Effect of dissipation on the collapse of a solitary wave in a nonlinear weakly dispersive medium

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Collapse (self-compression over a finite time) of a solitary wave is studied. The process is described by a one-dimensional nonlinear Schrödinger equation, in which terms corresponding to damping and fifth-order nonlinearity are included. Numerical solution of a system of equations for the width, amplitude, and degree of modulation reveals that the dissipation can suppress collapse if it is strong enough. Analytical expressions for the threshold values of the linear and nonlinear absorption coefficients are found by means of qualitative arguments. © *1995 American Institute of Physics.*

1. INTRODUCTION

The nonlinear Schrödinger (NLS) equation[1–8]

$$iq_{,t}+\sigma\Delta q+\mu|q|^2q=0 \tag{1}$$

is widely used to describe the evolution of nonlinear waves in dispersive media, where $q(t,r)$ is the complex envelope of the wave. The term containing the Laplacian Δ represents the diffractional (dispersive) broadening, while the last term in (1) describes the nonlinear action that limits spreading of the wave.

In the one-dimensional case ($\Delta=\partial_x^2$) the NLS equation is an example of a completely integrable evolution equation that can be solved by the inverse scattering method.[9,10] In this case Eq. (1) possesses particular solutions in the form of steady solitary waves, called solitons, and unsteady multi-soliton solutions describing the interactions between solitons. Besides these there are many others, including periodic solutions[11] and solutions related to the transcendental Painlevé functions.[12,13]

In d-dimensional space ($d>1$) the NLS equation has been used to describe self-focusing of light beams[3–8] and optical collapse.[14,15] In these cases the NLS equation cannot be integrated by the inverse scattering method, and it becomes quite complicated. In addition to numerical integration, the variational method first developed by Anderson for the one-dimensional case[16] has been used successfully.[8]

As shown in Refs. 17 and 18, in the one-dimensional case a soliton can collapse if a higher-order nonlinearity is included in (1):

$$iq_{,t}+\sigma q_{,xx}+\mu|q|^2q+\beta|q|^4q=0. \tag{2}$$

This phenomenon can be described analytically by use of either the variational method or the generalized moment technique.[19] These approaches, as well as the adiabatic perturbation theory for solitons,[20,21] have been employed frequently in analytical studies of the effect of small perturbations on NLS solitons. In particular, it has been found that dissipation associated with linear[21] and nonlinear[22] absorption causes broadening of the soliton. Here it should be mentioned that the variational approach provides a picture of the evolution of the soliton width that is closer to that observed in numerical simulations. Strictly speaking, the term "soliton" should not be used; it is more accurate to say "solitary wave," since the perturbed NLS equation is in general not integrable. But for brevity one often refers to a solution as a soliton if its behavior resembles that of solitons. With this in mind one can conclude that dissipation causes an increase in the width of a soliton, while a higher-order nonlinearity [like that in Eq. (2)] can produce the opposite effect (self-compression and collapse of the soliton) for $\beta>0$ and $\sigma>0$.

The purpose of the present work was to study the effect of linear and nonlinear absorption on the change in width of a solitary wave (soliton) propagating in a medium having a fifth-order nonlinearity as in Refs. 17 and 18. When the soliton begins to compress, so that its width approaches zero, a theory based on the NLS equation (1) or (2) becomes inadequate. In this case it is necessary to take into account higher-order derivatives of the wave envelope $q(t,x)$, and the same is true of nonlinear effects. But if we are interested in the time dependence of the soliton parameters close to the threshold determining collapse, then it is permissible to restrict the treatment to this model.

Using the formalism of the reduced description of soliton evolution,[19] which is a simple generalization of the adiabatic perturbation theory for solitons,[20,21] we will find approximate expressions for the absorption coefficients above which collapse becomes impossible. The derivation of these coefficients is based on results found by numerical solution of the system of equations describing the behavior of the width of the solitary wave (soliton).

2. BASIC EQUATIONS OF THE MODEL

We assume that the complex envelope $q(t,x)$ of the solitary wave satisfies the perturbed NLS equation

$$iq_{,t}+\sigma q_{,xx}+\mu|q|^2q=R[q], \tag{3}$$

which includes Eq. (2) as a special case. If we set ourselves the task of studying the behavior of the wave amplitude $A(t)$, its width $x_p(t)$, the corrections to the phase velocity $C(t)$, the position of the center of mass $x_c(t)$, and the degree of phase modulation $B(t)$, then from (3) we can derive a

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system of equations for these variables.[19] For this it is necessary to assume that the envelope has some definite shape, which is equivalent to the choice of a test function in the variational method. If we take $q(t,x)$ in the form

$$q(t,x)=A(t)\operatorname{sech}[Y(t,x)]\exp\{i\Phi(t,x)\}$$

$$Y(t,x)=\frac{x-x_c(t)}{x_p(t)},$$

$$\Phi(t,x)=\phi(t)+C(t)[x-x_c(t)]+B(t)[x-x_c(t)]^2, \quad (4)$$

then a system of equations can be derived which provides a coarsened description of the time dependence of the solution of Eq. (3) in terms of the variables $A(t)$, $x_p(t)$, $x_c(t)$, $C(t)$, and $B(t)$:

$$\frac{d(x_pA^2)}{dt}=x_pA\int_{-\infty}^{\infty}\operatorname{sech} y \operatorname{Im} \rho\, dy, \quad (5.1)$$

$$\frac{dC}{dt}=-(x_pA)^{-1}\int_{-\infty}^{\infty}\tanh y \operatorname{Re} \rho\, dy$$

$$+2Bx_pA^{-1}\int_{-\infty}^{\infty} y \operatorname{sech} y \operatorname{Im} \rho\, dy, \quad (5.2)$$

$$\frac{x_c}{dt}=2\sigma C+x_pA^{-1}\int_{-\infty}^{\infty} y \operatorname{sech} y \operatorname{Im} \rho\, dy, \quad (5.3)$$

$$\frac{dx_p}{dt}=4\sigma Bx_p-x_p(2A)^{-1}\int_{-\infty}^{\infty}\left(1-\frac{12y^2}{\pi^2}\right)\operatorname{sech} y \operatorname{Im} \rho\, dy, \quad (5.4)$$

$$\frac{dB}{dt}=\frac{4\sigma}{\pi^2}(x_p^{-4}-B^2\pi^2)-2\mu\left(\frac{A}{\pi x_p}\right)^2-6(A\pi^2x_p^2)^{-1}$$

$$\times\int_{-\infty}^{\infty}\left[y\tanh y-\frac{1}{2}\right]\operatorname{sech} y \operatorname{Re} \rho\, dy, \quad (5.5)$$

where $\rho=R[q]\exp(-i\Phi)$. The equation for $\phi(t)$ is omitted, since this variable plays no role in the problem.

For the perturbation in (3) we take a sum of terms describing the effects of dissipation and collapse of the solitary wave, viz.,

$$R[q]=-\beta|q|^4q-i\Gamma[q], \quad (6)$$

where the choice $\Gamma[q]=\Gamma_1 q$ corresponds to linear damping ($\Gamma_1>0$) or growth ($\Gamma_1<0$), and the term $\Gamma[q]=\Gamma_2|q|^2q$ describes nonlinear damping. This may be, e.g., two-photon absorption for an optical pulse in a nonlinear dispersive medium, as in Ref. 22.

3. ROLE OF LINEAR DAMPING

For $\Gamma[q]=\Gamma_1 q$ we can derive a system of equations from (5) for the variables that are significant in this case:

$$\frac{d(x_pA^2)}{dt}=-2\Gamma_1x_pA^2, \quad (7.1)$$

$$\frac{dx_p}{dt}=4\sigma Bx_p, \quad (7.2)$$

$$\frac{dB}{dt}=\frac{4\sigma}{\pi^2}(x_p^{-4}-B^2\pi^2)-2\mu\left(\frac{A}{\pi x_p}\right)^2-\frac{2\beta_nA^4}{\pi^2x_p^2}, \quad (7.3)$$

where $\beta_n=16\beta/15$. Here we have written $C=C_0=\text{const}$ and $x_c(t)=2\sigma C_0$.

From (7.1) we find

$$x_pA^2=W_0\exp(-2\Gamma_1t), \quad (8)$$

where W_0 is an integration constant determined by the initial conditions at $t=0$. Eliminating $B(t)$ from (7.2) and (7.3), we find the following equation for the soliton width $x_p(t)$:

$$\frac{d^2x_p}{dt^2}=\left[\left(\frac{4\sigma}{\pi}\right)^2-8\sigma\beta_n\left(\frac{W_0}{\pi}\right)^2\exp(-4\Gamma_1t)\right]x_p^{-3}$$

$$-\frac{8\sigma\mu W_0}{\pi^2}\exp(-2\Gamma_1t)x_p^{-2}.$$

This equation can usefully be written in dimensionless form for the variable $r=x_p/x_p(0)$, which is a function of $\tau=t[4\sigma/\pi x_p^2(0)]$, where we assume $\sigma>0$ (the case $\sigma\beta<0$ does not give rise to collapse):

$$\frac{d^2r}{d\tau^2}=[1-\delta W_0^2\exp(-2\gamma\tau)]r^{-3}-\alpha W_0\exp(-2\gamma\tau)r^{-2}, \quad (9)$$

where $\gamma=\Gamma_1\pi x_p^2(0)/2\sigma$ is the scaled absorption coefficient and we have introduced the parameters $\alpha=\mu x_p(0)/2\sigma$ and $\delta=\beta_n/2\sigma$.

Since the coefficients in (9) depend on τ, we took a numerical approach in studying the solutions of this equation. But some preliminary conclusions of a qualitative nature can be drawn.

When there is no dissipation in (9) ($\gamma=0$), for $\delta W_0^2<1$ either the width $x_p(\tau)$ of the solitary wave grows monotonically or $x_p(\tau)$ varies periodically about a steady value that coincides with the soliton width.[16,17]. The absence of damping is a consequence of the approximation we have used: the nonsoliton part of the solution of the NLS equation has been dropped from (4). For $\delta W_0^2\geqslant 1$ both variables on the right-hand side of (9) are negative and the soliton width decreases monotonically, becoming equal to zero after a finite time τ_{col}, i.e., the wave collapses.[23] If we write

$$u_0=1-\delta W_0^2-2\alpha W_0+[\pi B(0)x_p^2(0)]^2,$$

$$a=\frac{\alpha W_0}{|u_0|}, \quad p=\left[1+\frac{|1-\delta W_0^2||u_0|}{(\alpha W_0)^2}\right]^{1/2}\geqslant 1,$$

then the dependence $r=r(\tau)$ can be written parametrically:[23]

$$r=a(1+p\sin\psi), \quad \tau=a|u_0|^{-1/2}(\psi-p\cos\psi)+\tau_0, \quad (10)$$

where τ_0 is chosen to satisfy $r(\tau=0)=1$.

In the parameter space associated with Eq. (9) the surface $\delta W_0^2=1$ separates the range of parameters for which collapse can occur from that for which collapse is impossible. We will call the values of δ and W_0 lying on this surface "critical." For such δ and W_0 the time τ_{col} at which collapse occurs is given by

$$\tau_{\text{col}}=(\pi/2)(2\alpha W_0)^{-1/2}, \quad (11)$$

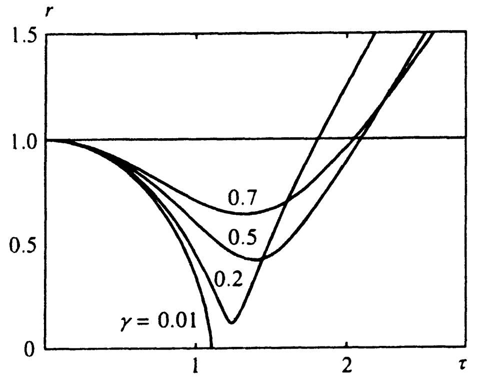

FIG. 1. Dimensionless width of the solitary wave as a function of time for different values of the linear absorption coefficient γ (for $\delta W_0 > 1$).

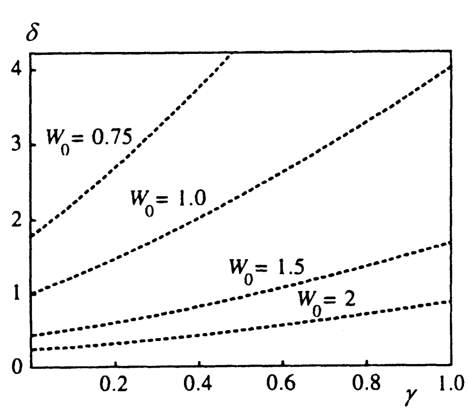

FIG. 2. Calculated $\delta = \delta_c(\gamma)$ for $\alpha = 1$ and different values of the soliton energy W_0.

which follows from (10).

If dissipation is present ($\gamma \neq 0$), then the equation for the critical surface changes. To derive an approximate equation relating the critical parameters, we can argue as follows. In Eq. (9) the first term on the right-hand side (which varies as τ increases) should remain negative until $r(\tau)$ goes to zero. The characteristic time for this process can be taken to be on the order of the time τ_{col} given by (11): $\tau = m\tau_{col}$, where m is a correction factor of order unity, which must be found numerically. Consequently, the approximate equation for the critical surface can be written as

$$1 = \delta\, W_0^2 \exp(-2\gamma m \tau_{col}).$$

Assuming $\gamma \ll 1$ and using (11), we can derive an expression for the critical value $\delta_c = \delta_c(\gamma, W_0)$ or $\gamma_c = \gamma_c(\delta, W_0)$:

$$\delta_c(\gamma, W_0) = W_0^{-2} + (\pi m/\sqrt{2})\alpha^{-1/2} W_0^{-5/2}\gamma, \qquad (12.1)$$

$$\gamma_c(\delta, W_0) = (\sqrt{2}/\pi m)\alpha^{1/2} W_0^{1/2}(\delta W_0^2 - 1). \qquad (12.2)$$

From these expressions it is clear that the energy W_0 of the original solitary wave must exceed a value of order $\delta^{-1/2}$, or the collapse regime will not be reached. But in the presence of damping the energy must be even larger than the value determined by Eqs. (12).

The evolution of the width of the solitary wave as a function of the scaled absorption coefficient γ was also studied via numerical solution of Eq. (9). We took $r = 1$ and $dr/d\tau = 0$ as the initial conditions at $\tau = 0$ and used a fourth-order Runge–Kutta technique.

In the absence of dissipation the solitary wave underwent collapse when its energy exceeded $\delta^{-1/2}$, which is consistent with the known facts. For small γ and sufficiently large energy W_0 the soliton width $r(\tau)$ also decreased to zero. But as the absorption coefficient increased the collapse regime disappeared and the compression of the solitary wave in the initial stage was replaced by unrestricted spreading (Fig. 1). Repeating the numerical solution of Eq. (9) for different choices of the parameters, we find the threshold (critical) value of the scaled absorption coefficient γ_c as a function of δ, W_0, and α.

In the course of our investigation it was found to be more convenient to specify W_0 and α and determine the curve $\delta = \delta_c(\gamma)$ that divides the (δ, γ) plane into a region where collapse takes place and a region where it does not. Under the condition $\alpha = 1$ the function $\delta = \delta_c(\gamma, W_0)$ was plotted for different values of the initial energy W_0. The points lying above the curves in Fig. 2 labeled with values of W_0 belong to the collapse region. It turned out that in the ranges $0 \leq \gamma \leq 1$, $0.2 \leq W_0 \leq 5$ the behavior of $\delta = \delta_c(\gamma, W_0)$ is approximated well by the function

$$\delta_c(\gamma, W_0) = \delta_{c0}(W_0) + \delta_{c1}(W_0)\gamma. \qquad (13)$$

If there is no damping we recover the well known result

$$\delta_{c0}(W_0) = \frac{1}{W_0^2}.$$

The slope $\delta_{c1}(W_0)$ in Eq. (13) decreases as a function of W_0 (Fig. 3), and this behavior can be fitted well by a function of the form

$$\delta_{c1}(W_0) = 1.42 W_0^k$$

with $k = -2.28$. Thus, by using (13) we have derived an approximate formula for the critical curve:

$$\delta_c(\gamma, W_0) = W_0^{-2} + 1.42 W_0^{-2.28}\gamma. \qquad (14)$$

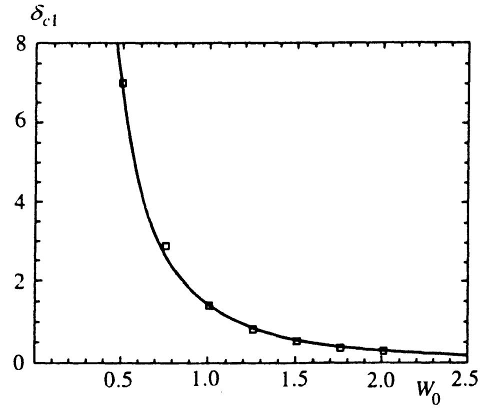

FIG. 3. Dependence of the slope δ_{c1}[Eq. (13)] on the soliton energy for $\alpha = 1$.

Comparing this expression with (12.1), we can discern better agreement between these results by recalling that $\alpha = 1$ was assumed for (14) and taking the correction factor m to be 0.64 in (12.1).

Thus, using either (12.2) or (14) we can find an approximate value of the threshold (critical) scaled absorption coefficient γ_c as a function of the rest of the parameters of the problem, i.e., either

$$\gamma_c(\delta,W_0)=0.704\alpha^{1/2}W_0^{1/2}(\delta W_0^2-1), \qquad (15.1)$$

or

$$\gamma_c(\delta,W_0)=0.704\alpha^{1/2}W_0^{0.28}(\delta W_0^2-1), \qquad (15.2)$$

respectively. When the linear damping is characterized by a coefficient $\gamma \geqslant \gamma_c$, then the solitary wave is not subject to collapse.

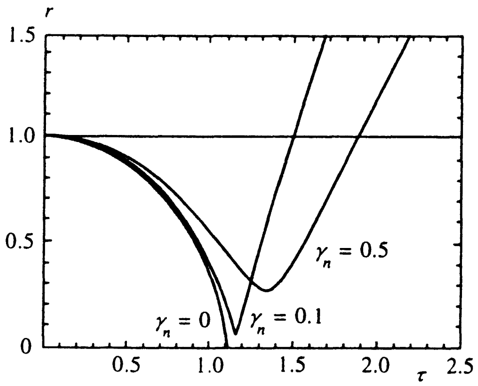

FIG. 4. Dimensionless width of the solitary wave as a function of time for different values of the nonlinear absorption coefficient γ_n.

4. ROLE OF NONLINEAR DAMPING

When the dissipation results from nonlinear damping, assumed to be describable by putting $\Gamma[q]=\Gamma_2|q|^2q$ in Eq. (6), we can derive the following system from Eq. (5):

$$\frac{d(x_pA^2)}{dt}=-\frac{4}{3}\Gamma_2x_pA^4, \qquad (16.1)$$

$$\frac{dx_p}{dt}=4\sigma Bx_p+\frac{4}{\pi^2}\Gamma_2x_pA^2, \qquad (16.2)$$

$$\frac{dB}{dt}=\frac{4\sigma}{\pi^2}(x_p^{-4}-B^2\pi^2)-2\mu\left(\frac{A}{\pi x_p}\right)^2-\frac{2\beta_nA^4}{\pi^2x_p^2}, \qquad (16.3)$$

which now determines the variation of the parameters of the solitary wave. This system is more complicated than (7), so we went immediately to numerical solution in order to determine the critical curve. It is convenient to rewrite Eqs. (16) by introducing variables similar to those used in the previous section. Accordingly, for $W=x_pA^2$, $r=x_p(\tau)/x_p(0)$, and $y=\pi x_p^2(0)B(\tau)$ as functions of $\tau=t[4\sigma/\pi x_p^2(0)]$ we derive the following system of equations from (16):

$$\frac{dW}{d\tau}=-\gamma_nW^2r^{-1}, \qquad (17.1)$$

$$\frac{dr}{d\tau}=yr+\frac{3}{\pi^2}\gamma_nW, \qquad (17.2)$$

$$\frac{dy}{d\tau}=-y^2+[1-\delta W^2]r^{-4}-\alpha Wr^{-3}, \qquad (17.3)$$

where $\alpha=\mu x_p(0)/2\sigma$, $\delta=\beta_n/2\sigma$, and the scaled (nonlinear) absorption coefficient is $\gamma_n=\pi\Gamma_2x_p(0)/3\sigma$.

Numerical solution of this system demonstrated that the soliton can collapse until γ_n exceeds some threshold value, just as in the case of linear damping (Fig. 4). By repeating the numerical solution of (17) for different values of the parameters and the initial soliton energy W_0 we can find this threshold value γ_{nc} of the absorption coefficient as a function of δ, α, and W_0. The search procedure for the function $\delta=\delta_c(\gamma_n,W_0)$ is the same as that described earlier, and for small values of γ_n and W_0 and for $\alpha=1$ we can immediately write down a result analogous to (14):

$$\delta_c(\gamma_n,W_0)=W_0^{-2}+3.84W_0^{-1.28}\gamma_n. \qquad (18)$$

This approximate formula can be derived semi-analytically by arguing as follows. When collapse begins to occur the derivatives satisfy $dr/d\tau<0$ and $|dr/d\tau|\gg 1$ and the second term on the right-hand side of (17.2) is positive, so the first term makes the main contribution to $dr/d\tau$. Moreover, it is negative, as can be seen from Eq. (17.3) close to collapse. In this case Eq. (17.2) can be replaced by an equation similar to (9):

$$\frac{d^2r}{d\tau^2}=[1-\delta W^2(\tau)]r^{-3}-\alpha W(\tau)r^{-2}, \qquad (19)$$

Assuming that the damping is weak, we can find the function $W(\tau)$ approximately from (17.1):

$$W(\tau)\cong W_0-\gamma_nW_0^2\tau, \qquad (20)$$

which can be used to write down an approximate equation for the critical surface:

$$1=\delta W_0^2(1-\gamma_nW_0m\tau_{\mathrm{col}})^2,$$

where m is a correction factor. Assuming $\gamma_nW_0\ll 1$, we can find from this an expression for the critical curve $\delta=\delta_c(\gamma_n,W_0)$ on the (γ_n,W_0) surface:

$$\delta_c(\gamma_n,W_0)\cong W_0^{-2}+(\pi m/\sqrt{2})\alpha^{-1/2}W_0^{-3/2}\gamma_n. \qquad (21)$$

Comparing this expression with (18) we find $m=1.73$ for the correction factor.

An approximate value of the threshold scaled nonlinear absorption coefficient as a function of the parameters of the problem can be obtained either from (19),

$$\gamma_{nc}(\delta,W_0)=0.26\alpha^{1/2}W_0^{-0.63}(\delta W_0^2-1), \qquad (22.1)$$

or from (21),

$$\gamma_{nc}(\delta,W_0)=0.26\alpha^{1/2}W_0^{-1/2}(\delta W_0^2-1). \qquad (22.2)$$

5. CONCLUSION

In this work we have studied the effect of linear and nonlinear absorption on the collapse of a solitary wave that can occur in a one-dimensional nonlinear dispersive me-

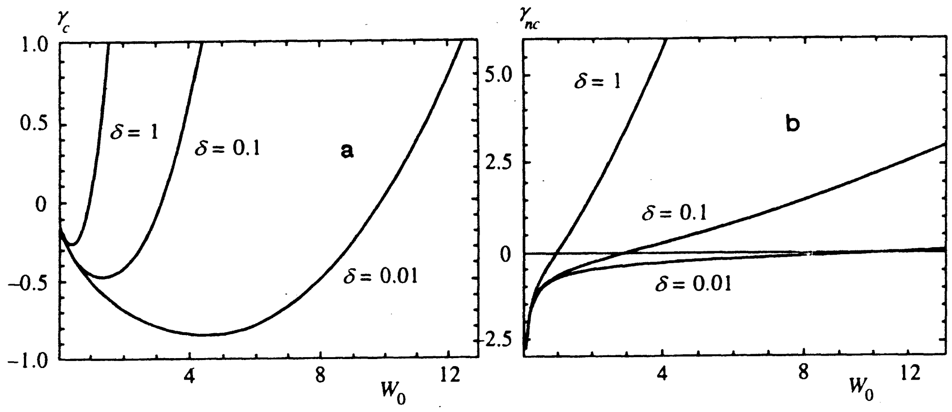

FIG. 5. Critical absorption coefficient as a function of the energy W_0 for $\alpha = 1$: a) linear absorption; b) nonlinear absorption.

dium. In the absence of dissipation this phenomenon is due to nonlinearities of higher order than the Kerr nonlinearity. Rather than solve the partial differential equation directly, we used a set of ordinary differential equations which yields the time dependence of the width of the solitary wave (sometimes called a soliton merely for brevity). This simpler approach enabled us to derive approximate formulas for the threshold values of the scaled absorption coefficients. If the dissipation is characterized by values greater than these coefficients, then there is no collapse and the soliton width increases in time without bound.

Dissipation is not the only mechanism that prevents collapse of a solitary wave. Diffraction or dispersive effects can also enter in this role. But in contrast to the other mechanisms, the one treated here does not depend on the soliton width. This is important.

It should be noted that the time dependence of the dimensionless width of the solitary wave exhibits little qualitative change when we go from linear to nonlinear absorption (compare Figs. 1 and 4). But these two cases lead to very different behavior for the critical absorption coefficient as a function of the initial soliton energy [Figs. 5(a,b)]. This is reflected in Eqs. (15) and (22). The difference is greatest for negative γ and γ_n, i.e., when the solitary wave is growing. In our view this case requires further study, preferably through numerical solution of the NLS equation itself with the correction terms (6).

Analysis of the effect of dissipation on the collapse of two- and three-dimensional solitary waves is also of independent interest. Those results may turn out to be useful for the "optical bullet" theory of Ref. 14, aside from their general physical significance.

[1]G. B. Whitham, *Linear and Nonlinear Waves*, Wiley–Interscience, New York (1974).
[2]B. B. Kadomtsev and V. I. Karpman, Uspekhi Fiz. Nauk **103**, 193 (1971) [Sov. Phys. Uspekhi **14**, 40 (1971)].
[3]V. E. Zakharov, B. V. Sobolev, and V. S. Synakh, Zh. Éksp. Teor. Fiz. **60**, 136 (1971) [Sov. Phys. JETP **33**, 77 (1971)].
[4]N. N. Rozanov and V. A. Smirnov, Zh. Éksp. Teor. Fiz. **70**, 2060 (1976) [Sov. Phys. JETP **43**, 1075 (1976)].
[5]K. Konno and H. Suzuki, Phys. Scr. **20**, 382 (1979).
[6]D. W. McLaughlin, G. C. Papanicolau, C. Sulem, and P. L. Sulem, Phys. Rev. A **34**, 1200 (1986).
[7]L. Gagnon, J. Opt. Soc. Amer. B **7**, 1098 (1990).
[8]M. Karlsson, D. Anderson, M. Desaix, and M. Lisak, Opt. Lett. **16**, 1373 (1991); **17**, 22 (1992).
[9]V. E. Zakharov and A. B. Shabat, Zh. Éksp. Teor. Fiz. **61**, 1188 (1971) [Zh. Éksp. Teor. Fiz. **34**, 62 (1972)].
[10]S. P. Novikov, S. V. Manakov, L. P. Pitaevskiĭ, and V. E. Zakharov, *Theory of Solitons: the Inverse Scattering Method*, Plenum, New York (1984); M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia (1981).
[11]A. R. Its and V. P. Kotlyarov, Dokl. Akad. Nauk UkrSSR A **11**, 965 (1976); Y. C. Ma and M. J. Ablowitz, Stud. Appl. Math. **65**, 113 (1981); E. R. Trecy, H. H. Chen, and Y. C. Lee, Physica D **18**, 481 (1986).
[12]M. J. Ablowitz, A. Ramani, and H. Segur, Lett. Nuovo Cimento **23**, 333 (1979); J. Math. Phys. **21**, 715, 1006 (1980).
[13]M. Boiti and F. Pempinelli, Nuovo Cimento B **59**, 40 (1980).
[14]Y. Silberberg, Opt. Lett. **15**, 1282 (1990).
[15]Yu. S. Kivshar and S. K. Turitsyn, Phys. Rev. E **49** (1994).
[16]D. Anderson, Phys. Rev. A **27**, 3135 (1983); D. Anderson, M. Lisak, and T. Reichel, Phys. Rev. A **38**, 1618 (1988).
[17]R. A. Mishaev and É. Teplitskiĭ, Trans. Inst. Appl. Math. Tbilisi State Univ. **32**, 196; 215 (1989).
[18]B. C. Azimov, M. M. Sagatov, and A. P. Sukhorukov, Kvantov. Elekton. **18**, 104 (1991) [Sov. J. Quantum Electron. **21**, 93 (1991)].
[19]A. I. Maĭmistov, Zh. Éksp. Teor. Fiz. **104**, 3620 (1993) [Zh. Éksp. Teor. Fiz. **77**, 727 (1993)].
[20]V. I. Karpman and E. M. Maslov, Zh. Éksp. Teor. Fiz. **73**, 537 (1977) [Zh. Éksp. Teor. Fiz. **46**, 281 (1977)].
[21]Yu. Kivshar and B. Malomed, Rev. Mod. Phys. **61**, 763 (1989).
[22]Y. Silberberg, Opt. Lett. **15**, 1005 (1990).
[23]L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd ed., Pergamon, Oxford (1976).

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