z}+\frac{n_0}{c}\frac{\partial E_+(z,t)}{\partial t}\right\}\\ &=-\frac{i}{2\pi}\frac{2\pi\omega}{cn_0}\int_{-\pi}^{+\pi} d\eta\,\exp(-i\eta)\tilde{P}(z,t,\eta). \end{aligned} \tag{6}$$

Note that the field amplitudes of the counterpropagating waves are coupled, and are determined by the common polarization $\tilde{P}(z,t,\eta)$.

In deriving the expression for the macroscopic polarization

$$P(z,t)=\tilde{P}(z,t,\eta)\exp(i\omega t)+\text{c.c.}$$

one must take into account the possibility of inhomogeneous broadening of the resonance transition line

$$P(z,t,\eta)=N_0\langle p_\varepsilon(z,t,\eta)\rangle. \tag{7}$$

Here N_0 is the number of atoms per unit volume that participate in resonance transitions, and angle brackets denote frequency averaging over the offset $\varepsilon=\omega_{21}-\omega$ with a weighting function $g(\varepsilon)$ that characterizes the inhomogeneously broadened line profile:

$$\langle p_\varepsilon(z,t,\eta)\rangle=\int_{-\infty}^{+\infty} d\varepsilon g(\varepsilon)p_\varepsilon(z,t,\eta). \tag{8}$$

In the following, we start with the equation of motion for a quantized oscillator (see, for example, Ref. 27) interacting with an external field,

$$\begin{aligned} &\ddot{p}+2\gamma_\perp\dot{p}+\omega_{21}^2 p=-\frac{2|d_{12}|^2\omega_{21}}{\hbar}(nE),\\ &\dot{n}+\gamma_\parallel(n-n^{(0)})=\frac{2}{\hbar\omega_{21}}(E\dot{p}), \end{aligned} \tag{9}$$

where $\gamma_\perp$ and $\gamma_\parallel$ are phenomenological damping coefficients that govern the relaxation of the polarization and the population, p is the dipole moment, and n the inversion per resonant atom.

Representing $P(z,t)$ as

$$P(z,t)=N_0\{\langle p_\varepsilon(z,t,\eta)\rangle\exp(i\omega t)+\text{c.c.}\}$$

we obtain the equations of motion of the quantized oscillator in the form

$$\begin{aligned} &\frac{\partial}{\partial t}p_\varepsilon+\gamma_\perp p_\varepsilon-i\varepsilon p_\varepsilon=\frac{i|\mathbf{d}_{21}|^2}{\hbar}n_\varepsilon[E_+\exp(-i\eta)+E_-\exp(i\eta)],\\ &\frac{\partial}{\partial t}n_\varepsilon+\gamma_\parallel(n_\varepsilon-n_\varepsilon^{(0)})=\frac{i}{2\hbar}\{p_\varepsilon[E_+^*\exp(i\eta)+E_-^*\times\exp(-i\eta)]-\text{c.c.}\}. \end{aligned} \tag{10}$$

Equations (6)–(8) and (10) form a closed system describing the dynamics of the field and the material medium in PRS with the two counterpropagating waves taken into account. Naturally, it must be supplemented by appropriate boundary conditions relating the field amplitudes at the boundaries of the resonant medium, i.e., the resonator mirrors. We assume that reflecting surfaces with reflection coefficients R_1 and R_2 close to unity are located at $z=0$ and $z=L$ (Fig. 1).

Then

$$\begin{aligned} &E_r(0,t)=-\sqrt{R_1}E_0+\sqrt{1-R_1}E_-(0,t),\\ &E_+(0,t)=\sqrt{1-R_1}E_0+\sqrt{R_1}E_-(0,t),\\ &E_t(L,t)=\sqrt{1-R_2}E_+(L,t),\\ &E_-(L,t)=\sqrt{R_2}E_+(L,t)\exp(-2ikL), \end{aligned} \tag{11}$$

where E_0 is the amplitude of the external field and E_r and E_t are the amplitudes of the reflected and transmitted fields.

Note that an analogous statement of the problem for a planar Fabry–Perot resonator appeared under certain simplifying assumptions in the work of Lugiato and Narducci.[2] By transforming the field, they were able to reduce the system of equations to a form typical of an ideal resonator with no losses at the boundaries that is often utilized in laser theory.[28]

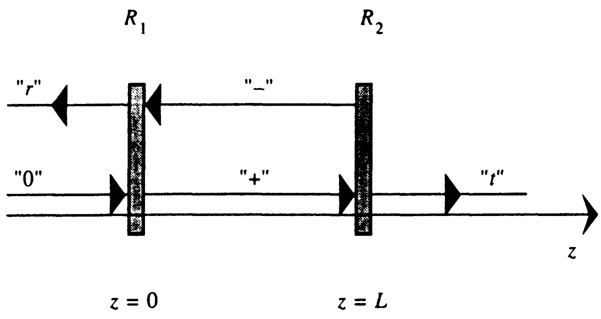

FIG. 1. Fabry–Perot resonator formed by plane reflectors with reflection coefficients R_1 and R_2. The subscripts + and − refer to the forward and backward waves within the resonator, and 0, r, and t refer to the incident, reflected, and transmitted waves.

The system of equations (6)–(8), (10) and (11) turns out to be convenient for the solution of a number of different problems of coherent and incoherent interaction of light with a resonant medium in a planar resonator, including the generation of laser radiation by an active medium.

To obtain the general solution of Eqs. (6)–(8), (10), and (11), one must solve a complex nonlinear electrodynamic problem. The point is that the interference of the forward and backward waves in the nonlinear medium leads to terms involving various powers of the parameter $i\eta$ in the expansions of the polarization and inversion (odd powers for the polarization, even powers for the inversion); those terms characterize the rapidly oscillating factors. The appearance of "high" spatial frequencies in the problem is due to strong spatial inhomogeneities. The result is an infinite set of coupled equations for the polarization, inversion, and field, which can only be solved numerically, having first truncated the infinite set of equations at some point.[6]

Henceforth, we retain only the leading terms in the expansion of p_ε in the parameter η assuming that the z dependence of $p_\varepsilon^\pm$ can be neglected:

$$p_\varepsilon(z,t,\eta)=p_\varepsilon^+(t)\exp(-i\eta)+p_\varepsilon^-(t)\exp(i\eta),$$

This can obviously be done only if $\alpha L\ll 1$, which means that absorption is small over the length L of the resonant medium (α is the absorption coefficient of the medium at field frequency ω).

Integrating over η in Eq. (6) with Eq. (5) taken into account, we obtain

$$i\left(\frac{n_0\omega}{c}-k\right)E_\pm\pm\frac{\partial E_\pm}{\partial z}+\frac{n_0}{c}\frac{\partial E_\pm}{\partial t}=-\frac{2\pi\omega N_0}{cn_0}\langle p_\varepsilon^\pm\rangle,$$

$$\frac{\partial}{\partial t}p_\varepsilon^\pm+\gamma_\perp p_\varepsilon^\pm-i\varepsilon p_\varepsilon^\pm=\frac{i|d_{12}|^2}{\hbar}n_\varepsilon E_\pm,$$

$$\frac{\partial}{\partial t}n_\varepsilon+\gamma_\parallel(n_\varepsilon-n_\varepsilon^{(0)})=\frac{i}{2\hbar}\{p_\varepsilon^+E_+^*+p_\varepsilon^-E_-^*-\text{c.c.}\}. \quad (12)$$

On the other hand, neglecting the z dependence of the fields $E_\pm$ should mean that the relation

$$\tau\gg\frac{n_0L}{c} \quad (13)$$

between the pulse width and light propagation time within the resonator is satisfied, and that also means that it is possible to neglect the spectral width of the pulse τ^{-1} compared to the separation $c/2Ln_0$ between longitudinal modes of the resonator or, in other words, to make use of the single-mode approximation.

Then integrating the first equations for $E_\pm$ over z, we obtain

$$i\left(\frac{n_0\omega}{c}-k\right)LE_\pm\pm E_\pm(L)-(\pm E_\pm(0))$$
$$+\frac{n_0L}{c}\frac{\partial E_\pm}{\partial t}=-\frac{2\pi\omega N_0}{cn_0}\langle p_\varepsilon^\pm\rangle. \quad (14)$$

Finally, eliminating E_- from Eq. (14) with the help of the boundary conditions Eq. (10), we obtain an equation for the direct wave,

$$i\left(\frac{n_0\omega}{c}-k\right)Lr_+(\beta)E_++r_-(\beta)E_+$$
$$+\frac{n_0L}{c}r_+(\beta)\frac{\partial E_\pm}{\partial t}-\sqrt{1-R_1}E_0$$
$$=-\frac{2\pi\omega N_0L}{cn_0}r_+(\beta)\langle p_\varepsilon^\pm\rangle, \quad (15)$$

where

$$r_\pm(\beta)=1\pm\sqrt{R_1R_2}\exp(-i\beta),$$

$$\beta=2kL.$$

Note that if we neglect the first term in Eq. (15), which results, as will be seen below, in a correction to the dispersion law $k(\omega)$, even for the steady-state solutions with $R_1=R_2=R$, Eq. (15) acquires a form often used in the theory of optical bistability.[13]

We now introduce the characteristic lifetime of the photon in the resonator, which governs the loss of stored energy over time due to emitted radiation:

$$W(t)=W_0\exp\left(-\frac{t}{\tau_c}\right).$$

In time $2Ln_0/c$, the energy will change in accordance with the expression

$$R_1R_2W_0=W_0\exp\left(-\frac{2Ln_0}{c\tau_c}\right),$$

whence

$$\tau_c=-\frac{2Ln_0}{c\ln(R_1R_2)}. \quad (16)$$

For values of R_1, R_2 close to unity, one can use in place of Eq. (16) the relation

$$\tau_c=\frac{2Ln_0}{c(1-R_1R_2)}.$$

For the following, isolating the real quantities $E_+=\mathscr{E}_+\exp(i\varphi)$ in (15) as well as the reactive and active components of the dipole moment $p_\varepsilon^+=(u_\varepsilon^+ + iv_\varepsilon^+)\exp(i\varphi)$, which describe the contribution to dispersion and absorption of light by the resonant medium (for weak fields), we obtain

$$\frac{i\left(\frac{n_0\omega}{c}-k\right)L}{1-R_1R_2}\, r^s(\beta)\mathscr{E}_+ + \frac{r_-^c(\beta)}{1-R_1R_2}\,\mathscr{E}_+ + r_+^c(\beta)\,\frac{\tau_c}{2}\frac{\partial\mathscr{E}_+}{\partial t} + r^s(\beta)\,\frac{\tau_c}{2}\frac{\partial\varphi}{\partial t}\,\mathscr{E}_+ + C'[r^s\langle u_\varepsilon^+\rangle - r_+^c\langle v_\varepsilon^+\rangle] = \mathscr{E}_0'\cos(\varphi_0-\varphi), \tag{17a}$$

$$\frac{i\left(\frac{n_0\omega}{c}-k\right)L}{1-R_1R_2}\, r_+^c(\beta)\mathscr{E}_+ + \frac{r^s(\beta)}{1-R_1R_2}\,\mathscr{E}_+ - r^s(\beta)\,\frac{\tau_c}{2}\frac{\partial\mathscr{E}_+}{\partial t} + r_+^c(\beta)\,\frac{\tau_c}{2}\frac{\partial\varphi}{\partial t}\,\mathscr{E}_+ + C'[r_+^c\langle u_\varepsilon^+\rangle + r^s\langle v_\varepsilon^+\rangle] = \mathscr{E}_0'\sin(\varphi_0-\varphi), \tag{17b}$$

$$\frac{\partial u_\varepsilon^\pm}{\partial t}+\gamma_\perp u_\varepsilon^\pm+\varepsilon v_\varepsilon^\pm=0,$$

$$\frac{\partial v_\varepsilon^\pm}{\partial t}+\gamma_\perp v_\varepsilon^\pm-\varepsilon u_\varepsilon^\pm=\frac{|d_{12}|^2}{\hbar}\,n_\varepsilon\mathscr{E}_\pm, \tag{17c}$$

$$\frac{\partial n_\varepsilon}{\partial t}+\gamma_\parallel(n_\varepsilon-n_\varepsilon^{(0)})=-\frac{1}{\hbar}\,(\mathscr{E}_+v_\varepsilon^+ + \mathscr{E}v_\varepsilon^-),$$

where

$$r^s(\beta)=\sqrt{R_1R_2}\,\sin\beta,\quad r_\pm^c(\beta)=1\pm\sqrt{R_1R_2}\,\cos\beta,$$

$$C'=\frac{2\pi\omega N_0L}{cn_0(1-R_1R_2)},\quad \mathscr{E}_0'=\frac{\sqrt{1-R_1}}{(1-R_1R_2)}\,\mathscr{E}_0.$$

We represent the phase shift β in the form $\beta=2kL=2\pi m+\beta_0$ (integer m), where

$$\beta_0=2kL-2\pi m=\frac{\omega-\omega_c}{c/2Ln_0},$$

and ω_c is one of the resonator eigenfrequencies in the uniformly spaced spectrum.

In what follows, we confine any detailed discussion to the "tuned" resonator, for which we can set $\beta_0=0$, which means that the traditional equality $L=m\lambda/2$ is satisfied. In that case, in the absence of phase modulation of the external field at the "entry point" (φ_0=const), such modulation can also be ignored inside the resonator, and we can assume that $\varphi=\varphi_0$. Then the second equation in (17), as in the case of an infinite medium, can be transformed into a constraint on the dispersion law $k(\omega)$, and the first of the equations in (17) can be significantly simplified:

$$\mathscr{E}_+ + \frac{\tau_c}{2}\frac{\partial}{\partial t}\,\mathscr{E}_+ - C''\langle v_\varepsilon'\rangle=\mathscr{E}_0''(t), \tag{18}$$

where

$$\tau_c=(1+\sqrt{R_1R_2})^2\tau_c,$$

$$C''=\frac{2\pi\omega LN_0(1+\sqrt{R_1R_2})}{cn_0(1-\sqrt{R_1R_2})}.$$

Note that an "offset" resonator can be treated in a similar manner.

This last Eq. (18), together with the constitutive equations for the medium (17c), determines the dynamics of the field inside the resonator system and provides the starting point for the study of a number of phenomena: transmission and reflection of USP of light, photon echoes, and optical bistability. Although Eq. (18) is written for the forward wave in the resonator, the expression for the backward, transmitted, and reflected waves is not hard to obtain by utilizing the corresponding boundary conditions.

3. INTEGRAL OF THE MOTION UPON INTERACTION OF ULTRASHORT LIGHT PULSES WITH PLANAR RESONANT STRUCTURES

Ultrashort pulses of light, as before, are pulses whose duration satisfies

$$\tau\ll T_2\cong\gamma_\perp^{-1}\ll T_1\cong\gamma_\parallel^{-1}. \tag{19}$$

At the same time, we assume, as before, that the inequality (13) is satisfied and, since $(1-R_1R_2)\ll1$ and $\tau\gg\Delta\nu_c^{-1}$, the relation between τ and τ_c can be arbitrary.

In the approximation of Eq. (19), the constitutive equations for the reactive and active components of the dipole moment, as well as for the inversion of an individual resonant atom, take the form.

$$\frac{\partial u_\varepsilon^\pm}{\partial t}+\varepsilon v_\varepsilon^\pm=0,\quad \frac{\partial v_\varepsilon^\pm}{\partial t}-\varepsilon u_\varepsilon^\pm=\frac{|d_{12}|^2}{\hbar}\,n_\varepsilon\mathscr{E}_\pm,$$

$$\frac{\partial n_\varepsilon}{\partial t}+\gamma_\parallel(n_\varepsilon-n_\varepsilon^{(0)})=-\frac{1}{\hbar}\,(\mathscr{E}_+v_\varepsilon^+ + \mathscr{E}_-v_\varepsilon^-). \tag{20}$$

Noting that $\alpha L\ll1$, making use of the last boundary condition in (10) and the condition $\beta_0=0$ in the resonator, we can confine our attention solely to the field of the direct wave $\mathscr{E}_+$. Then the last equation in (20) takes the form.

$$\frac{\partial n_\varepsilon}{\partial t}=-\frac{1}{\hbar}\,(R_2+1)\mathscr{E}_+v_\varepsilon^+. \tag{21}$$

Equations (18), (20), and (21) admit the existence of an integral of the motion for the system, which is the analog of the McCall–Hahn "areas" theorem in a semi-bounded extended medium.[26] To derive it, we introduce the "area" of a coherent light pulse

$$\psi(t)=\frac{|d_{12}|}{\hbar}\int_{-\infty}^{+\infty}\mathscr{E}_+(t)dt,$$

and its asymptotic value $\theta=\psi(t=\infty)$.

After integrating Eq. (18) over time between $-\infty$ and $+\infty$, we obtain

$$\theta-C'\int_{-\infty}^{+\infty}\langle v_\varepsilon^+(t)\rangle dt=\theta_0, \tag{22}$$

where we have used the obvious relation

$$\mathscr{E}_+(\pm\infty)=0.$$

Based on the properties of the solution of the first equations in (20) and (21), we can integrate the active component of the resonant medium's dipole moment (averaged over the inhomogeneously broadened line profile). Indeed, it is not hard to show that

$$\int_{-\infty}^{+\infty}\langle v_\varepsilon^+(t)\rangle dt=\left\langle -\frac{u_\varepsilon^+(+\infty)}{\varepsilon}\right\rangle=\frac{|d_{12}|^2}{\hbar}\int_{-\infty}^{+\infty}\mathscr{E}_+(t')\times\left\langle n_\varepsilon\frac{\sin[\varepsilon(t-t')]}{\varepsilon}\right\rangle dt, \tag{23}$$

and since for long times (exceeding T_2^*, which characterizes the inhomogeneous broadening of the energy levels) we have

$$\lim_{t-t'\to\infty}\frac{\sin[\varepsilon(t-t')]}{\varepsilon}=\pi\delta(\varepsilon), \tag{24}$$

we find that

$$\begin{aligned}\int_{-\infty}^{+\infty}\langle v_\varepsilon^+(t)\rangle dt&=\frac{|d_{12}|^2}{\hbar}\pi g(0)\int_{-\infty}^{+\infty}\mathscr{E}_+(t')n_{\varepsilon=0}(t')dt\\&=\pi g(0)v_0^+(\infty)\\&=-\pi g(0)\frac{|d_{12}|}{\sqrt{R_2+1}}\sin(\sqrt{R_2+1}\,\theta),\end{aligned} \tag{25}$$

(here we have made use of the fact that $n_{\varepsilon=0}(-\infty)=-1$).

Ultimately, the relation (22) takes the form

$$\tilde{\theta}+C\sin\tilde{\theta}=\tilde{\theta}_0, \tag{26}$$

where the constant C turns out to be

$$C=\frac{\alpha_0L}{2}\frac{(1+R_1R_2)}{(1-R_1R_2)}, \tag{27}$$

and α_0 is the absorption coefficient for a weak signal,

$$\alpha_0=\frac{4\pi^2\omega|d_{12}|^2N_0g(0)}{hcn_0},$$

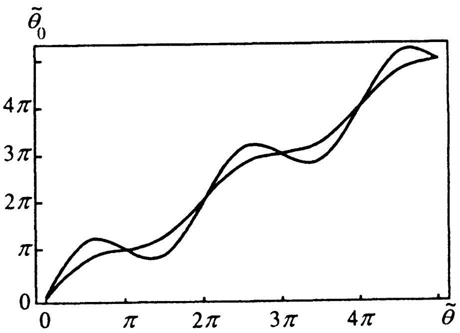

FIG. 2. Incident pulse "area" θ_0 as a function of the "area" θ of the light pulse of the forward wave inside the resonator for $C=2.0$ and 0.8.

and the "areas" of the pulses within the resonator and outside it are respectively

$$\tilde{\theta}=\sqrt{R_2+1}\,\theta,\quad \tilde{\theta}_0=\frac{\sqrt{(R_1-1)(R_2+1)}}{(1-\sqrt{R_1R_2})}\theta_0.$$

Note that Eq. (26) was obtained asymptotically, i.e., in the limit $t\to\infty$. From a frequency point of view, infinitely long times (long compared to the characteristic time T_2^*) correspond to infinitesimal frequencies (a frequency range significantly narrower than the characteristic inhomogeneous spectral line width $(T_2^*)^{-1}$ for the resonant transition). It is for this reason that Eq. (26) contains the value of the absorption coefficient α_0 at the frequency $\omega_{21}=\omega$ $(\varepsilon=0)$.

4. "BISTABLE" AND SOLITON SOLUTIONS. STABILITY REGION

Expression (26) represents the analog of the McCall–Hahn "areas" theorem in the case of the interaction of USP of light with thin-film Fabry–Perot resonator structures, and establishes the correspondence between the "areas" of the pulses of light within the resonator θ and the external incident pulses θ_0. There is a similar correspondence between the "areas" of the transmitted and reflected pulses.

The relation (26) can be conveniently analyzed graphically by treating the "area" of the external incident pulse $\tilde{\theta}_0$ as a function of the "area" of the pulse of the field of the direct wave within the resonator $\tilde{\theta}$ (Fig. 2). As can be seen from the figure, the character of possible solutions for $\tilde{\theta}$ is determined by the nonlinearity parameter C, which is the ratio of two small quantities (α_0L and $1-\sqrt{R_1R_2}$) and can therefore take on various values (both greater and smaller than unity). When $C\leqslant1$, there is but one unique solution for $\tilde{\theta}$ and $\tilde{\theta}_0$ for arbitrary values of $\tilde{\theta}_0$.

For $C>1$, the dependence between $\tilde{\theta}_0$ and $\tilde{\theta}$ ceases to be single-valued—near $\tilde{\theta}_0=m\pi$ (odd m), there are regions with three solutions $\tilde{\theta}$ that satisfy Eq. (26).

Values of $\tilde{\theta}$ for which the derivative $\partial\tilde{\theta}_0/\partial\tilde{\theta}$ vanishes can be determined by solving the equation

$$\cos\tilde{\theta}=-\frac{1}{C},$$

whence

$$\tilde{\theta}_1=\pi-\arcsin\left(\frac{\sqrt{C^2-1}}{C}\right),$$

$$\tilde{\theta}_2=\pi+\arcsin\left(\frac{\sqrt{C^2-1}}{C}\right),$$

and values of $\tilde{\theta}_0$ at these points turn out to be

$$\tilde{\theta}_{01}=m\pi+\sqrt{C^2-1}-\arcsin\left(\frac{\sqrt{C^2-1}}{C}\right),$$

$$\tilde{\theta}_{02}=m\pi-\sqrt{C^2-1}+\arcsin\left(\frac{\sqrt{C^2-1}}{C}\right).$$

In regions where the solutions are multivalued, we obtain

$$\Delta\tilde{\theta}_0=2\sqrt{C^2-1}-2\arcsin\left(\frac{\sqrt{C^2-1}}{C}\right).$$

The stability of the solutions $\tilde{\theta}$ are of interest when they are not single-valued. To this end, one must know the exact equation for the area of the pulse $\tilde{\psi}(t)$, which goes over asymptotically into $\tilde{\theta}$. We are not able to obtain an equation describing the evolution of $\tilde{\psi}(t)$ for arbitrary t, but we can make use as before of the properties of the solutions (23)–(25) for large but finite t.

As a result, we obtain an asymptotic equation valid for large t ($t\gg T_2^*$):

$$\tilde{\psi}+\frac{\tilde{\tau}_c}{2}\frac{\partial}{\partial t}\tilde{\psi}+C\sin\tilde{\psi}=\tilde{\psi}_0(t). \tag{28}$$

We linearize Eq. (28) and introduce a new variable

$$\delta\tilde{\psi}(t)=\tilde{\theta}-\tilde{\psi}(t).$$

Then

$$\delta\tilde{\psi}+\frac{\tilde{\tau}_c}{2}\frac{\partial}{\partial t}\delta\tilde{\psi}+C\cos\tilde{\theta}\delta\tilde{\psi}=\delta\tilde{\psi}_0, \tag{29}$$

where $\delta\tilde{\psi}_0\to 0$ as $t\to\infty$.

We seek a solution of the homogeneous Eq. (29) as $t\to\infty$ in the standard manner:

$$\delta\tilde{\psi}(t)=\delta\tilde{\psi}(t')\exp[\chi(t-t')],$$

where upon we obtain the characteristic equation for χ:

$$\frac{\tilde{\tau}_c}{2}\chi=-(1+C\cos\tilde{\theta})=-\frac{d\tilde{\theta}_0}{d\tilde{\theta}}. \tag{30}$$

It follows from Eq. (30) that stable solutions for $\tilde{\theta}$ correspond to $\chi<0$, which in turn means a positive slope for the tangent to the graph of $\tilde{\theta}_0(\tilde{\theta})$. Conversely, for $d\tilde{\theta}_0/d\tilde{\theta}<0$, the sign of χ is positive, and an infinitesimal deviation $\tilde{\psi}(t)$ from $\tilde{\theta}$ will grow with time. Thus, unstable solutions $\tilde{\theta}(\tilde{\theta}_0)$ correspond to decreasing, interior regions of the dependence $\tilde{\theta}_0(\tilde{\theta})$ where the derivative of the function $\tilde{\theta}_0$ is negative.

This behavior of $\tilde{\theta}$ and $\tilde{\theta}_t$ is reminiscent of intraresonator absorptive optical bistability in the region of quasistationary solutions for E_+, for which the presence of dissipative relaxation processes for the polarization and inversion of the resonant medium is important.[13] Transitions between branches of solutions characterized by differing intensities of the transmitted light ("optical" switching) turns out to be possible. In the case of ultrashort pulses, the "area" of the light pulse which determines the field within the resonator behaves similarly.

Note that the integral of the motion Eq. (26) retains its physical meaning regardless of whether the duration of the USP of light exceeds the lifetime of the photon in the resonator or not. Furthermore, (26) has precisely the same form as in the case of interaction of USP with a thin surface layer of resonant atoms whose thickness is significantly less than the wavelength of light.[24]

In contrast to Eq. (26), the fundamental equation of motion (18) for the field within the resonator coincides with that for a thin film of resonant atoms only if $\tau\gg\tau_c$, whereupon the second term with the time derivative in Eq. (18) can be neglected. It is known that the problem of transmission of USP of light through a thin resonant layer admits of a so-called "soliton" solution, for which all fields have a similar time dependence and $\tilde{\theta}_0=2\pi$. Such a solution was obtained in the case of a thin surface layer of resonant atoms for the first time in Ref. 20 by the inverse scattering method. In particular, it was shown that the field of the transmitted wave has a two-soliton character.

Similar conclusions regarding the soliton nature of the propagation of USP of light through a resonant planar structure, containing resonant atoms, will be valid under the conditions of quasistationarity $\tau\gg\tau_c$: the field in the resonator will be able to "track" variations in the external field.

5. CONCLUSION

The nonlinear interaction of light with planar resonator structures Fabry–Perot filled with a resonant medium displays features in many ways similar to those seen in infinite media. The special features of this interaction are determined by the existence of forward and backward waves, resulting in the creation of a standing wave. For large values of the parameter αL (where α is the absorption coefficient and L is the distance between the resonator reflectors), the problem becomes inhomogeneous in space and reduces to an infinite system of coupled equations. For values $\alpha L<1$ and pulse widths greater than the light transit time across the resonator, solutions of the spatially homogeneous problem are possible with dynamical evolution in time.

Coherent interaction of light pulses of ultrashort duration with PRS demonstrates properties possessed by semi-bounded resonant media. Under conditions in which the excitation pulses are significantly narrower than typical transverse and longitudinal relaxation times of the atoms in the resonant medium, but exceed the light-travel time between the resonator reflectors, in the approximation of slowly varying amplitudes, we obtain spatially homogeneous equations of motion referring to the appearance of one longitudinal mode of the resonator. In particular, we have shown that the problem then admits of an integral of the motion analogous to the McCall–Hahn "areas" theorem, regardless of the relation between the excitation pulse width and the lifetime of the photon within the resonator. We have investigated regions for the appearance of bistable and soliton solutions. We note that the character of multivalued solutions for the "areas" of USP of light is largely similar to the phenomenon of optical bistability in the case of quasistationary interaction of light with the resonant medium in a PRS.

It would be of interest to study the properties of an important physical phenomenon in the field of interaction of USP of light with a PRS–namely the photon echo, to consider the ultimate possibilities of dynamic echo-holograms, and in particular to compare the dynamic efficiency of a hologram based on a PRS with the efficiency in the case of semibounded resonant media and neglect of standing waves.

The author expresses his gratitude to E. A. Manykin for fruitful discussion of a number of results.

The present work was supported in part by the International Science Foundation (grant NE8000).

[1] L. A. Lugiato and C. Oldano, Phys. Rev. A **37**, 3896 (1988).
[2] L. A. Lugiato and L. M. Narducci, Z. Phys. B **71**, 129 (1988).
[3] M. Haelterman, G. Vitrant, and R. Reinisch, J. Opt. Soc. Am. B **7**, 1309, 1319 (1990).
[4] K. C. Ho and G. Indebetouw, Appl. Opt. **30**, 2437 (1991).
[5] J. Danckaert, G. Vitrant, R. Reinisch, and M. Georgiu, Phys. Rev. A **48**, 2324 (1993).
[6] A. J. van Wonderen and L. G. Suttorp, Z. Phys. B **83**, 135, 143 (1991).
[7] J. Danckaert and G. Vitrant, Opt. Commun. **104**, 196 (1993).
[8] J. L. Jewell, A. Scherer, S. L. McCall *et al.*, Appl. Phys. Lett. **51**, 94 (1987).
[9] S. L. McCall and J. Jewell, in *Laser Optics of Condensed Media*, Plenum Press; New York (1988), p. 449.
[10] J. L. Jewell, A. Scherer, S. L. McCall *et al.*, Appl. Phys. Lett. **55**, 22 (1989).
[11] Y. Yamamoto and R. E. Slusher, Physics Today, No. 6, 66 (1993).
[12] H. M. Gibbs, G. Hitrova, and N. Peyghambarian (eds.), *Nonlinear Photonics Springer Series in Electronics and Photonics*, Vol. 30, Springer-Verlag, Berlin (1990).
[13] H. M. Gibbs, *Optical Bistability: Controlling Light with Light*, Academic Press, Orlando (1985).
[14] L. A. Lugiato, in *Progress in Optics*, E. Wolf (ed.), Vol. 21, p. 69, North-Holland, Amsterdam (1984).
[15] L. A. Lugiato, L. M. Narducci, D. K. Bandy, and C. A. Pennise, Opt. Commun. **43**, 281 (1982).
[16] M. Haelterman and P. Mandel, Opt. Lett. **15**, 1412 (1990).
[17] E. Abraham, R. K. Bullough, and S. S. Hassan, Opt. Commun. **29**, 109 (1979).
[18] P. Meystre, Opt. Commun. **26**, 277 (1978).
[19] R. Bonifacio and P. Meystre, Opt. Commun. **27**, 147 (1978).
[20] V. I. Rupasov and V. I. Yudson, Kvant. Elektron **9**, 2179 (1982) [Quantum Electron. **12**, 1415 (1982)].
[21] V. I. Rupasov and V. I. Yudson, Zh. Éksp. Teor. Fiz. **93**, 494 (1987) [Sov. Phys. JETP **66**, 282 (1987)].
[22] M. G. Benedict and E. D. Trifonov, Phys. Rev. A **38**, 2854 (1988).
[23] S. M. Zakharov, A. I. Maimistov, and E. A. Manykin, Poverkhnost' **12**, 60 (1989).
[24] S. M. Zakharov and E. A. Manykin, Poverkhnost' **7**, 68 (1989).
[25] S. M. Zakharov and E. A. Manykin, JETP **78**, 566 (1994).
[26] S. L. McCall and E. L. Hahn, Phys. Rev. **183**, 457 (1969).
[27] R. Pantell and H. Puthoff, *Fundametals of Quantum Electronics*, Wiley, New York (1969).
[28] H. Haken, *Laser Theory*, Handbuch der Physik, Springer-Verlag, Heidelberg–New York (1970).

Translated by Adam M. Bincer