Theory of elementary atomic processes in an ultracold plasma

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Recombination mechanisms in an ultracold hydrogen (e^-p^+ or e^+p^-) or electron-positron plasma ($T_e \lesssim 100$ K, $N_e \sim 10^8$ cm^{-3}) produced in experiments with magnetic traps are considered. The recombination time in this case is determined by the deexcitation of an atom in collisions with electrons, not by initial capture into the bound state. In a sufficiently strong magnetic field the recombination time increases as the square of the field strength, which is explained by the magnetization of the electrons, as a result of which their motion across the magnetic field is inhibited. It is shown that the external electric field does not affect the recombination time. The possibility of experimentally verifying these effects is discussed. © *1995 American Institute of Physics.*

1. INTRODUCTION

The elementary atomic processes in an ultracold plasma have attracted considerable attention in connection with the possibility which has recently arisen of experimentally producing cold antihydrogen atoms and confining them in Penning magnetic traps.[1] This opens up prospects for an additional way of verifying the invariance of the electromagnetic interaction with respect to charge conjugation, and also for verifying the equivalence principle for antimatter.

Low temperatures allow one to compare the spectra of hydrogen and antihydrogen atoms with high precision ($\sim 10^{18}$–10^{15}),[2] i.e., to test charge symmetry. Measurement of the hyperfine splitting of the hydrogen ground state permits the proton magnetic moment to be determined with a precision of 10^{-8}. If measurements can be made with the same precision for antihydrogen, then the magnetic moment of the antiproton will be found with an error five orders of magnitude smaller than in accelerator experiments.

Experiments in the spirit of the Eötös experiment leave practically no doubt about the validity of the weak equivalence principle for matter; however, no tests have been carried out thus far for antimatter. Recently theories have been proposed according to which particles and antiparticles undergo different accelerations in a gravitational field. Charged-particle experiments cannot achieve the necessary precision, since the slightest parasitic electromagnetic fields strongly distort the trajectories of such particles. The experiment with antihydrogen is free of these difficulties. An even more precise test of the equivalence principle results from comparing the gravitational red shift for hydrogen and antihydrogen. Such tests can of course be carried out with positronium or muonium, but they are of somewhat less value since these systems consist exclusively of leptons.

Antihydrogen atoms have been produced[3,4] by stopping previously slowed antiprotons in a positron cloud confined in a magnetic trap, with subsequent recombination. Antiprotons with an energy of 5.9 MeV from the accelerator at CERN are slowed down in matter[3] to an energy of 3 keV, trapped in a hybrid Penning magnetic trap, and thermalized in a cloud of cold positrons to an energy of 1 meV. The antiproton confinement time at this energy is several months.[5] A Penning trap consists of a region filled with magnetic field and a superposed electrostatic quadrupole potential. The trapped particles gyrate on cyclotron orbits, oscillate in the direction of the magnetic field, and slowly drift in the crossed magnetic and electric fields (for more detail see Ref. 1). The antihydrogen atoms resulting from recombination of e^+ and p^- are confined in the same trap due to the magnetic moment of the atom. As an atom moves its spin rotates adiabatically in the direction of the magnetic field, so the potential energy of the atom is equal to $-\mu_B H$ if the positron spin is parallel to the field and $\mu_B H$ if it is antiparallel. [Because of the slow variation in the direction of the magnetic field in the trap the spin wave function of the electron in an atom at any time is an eigenfunction of the Hamiltonian $2\mu_B \sigma \mathbf{H}(t)$ to a good approximation.] The magnetic field in the center of the Penning trap has a minimum, so atoms with antiparallel spin are confined in the trap while those with parallel spin are expelled. For the trapped fields $H \sim 5$T that can be achieved in the laboratory the atoms are confined with a temperature $T \lesssim 5$ K. For recombination studies at the present time a device is used consisting of two coaxial Penning traps which separately confine antiprotons and positrons.[6] Variation of the potential in the antiproton part creates conditions which enable the antiprotons to get over the potential barrier by means of collisions and reach the positron part of the device with a small kinetic energy, whereupon recombination occurs.

To determine the optimum positron densities in the trap one needs to know the recombination time. Recombination, in turn, proceeds differently for different gas temperatures, as will be shown below.

In order to use more familiar language, in what follows we will talk about an ordinary hydrogen (e^-p^+) plasma.

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The theory is readily generalized to the case of a nucleus with charge state greater than unity.

In the second section we treat the collisional stage in an ultracold plasma with no external fields. It is shown that in addition to inelastic collisions, collisions in which the angular momentum changes with no accompanying change in the energy of the atom are important (the so-called Stark mixing; see Appendix). We calculate the recombination time. In the third section recombination in a strong magnetic field is studied, and in the fourth section the effect of an electric field is taken into account.

In the calculations we use atomic units ($\hbar = m_e = e = 1$).

2. RECOMBINATIONS IN THE ABSENCE OF EXTERNAL FIELDS

This section is devoted to recombination kinetics in a plasma with parameters typical of a magnetic trap ($N_e = 10^8$ cm^{-3}, $T = 4K$, $N_p = 10^4 - 10^8$ cm^{-3}).[3,5]

Recombination takes place in two stages. First the electron is captured in a highly excited orbit with binding energy on the order of the temperature according to the Thomson scheme:

$$p + e + e \to H_{n_0 l} + e, \tag{1}$$

into states with principal quantum numbers $n \sim n_0$ and orbital quantum numbers $l \sim n_0$. The magnitude of the initial quantum number n_0 can readily be estimated from the relationships

$$\frac{1}{2n_0^2} \sim T, \quad n_0 \sim \frac{1}{v_T} = 300,$$

where $v = \sqrt{8T/\pi}$ is the electron thermal velocity.

Since we have $n \gg 1$ and $l \gg 1$, in order to describe the recombination process and the subsequent "motion" of the electron through energy levels (with $n \gg 1$) a purely classical treatment is applicable. An estimate of the time for this capture is obtained from the following well-known simple argument.[7] In order that one of the two colliding electrons be left in a bound state, it is necessary that the collision take place at a distance R_T from the proton no greater than

$$\frac{1}{R_T} \sim T.$$

The reciprocal of the capture time $1/\tau_T$ is the product of the probability w for one of the electrons to be within a sphere of radius R_T about the proton,

$$w = \frac{4}{3}\pi R_T^3 N_e$$

times the frequency with which an electron passes through the Thomson sphere:

$$\frac{1}{\tau_T} \sim w N_e v_T \pi R_T^2 \sim \frac{N_e^2}{T^{9/2}}. \tag{2}$$

At low temperatures the capture time is short. Thus, for the conditions specified above it is equal to 10^{-6} s.

After the initial capture the bound electron begins to move down through the energy levels as a result of collisions with electrons and radiative transitions, i.e., through a multistage deexcitation process. In the upper levels ($10 \leqslant n \leqslant n_0$) collisional deexcitation dominates over radiative (see below), while for $n < 50$ radiative relaxation dominates. There is some risk[1)] that if an electron gets into circular (i.e., with $l = n - 1$) excited orbits it will remain there for an extended time, since the radiative lifetime of circular orbits is[8]

$$\tau_{cir} = 0.3 \cdot 10^{-9} n^5 \text{ s}, \tag{3}$$

which is large in comparison with the time for the $np \to 1s$ transition:[8]

$$\tau_{np \to 1s} = 10^{-10} n^3 \text{ s}, \tag{4}$$

and according to the selection rules only a transition in which the principal quantum number decreases by unity is possible from a circular orbit into a lower-lying circular orbit, $|n, l = n-1\rangle \to |n' = n-1, l' = n'-1\rangle$). The physical reason for the long radiative lifetime of an electron in circular orbits is that the acceleration experienced by an electron in such orbits is small.

It turns out, however, that there exists an exceptionally fast mechanism (Stark mixing in angular momentum) which eliminates this delay. As shown in the Appendix, the electric field of electrons passing at large distances from an atom induces precession of the orbit of the atomic electron due to collisions between the atom and the electrons. Then the atomic distribution function $f(l)$ over orbital angular momentum l:

$$f(l) = \frac{2l+1}{n^2} \tag{5}$$

becomes equal to the statistical weight after a typical time:

$$\tau_{st} \sim \frac{\sqrt{T}}{N_e n^2}. \tag{6}$$

Then states with different values of l and m and fixed n become equally probable. The time τ_{st} is the shortest time in the problem, and for the specified values of density and temperature is equal to $1.5 \cdot 10^{-2} n^{-2}$ s.

The angular relaxation time (6) is shorter than the lifetime (3) of a circular orbital if $n \geqslant 9$ holds and shorter than the lifetime of the np state if $n \geqslant 13$. Consequently, the radiative deexcitation rate from levels with $n \geqslant 13$ is equal to the probability of being in the state np [for the distribution function (5) this is $3/n^2$], by inverse time (4):

$$\tau_{n \to 1s}^{rad} = 3 \cdot 10^{-10} n^5 \text{ s}. \tag{7}$$

Thus, for values of n that are not too small a very fast mechanism operates that produces equilibrium inside a single energy level and does not allow long-lived states with circular orbits to survive.

Let us calculate the rate at which the atomic electron moves down through the levels as a result of collisions. For sufficiently large principal quantum numbers $n > 50$ inelastic processes dominate radiative processes. This means that the limiting (slowest) stage of recombination is the motion of the electron through levels with $n \gg 1$, i.e., to determine the typi-

cal recombination time a classical treatment is completely applicable (the range of applicability of the results is specified in detail in the Appendix).

The classical equations of motion turn out to be too complicated for analytical solution, except for the important case of distant collisions with impact parameters $\rho \sim \rho_0 = n/v_T$, for which the energy of the atom is an adiabatic invariant and is therefore conserved.

For $\rho < \rho_0$ the electron can fall to the atom (in the Appendix such collisions are called hard). This gives rise to effective exchange of energy between electrons, after which the outgoing electron carries away an energy on the order of the binding energy in the original atom.

The expression for the motion of the atom through the energy scale that results from these inelastic collisions takes the form

$$\dot{E} = \left\langle \frac{dE}{dt} \right\rangle = \sum_i \nu_i \Delta E_i, \tag{8}$$

where the subscript i labels the intervals into which the scale is broken by changes ΔE in the atomic energy, and the sum is over these intervals with a weight equal to the frequency ν_i of the corresponding inelastic collisions. For hard collisions (distant collisions contribute almost nothing to $\dot{E}$) we have

$$\Delta E \sim -\frac{1}{n^2}$$

and the frequency is

$$\nu \sim \pi \rho_0^2 N_e v_T.$$

Hence

$$\dot{E} = -\frac{CN_e}{v_T}, \tag{9}$$

where C is a coefficient of order unity and $v = (8T/\pi)^{1/2}$.

To find C numerically we solve the system of equations of motion:

$$\begin{aligned} \ddot{\mathbf{r}} &= -\frac{\mathbf{r}}{r^3} - \frac{\mathbf{R}-\mathbf{r}}{|\mathbf{R}-\mathbf{r}|^3}, \\ \ddot{\mathbf{R}} &= -\frac{\mathbf{R}}{R^3} + \frac{\mathbf{R}-\mathbf{r}}{|\mathbf{R}-\mathbf{r}|^3}. \end{aligned} \tag{10}$$

Here $\mathbf{r}$ and $\mathbf{R}$, respectively, are the radius vectors of the atomic and incident electron, and the proton is assumed to be at rest at the origin. The velocities of the electrons incident on the atom were chosen randomly according to a Maxwell distribution, and the atom was "prepared" in a state with a specified energy but with random values of l and m, in view of the high rate of Stark mixing mentioned above.

Figure 1 shows the collision frequency ν_i (in arbitrary units) as a function of the energy ΔE_i transferred to the atom in a collision with an electron. The initial atomic energy in the figure corresponds to $n = 50$ and is equal to -0.0002 a.u. The sharp peak associated with zero energy transfer results from distant collisions and does not contribute to the sum (8). On the left can be seen "hard" collisions, for which the energy transfer is on the order of the initial atomic energy.

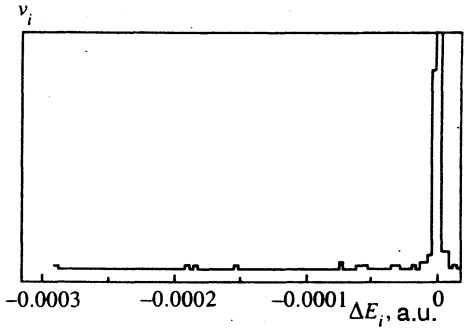

FIG. 1. Collision frequency as a function of energy transfer.

The numerical calculation yielded $C \approx 0.2$. For the parameters given above the rate of collisional relaxation (9) is equal to 24 a.u. per second.

Thus, recombination in an ultracold plasma takes place in the following way. First the electron is captured rapidly (after a typical time 10^{-6} s) into a bound state by the Thomson mechanism. Then deexcitation begins through inelastic processes with a rate

$$\dot{E} = -0.2 \frac{N_e}{v_T}, \tag{11}$$

where the angular momentum distribution is close to the equilibrium distribution (5).

In lower levels ($n \lesssim 10$) the deexcitation mechanism changes: there it occurs as a result of radiative processes taking place over the time (7). Thus, the total deexcitation time is given by

$$\tau \approx \tau_{\text{col}} = \frac{1}{2|\dot{E}|},$$

and under the experimental conditions is equal to 10^{-5} s. From this it can be seen that in an ultracold plasma the limiting (slowest) stage of recombination is deexcitation, not the initial capture into a bound state as in an ordinary plasma.

3. RECOMBINATION IN EXTERNAL FIELDS

Now we show that the strong magnetic field in the trap qualitatively changes the picture of recombination.

Consider the typical case in a magnetic trap, when the magnetic field is large and the electron Larmor radius r_{He} is small in comparison with the characteristic size R_T of the problem:

$$\frac{r_{He}}{R_T} \sim \frac{T^{3/2} C}{H} \ll 1, \tag{12}$$

or $T \ll 60 H^{2/3}$, where T is in kelvin and H in tesla. In a typical trap with $H = 5$ T this condition therefore takes the form $T \ll 200$ K. In this magnetic field at a temperature $T = 4$ K the Thomson radius is $R_T \sim 10^5$ a.u. and the Larmor radius is $r_{He} \sim 200$ a.u.

We write down equations of motion for a proton and electron in a uniform external magnetic field:

$$\ddot{\mathbf{r}}_e=-\frac{\mathbf{r}}{r^3}-\frac{1}{c}[\dot{\mathbf{r}}_e\mathbf{H}],$$
$$M\ddot{\mathbf{r}}_p=\frac{\mathbf{r}}{r^3}+\frac{1}{c}[\dot{\mathbf{r}}_p\mathbf{H}], \tag{13}$$

where M is the proton mass in atomic units, $\mathbf{r_e}$ and $\mathbf{r_p}$ are the radius vectors of the electron and proton, respectively, and we have written $\mathbf{r}=\mathbf{r_e}-\mathbf{r_p}$. We introduce the new variable

$$\mathbf{R}_0=\frac{M\mathbf{r}_p+\mathbf{r}_e}{M+1},$$

the coordinate of the atomic center of mass. We rewrite the system of equations (13):

$$(M+1)\ddot{\mathbf{R}}_0=\frac{1}{c}[\mathbf{rH}],$$
$$\ddot{\mathbf{r}}=-\frac{M+1}{Mr^3}\mathbf{r}-\frac{M+1}{Mc}[\dot{\mathbf{R}}_0\mathbf{H}]-\frac{M-1}{Mc}[\dot{\mathbf{r}}\mathbf{H}].$$

The first equation can be integrated directly:

$$(M+1)\dot{\mathbf{R}}_0=\frac{1}{c}[\mathbf{rH}]+(M+1)\mathbf{V},$$

where $\mathbf{V}$ is a constant vector. We substitute this solution into the second equation and obtain an equation for the relative motion:

$$\ddot{\mathbf{r}}=-\frac{M+1}{Mr^3}\mathbf{r}-\frac{1}{Mc}\left[\left(\frac{1}{c}[\mathbf{rH}]+(M+1)\mathbf{V}\right)\mathbf{H}\right]$$
$$-\frac{M-1}{Mc}[\dot{\mathbf{r}}\mathbf{H}]. \tag{14}$$

For the atom in a bound state $\langle\mathbf{r}\rangle=0$ holds (the angle brackets denote a time average), so we have $\mathbf{V}=\langle\dot{\mathbf{R}}\rangle$. From this the physical meaning of the vector $\mathbf{V}$ is clear: it is the velocity with which the atom moves and is constant in a magnetic field, since the atom is neutral. We go over to a coordinate frame moving with the velocity $\mathbf{V}$. Then in addition to the magnetic field in this frame there will be a transverse electric field

$$\mathbf{E}=\frac{1}{c}[\mathbf{VH}], \tag{15}$$

which explains the meaning of the term with $\mathbf{V}$ in Eq. (14). This electric field will alter the motion of the electron in the atom, but if it is small compared with the characteristic proton field $\sim n^{-4}$ (which means that the velocity $\mathbf{V}$ of the atom is small), then the atom will remain bound. It is obvious that $\mathbf{V}\sim v_i$ holds (here v_i is the ion thermal velocity), from which it follows that under the conditions specified the motion of the atom can be disregarded for $n<150$, which we will assume in what follows. In this connection we note that the electric field actually has no effect on the recombination time (see the next section). Thus, in what follows we take $\mathbf{V}=0$.

In the approximation of an infinitely massive proton we have from (14)

$$\ddot{\mathbf{r}}=-\frac{\mathbf{r}}{r^3}-\frac{1}{c}[\mathbf{rH}]. \tag{16}$$

Thus far we have made no use of the fact that the Larmor radius (12) is relatively small. If $\mathbf{H}$ is large, then the motion across the field is a drift and is associated with the velocity

$$\mathbf{v}_\perp=\frac{c}{H^2}[\mathbf{EH}],$$

where $\mathbf{E}$ is the electric field strength. The motion in the direction z of the magnetic field is given by the equation

$$\ddot{z}=-\frac{z}{r^3},$$

which follows from (16). This allows us to consider the system of equations in the drift approximation, which is much simpler than (16):

$$\dot{\mathbf{r}}_\perp=\frac{c}{r^3H^2}[\mathbf{rH}],$$
$$\dot{z}=-\frac{z}{r^3}. \tag{17}$$

Thus, the electron moves along the magnetic field line (to which it is "frozen") in an effective potential well produced by the attraction of the proton. In addition, it performs a slower drift motion rotating about the proton. The transverse separation between the particles does not change, and in the absence of collisions deexcitation occurs only as a result of radiation, i.e., it takes place exceedingly slowly. As it approaches the proton the electron orbit becomes highly elongated in the longitudinal direction and the radiative lifetime decreases, since the electron is subjected to greater acceleration.

As in the absence of a field, recombination occurs in two stages. First an electron from the plasma is captured into a weakly bound state in the Coulomb potential well on a field line in a process completely analogous to Thomson capture. The capture time τ_{Th} is given by the Thomson formula, in which the magnetic field only changes the numerical coefficient:[9]

$$\frac{1}{\tau_{Th}}\sim\frac{N_e^2}{T^{9/2}}.$$

Then the extended stage of collisional motion across H starts. We note that Glinsky and O'Neil[9] did not take this stage into account, although in our case it is the limiting factor. As a result of collisions with plasma particles the atomic electron undergoes random walk in the plane perpendicular to the magnetic field with a step size $\sim r_{He}$. The expression for the transverse diffusion coefficient $D_\perp$ was derived by Belyaev[10] in the general case for electron-ion collisions. The collision frequency of a relaxing bound electron with protons is always smaller than the electron-electron collision frequency due to the large difference in the thermal velocities of protons and electrons. This means that the diffusion coefficient of interest to us for the bound electron is mainly determined by collisions with plasma electrons. Because of the Coulomb repulsion two charged particles of the same sign cannot approach one another closer than the Thomson radius. Hence we conclude that electron collisions will be mostly distant. At such separations the particle mo-

tion is given by drifts. Moreover, as will be seen below, the main contribution to the diffusion coefficient comes from collisions with impact parameters much larger than the Thomson radius.

The electron collision time τ depends on the relative particle velocity v parallel to the magnetic field, while the drift velocity is given by the well-known formula

$$\mathbf{v}_D=\frac{c[\mathbf{EH}]}{H^2}=\frac{c[\boldsymbol{\rho}\mathbf{H}]}{(\rho^2+z^2)^{3/2}H^2},$$

where ρ is the transverse separation and z is the longitudinal separation between the particles. Assuming in advance that the impact parameters are large, we set $z\approx v_z t$, where v_z is the relative electron velocity in the z direction. The drift displacement of a particle over the time of such a collision is given by the expression

$$\delta\boldsymbol{\rho}_\perp=\int_{-\infty}^{\infty}\mathbf{v}_D dt,$$

from which we find

$$\delta\rho_\perp\frac{2c}{\rho H v}\sim v_D\tau,$$

where here and below the subscript "z" in v_z is omitted. The coefficient $D_\perp$ is by definition equal to[10,11]

$$D_\perp=\frac{1}{4}\langle\delta\rho_\perp^2\nu_{ee}\rangle,$$

where ν_{ee} is the collision frequency and the additional factor 1/2 results from the two-dimensionality of the transverse motion. We rewrite the last expression, substituting into it an estimate for $\delta\rho_\perp$ and averaging over the electron Maxwellian:

$$D_\perp\sim\int 2\pi\rho d\rho\int dv N_e v\frac{c^2}{\rho^2H^2v^2}\frac{e^{-v^2/2T}}{\sqrt{2\pi T}}$$

$$=\frac{\sqrt{2\pi}N_e c^2}{\sqrt{T}H^2}\int\frac{d\rho}{\rho}\int_{1/\sqrt{\rho}}^{\infty}\frac{dv}{v}e^{-v^2/2T}.$$

The lower limit in the integration over velocities is determined by the smallest velocity for which approach to a distance ρ is possible. Integrating once by parts we find a logarithmically divergent integral:

$$D_\perp=\sqrt{\frac{\pi}{2T}}\frac{N_e c^2}{H^2}\int_{\rho_{\min}}^{\rho_{\max}}\frac{\ln\rho d\rho}{\rho}e^{-1/2\rho T}.$$

Choosing as the minimum and maximum distances, respectively, the Thomson radius and the mean interparticle separation, we can evaluate the diffusion coefficient with logarithmic accuracy:

$$D_\perp=\sqrt{\frac{\pi}{2T}}\frac{N_e c^2\Lambda^2}{H^2},$$

where $\Lambda=\ln(T/N_e^{1/3})$ is the Coulomb logarithm. The mobility $b_\perp$ is obtained from the Einstein relation:

$$b_\perp=\frac{D_\perp}{T}=\sqrt{\frac{\pi}{2T^3}}\frac{c^2N_e\Lambda^2}{H^2}. \tag{18}$$

We enclose the atom with a cylindrical surface Σ with directrix parallel to the magnetic field. Plasma electrons move toward the atom from all directions. The current is determined by the mobility and is equal to

$$\mathbf{j}_\perp=N_e b_\perp\mathbf{E}=b_\perp N_e\frac{\mathbf{r}_\perp}{r^3}.$$

The number of particles per unit time intersecting the surface is given by

$$\frac{1}{\tau_{\text{col}}}=\oint(d\boldsymbol{\Sigma},\mathbf{j})=N_e b_\perp\oint d\boldsymbol{\Sigma}\cdot E_\perp,$$

since the magnetic field is perpendicular to the element of the surface chosen. Using Gauss's theorem to transform the integral, we arrive at the final expression for the deexcitation time:

$$\frac{1}{\tau_{\text{col}}}=4\pi N_e b_\perp,$$

i.e., the time τ_{col} for the collisional stage with mobility (18) is equal to

$$\tau_{\text{col}}=\sqrt{\frac{T^3}{(2\pi)^3}}\frac{H^2}{N_e^2c^2\Lambda^2}. \tag{19}$$

For the values of the plasma temperature and density and field strength given at the beginning of this paper we find

$$\tau_{\text{col}}\sim 0.1\,\text{s}.$$

It is clear that $\tau_{\text{col}}\gg\tau_{\text{Th}}$. Thus, collisional excitation, i.e., infrequent jumps of an electron from one magnetic field line to another approaching the proton, is the limiting stage and hence the recombination time is determined by the time (19) of the collisional stage.

4. EFFECT OF AN ELECTRIC FIELD ON THE RECOMBINATION PROCESS

Magnetic traps can have an electric field $\mathbf{E}\sim 100$ V/cm, consisting of a combination of the external field and the electric field resulting from redistribution of charge in the plasma.

In such a plasma the projection of the electric field on the direction of the magnetic field is obviously close to zero, since the electrons move without restriction in this direction, so that this component of the field damps out over distances on the order of the Debye radius r_D. At the same time the magnetic field in the trap is quite large, so the electrons are "magnetized," i.e., their motion across the magnetic field is inhibited. This means that the component of the electric field perpendicular to the magnetic field is not screened out. In the present work we consider the case of a dense plasma with dimension l sufficiently large that $l\gg r_D$ holds. This means that it is necessary to treat the problem of recombination in strong crossed magnetic $\mathbf{H}$ and electric $\mathbf{E}$ fields.

In crossed fields the magnetized plasma drifts as a whole with velocity $\mathbf{V}_D=c[\mathbf{EH}]/H^2$, where $\mathbf{E}$ is the electric field in the plasma. This description of the motion of the plasma particles refers only to scales much larger than the average distance $n^{-1/3}$ between plasma particles. For recombination

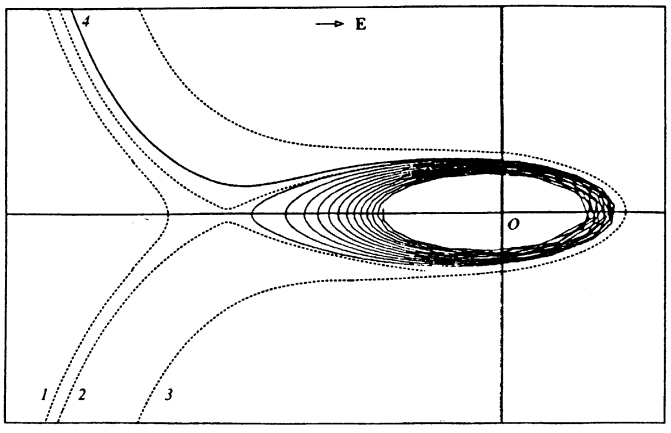

FIG. 2. Electron trajectories in the field of a proton (point O) and external crossed electric and magnetic fields (the latter is directed perpendicular to the plane of the figure toward the reader and is not shown): *1–3*) trajectories in the zero-mobility approximation; *4*) the trajectory that produces recombination.

the much smaller separations between electron and proton are important. At these distances the electric field of the other plasma particles can be disregarded in comparison with the field $\mathbf{E}$ and the field $\mathbf{r}/r^3$ of the proton. For the recombination process the relative electron-proton motion at distances $r \ll n^{-1/3}$ is important. The relative motion and the motion of the center of mass of this pair of particles are separated in analogy with the previous section. In the drift approximation ($r_H \ll r_n \sim n^2$) the equation of the relative electron-proton motion takes the form [cf. Eq. (17)]

$$\dot{\mathbf{r}}_\perp = \frac{c}{H^2}[\mathbf{E}_t\mathbf{H}] - b_\perp(\mathbf{E}_t)_\perp, \tag{20}$$

where $\mathbf{E}_t = \mathbf{r}/r^3 + \mathbf{E}$. The last term in (20) describes the collisional drift across the magnetic field. The second equation in (17), which describes the electron motion parallel to the magnetic field, remains unchanged. The term with $b_\perp$ in (20) is small and can be dropped in lowest order. Thus for $t \to -\infty$ the electron describes only a drift motion with velocity $\mathbf{v}_D = c[\mathbf{EH}]/H^2$. Figuratively speaking, the electric field $\mathbf{E}$ induces a "drift wind" in the electron component, which blows on the proton.

We shift the proton to the origin, and orient the z axis parallel to $\mathbf{H}$ and the y axis parallel to $\mathbf{E}$. Then the velocity $\mathbf{v}_D$ is in the x direction. Let us consider first the simplest case, in which the motion is confined to the $y=0$ plane (Fig. 2). In the zero-mobility approximation ($b_\perp = 0$, shown by the dots in Fig. 2) there can be no recombination (the energy of the system is greater than zero and in the absence of friction is conserved), and within the region bounded by trajectory 2 (the separatrix) there are no trajectories arriving from infinity. The size r_0 of this "forbidden" zone is determined by equating the external field to the Coulomb field of the proton:

$$\frac{1}{r_0^2} \sim E,$$

i.e., $r_0 \sim 1/\sqrt{E}$.

An electron can only penetrate into the forbidden region as a result of collisions with other electrons of the plasma. One of these "trapped" trajectories is shown by the heavy trace 4 in Fig. 2. Since the mobility is small, trajectories like these that penetrate into the forbidden region are concentrated first near the critical trajectory 2. Hence it would seem that the capture cross sections σ_c should be small, since at first glance all trapped trajectories should be localized within a thin tube surrounding the critical trajectory. However, numerical calculations reveal that recombination in crossed fields is not that simple. It was found that the main contribution to the capture cross section comes from trajectories lying outside the $y=0$ plane, at a considerable distance ($\sim r_0$) from the critical trajectory. As they move along these trajectories, electrons oscillate about the $y=0$ plane and at the same time drift in the x direction. As a result of these oscillations the trajectories, as it were, "embroider" the forbidden region; because of friction they are effectively slowed down and trapped. The capture cross section σ_c was determined by the Monte Carlo method. The results of the calculation can be approximated by the expression

$$\sigma_c = kr_0^2,$$

where the factor k is essentially constant; it changes from 0.6 to 0.9 as the fields E and H vary over a wide range, $E<300$ V/cm and $H=1-10$ T. The capture time τ_c is determined from the expression

$$\frac{1}{\tau_c} = N_e v_D \sigma_c = k\frac{N_e c}{H},$$

and for typical trap parameters is equal to 10^{-5}–10^{-4} s.

Capture into the forbidden region is the first and fastest stage of recombination in crossed fields, analogous to the Thomson stage for $E=0$. Then begins the longest (limiting) stage of recombination, the collisional drift of electrons toward the proton across the magnetic field lines. This process in no way differs from that treated in the previous section, since in the region $r \ll r_0$ the external electric field can be disregarded in comparison with the proton field. Consequently, the duration of this stage, and hence of the whole recombination process in crossed fields, is determined by the previous expression (19). Hence we conclude that even though the external electric field has a significant effect on the electron motion, it does not change the recombination time.

5. CONCLUSION

Thus, in this work we have developed a theory for recombination in an ultracold plasma in sufficient detail to optimize experiments (see Introduction). The recombination process in an ultracold plasma has a number of interesting features which distinguish it from recombination in an ordinary plasma. In the absence of external fields the limiting stage is that of collisional deexcitation rather than Thomson capture of an electron into the bound state. The magnetic field introduces an interesting effect. If the electron Larmor radius is less than the Thomson radius, then collisional de-

excitation is inhibited. As a result, the plasma recombination time increases in proportion to the square of the magnetic field and even in fields 5–10 T of laboratory scale can amount to times of order seconds, which may find practical application. Note, e.g., that the magnetic field can inhibit annihilation in an ultracold electron-positron plasma. Another conclusion derived in this work is that an electric field of reasonable magnitude does not change the recombination time.

To verify these effects we can propose two kinds of experiments. In the first case two γ waves are detected, resulting from annihilation of positrons stopped in an electron plasma. Annihilation "on the wing" is highly improbable. It occurs primarily in the $1s$ state, so the γ-ray counting rate will decrease inversely with the square of the magnetic field strength. We can also detect radiated photons from the $2p \rightarrow 1$ transition in an experiment with a hydrogen plasma. For this an electron cloud must be created in the magnetic trap and a small number of protons must be "planted" into the electrostatic potential well it creates.

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APPENDIX
STARK MIXING RATE

As mentioned Sec. 2, for $n \gg 1$ the electron motion in the atom is classical. In the classical problem of the collision of three particles the recombining $[e+p+e \rightarrow (pe)_n+e]$ electron "gets on" the classical bound orbit, which is a Kepler ellipse. For this orbit the time-averaged dipole moment of a hydrogen atom is nonzero, so the interaction of an electron with such an atom is long-range (charge–dipole), which introduces a number of interesting features into the process of an electron-atom collision. In particular, it will be shown below that the motion of all particles in this case is completely classical, which allows the problem to be solved in its entirety.

To avoid misunderstandings, we add that the most adequate quantum description of electron capture into an elliptical orbit is the transition from the continuous spectrum to a state of the parabolic basis, for which, as is well known,[12] the average dipole moment of the atom is nonzero. Below we will use the simpler classical description.

A review of work close to this topic is given in Ref. 13. The techniques used below to solve the classical equations of motion date back to Refs. 14 and 15, and in all probability go back to the work of Laplace on perturbations of planetary orbits. Stark transitions were probably first treated by Bethe and Leon.[16] In Ref. 17 it was shown that the quantum-mechanical calculation used in Ref. 16, based on the Born approximation, is inappropriate for this problem, and a more rigorous calculation of the cross sections for the Stark transitions was carried out. Below we will present a rigorous calculation, free of the restrictions assumed in Ref. 17.

Consider collisions between cold electrons and a Rydberg atom. From the very start we will restrict our treatment to the range of greatest interest:

$$1 \ll n \ll n_0 \sim 1/v = 300, \quad l \gg 1. \tag{21}$$

In this case the motion of the atomic electron is classical, and the incident particle moves in the field of a point dipole (with dipole moment n^2 in atomic units), since the impact parameter ρ is much larger than the atomic dimension r_{orb}. Specifically, the typical value of ρ can be obtained most simply from the estimate

$$T \sim \frac{n^2}{\rho^2} \text{ or } \rho \sim \frac{n}{v}. \tag{22}$$

Then by virtue of (21),

$$\frac{\rho}{r_{\text{orb}}} \sim \frac{1}{vn} = \frac{n_0}{n} \gg 1.$$

The angular relative motion can be treated classically, because the corresponding angular momentum ρv is large in accordance with (21) and (22):

$$\rho v \sim n \gg 1.$$

The de Broglie wavelength $\lambda = 1/v$ of the incident particle is also small:

$$\frac{\lambda}{\rho} \sim \frac{1}{n} \ll 1,$$

i.e., the radial relative motion is also classical. The motion of the atomic electron is fast in comparison with the relative motion of the incident electron. Specifically, the rotation period of the atomic electron is $\tau_{\text{orb}} \sim n^3$, and the collision time is $\tau_{\text{col}} \sim \rho/v = n/v^2$. Hence the relative motion (i.e., the motion of the particle passing by) is slow:

$$\frac{\tau_{\text{orb}}}{\tau_{\text{col}}} \sim (vn)^2 = \left(\frac{n}{n_0}\right)^2 \ll 1$$

in consequence of (21). This also implies that for distant collisions the probabilities of inelastic transitions are small, since the corresponding Massey parameters is large:

$$\xi = \tau_{\text{col}} \delta \mathrm{E} \sim \frac{\tau_{\text{col}}}{\tau_{\text{orb}}} \gg 1.$$

We displace the proton to the origin of the coordinate system. Let $\mathbf{r}$ and $\mathbf{R}$ be the radius vectors of the atomic and incident electrons, respectively. The equations of motion take the form

$$\ddot{\mathbf{r}} = -\frac{\hat{\mathbf{r}}}{r^2} - \frac{\hat{\mathbf{R}}}{r^2}, \quad \ddot{\mathbf{R}} = \frac{3\hat{\mathbf{R}}(\hat{\mathbf{R}}\mathbf{r}) - \mathbf{r}}{R^3}, \tag{23}$$

where $\hat{\mathbf{R}} = \mathbf{R}/R$ and $\hat{\mathbf{r}} = \mathbf{r}/r$.

We write the atomic angular momentum as $\mathbf{l} = [\mathbf{rr}]$ and the Runge–Lenz vector as $\mathbf{A} = \hat{r} + [\mathbf{lr}]$. Differentiating $\mathbf{l}$ and using Eqs. (23) we find

$$\dot{\mathbf{l}}=[\mathbf{r}\ddot{\mathbf{r}}]=\frac{1}{R^2}[\hat{\mathbf{R}}\mathbf{r}].$$

In one period τ_{orb} the angular momentum changes by

$$\delta\mathbf{l}=\int_0^{\tau_{orb}}dt\dot{\mathbf{l}}\approx\frac{1}{R^2}\left[\hat{\mathbf{R}}\int_0^{\tau_{orb}}dt\mathbf{r}(t)\right]. \tag{24}$$

Here we have used the slowness of the relative motion. The integral over the radius vector is expressed in terms of the Runge–Lenz vector through the well-known relation:[13]

$$\langle\mathbf{r}\rangle=\frac{1}{\tau_{orb}}\int_0^{\tau_{orb}}dt\mathbf{r}=\frac{3}{2}n^2\mathbf{A}. \tag{25}$$

Using (25) we readily find from (24)

$$\dot{\mathbf{l}}=\frac{3n^2}{2r^2}[\hat{\mathbf{R}}\mathbf{A}], \tag{26}$$

where $\dot{\mathbf{l}}$ describes the slow variation and is understood to mean $\dot{\mathbf{l}}=\delta\mathbf{l}/\tau_{orb}$. Similarly we find

$$\dot{\mathbf{A}}=\frac{3[\hat{\mathbf{R}}\mathbf{l}]}{2R^2}. \tag{27}$$

The relative motion is the motion in the electric field of the dipole (25),

$$\langle\mathbf{d}\rangle=-\langle\mathbf{r}\rangle=-\frac{3n^2\mathbf{A}}{2}$$

with field strength

$$\langle\mathbf{E}\rangle=\frac{1}{R^3}[3\hat{\mathbf{R}}(\hat{\mathbf{R}},\mathbf{d})-\mathbf{d}].$$

It is described by the equation

$$\ddot{\mathbf{R}}=\frac{3n^2}{2r^2}[3\hat{\mathbf{R}}(\hat{\mathbf{R}},\mathbf{A})-\mathbf{A}]. \tag{28}$$

From (28), (27), and (24) we find the system

$$\begin{aligned}&\ddot{\mathbf{R}}=\frac{3n^2}{2R^3}[3\hat{\mathbf{R}}(\hat{\mathbf{R}},\mathbf{A})-\mathbf{A}]\ ,\\&\dot{\mathbf{A}}=[\Omega\mathbf{B}],\ \dot{\mathbf{B}}=[\Omega\mathbf{A}],\end{aligned} \tag{29}$$

where $\mathbf{B}=\mathbf{l}/n$ and

$$\Omega=\frac{3n\hat{\mathbf{R}}}{R^2}. \tag{30}$$

Equations (29) constitute a closed self-consistent system for finding the trajectories $\mathbf{R}(t)$ of the passing electron, the atomic angular momentum $\mathbf{l}(t)=\mathbf{B}(t)n$, and the vector $\mathbf{A}(t)$. The initial conditions are the position and velocity of the incident particle along with the initial values of $\mathbf{l}$ and $\mathbf{A}$.

From (29) it follows that

$$(\mathbf{Al})=0,\quad \mathbf{A}^2+\mathbf{B}^2=1, \tag{31}$$

i.e., at any time the atomic electron moves in an ellipse.

Adding and subtracting the last two equations of Eqs. (29) we find

$$\frac{d\mathbf{j}_1}{dt}=[\Omega\mathbf{j}_1],\quad \frac{d\mathbf{j}_2}{dt}=-[\Omega\mathbf{j}_2],$$

where

$$\mathbf{j}_1=\frac{1}{2}(\mathbf{B}+\mathbf{A}),\quad \mathbf{j}_2=\frac{1}{2}(\mathbf{B}-\mathbf{A}).$$

From Eqs. (31) it follows that

$$\mathbf{j}_1^2=\mathbf{j}_2^2=1.$$

This means that the vectors $\mathbf{j}_1$ and $\mathbf{j}_2$ rotate with angular velocities $\boldsymbol{\Omega}$ and $-\boldsymbol{\Omega}$, respectively. In the important special case $\mathbf{R}=$const, for which it follows that $\boldsymbol{\Omega}=$const, the orbit varies periodically and shifts its orientation with period

$$T=\frac{2\pi}{\Omega}.$$

If $\mathbf{R}$ changes, then this precession is no longer periodic and the orbital angular momentum changes over a collision time. This is the physical essence of Stark mixing.

We can reduce the order of Eqs. (29) by using the equations of motion. In addition to (31) there are four other independent constants. These are the energy of the relative motion,

$$E=\frac{1}{2}\dot{\mathbf{R}}^2+\frac{3n^2}{2R^2}\mathbf{A}\hat{\mathbf{R}}, \tag{32}$$

and the total angular momentum

$$\mathbf{J}=\mathbf{l}+\mathbf{L}, \tag{33}$$

where $\mathbf{L}=[\mathbf{R}\dot{\mathbf{R}}]=r^2[\hat{\mathbf{R}}\dot{\hat{\mathbf{R}}}]$. There are two other constants of motion:

$$P=\mathbf{L}^2+3n^2\mathbf{A}\hat{\mathbf{R}}, \tag{34}$$

$$\lambda=\mathbf{l}\hat{\mathbf{R}}-\frac{2}{3}\mathbf{JA}. \tag{35}$$

The quantity P allows us to separate the radial and angular relative motion. Specifically, we can note that by using (34) we can rewrite Eq. (32) in the form

$$E=\frac{1}{2}\dot{\mathbf{R}}^2+\frac{P}{2R^2},$$

which is easily integrated:

$$R^2=\frac{P}{v^2}+v^2t^2,$$

where v is the collisional velocity (the velocity of the relative motion in the limit $t\to-\infty$). The quantity P is determined from the initial conditions for $t\to-\infty$. For $P<0$ the electron falls into the atom; in other words, a "hard" collision occurs. This happens if the impact parameter ρ is smaller than

$$\rho_0=\sqrt{3}n/v.$$

For $\rho>\rho_0$ we always have $P>0$, i.e., falling is not possible. Note that for $\rho<\rho_0$ trajectories with repulsion ($P>0$) are also possible when falling is impossible. The precession angle in the case $P<0$ for an electron moving over the portion of the trajectory with $R\gg n^2$ is equal to

$$\alpha(t)=2\int_{-\infty}^{t_0}dt'\Omega(t')=3n\int_{-\infty}^{t_0}\frac{dt'}{R^2(t')},$$

where $t_0=-|P|^{1/2}/v^2$ is the falling time and the factor 2 takes into account the return of the electron back to infinity. This integral diverges logarithmically at small distances and must be cut off for $R\sim r_{\rm orb}\sim n^2$. Simple calculations yield

$$\alpha\approx\ln\left(\frac{n^2}{v^2}\right)\sim 2-3,$$

i.e., total Stark mixing takes place even at large distances $R\gg n^2$.

Now let us consider the case $P>0$ and explicitly separate the angular from the radial motion. For this we rewrite the equation for the angular motion:

$$\frac{d\hat{\mathbf{R}}}{dt}=\frac{1}{R^2}[\mathbf{L}\hat{\mathbf{R}}],$$

using conservation of the total angular momentum (33):

$$\frac{d\hat{\mathbf{R}}}{dt}=\frac{1}{R^2}[(\mathbf{J}-\mathbf{l})\hat{\mathbf{R}}].$$

We introduce a new variable (the "precession angle"):

$$\alpha=\alpha(t)=2\int_{-\infty}^{t_0}dt'\Omega(t')$$

and again we write Eq. (29):

$$\frac{d\hat{\mathbf{R}}}{d\alpha}=\frac{2}{3n}[\mathbf{J}\hat{\mathbf{R}}]+\frac{2}{3}[\hat{\mathbf{R}}\mathbf{B}],\qquad \frac{d\mathbf{B}}{d\alpha}=[\hat{\mathbf{R}}\mathbf{A}],\quad \frac{d\mathbf{A}}{d\alpha}=[\hat{\mathbf{R}}\mathbf{B}]. \tag{36}$$

As t varies over the interval $(-\infty,\infty)$ the quantity α varies over $0<\alpha<\alpha_0$, where $\alpha_0=3\pi n/2\sqrt{P}$.

The slowest Stark mixing occurs from orbits which are close to circular, since in this case the dipole moment of the atom is close to zero there. This implies that only a hard collision with $\alpha(\infty)\sim 1$ can "push" the atom from such a state, and this is the limiting stage of Stark mixing. The characteristic impact parameter $\rho\sim\rho_0$ of such a collision is determined from the condition

$$3\pi n/2\sqrt{P}\sim 1.$$

The cross section for this process is

$$\sigma=\pi\rho_0^2\sim\frac{\pi^3n^2}{v_T^2},$$

and the rate of Stark mixing is

$$\frac{1}{\tau_{st}}=N_e\langle\sigma v\rangle\sim\frac{N_en^2}{v_T}. \tag{37}$$

To find the numerical factor we carried out a special computer experiment, based on the Monte Carlo technique. Relaxation of the angular part of the distribution function was studied. For this we solved Eq. (36) with various initial conditions. The result can be written

$$\tau_{st}=2\frac{v_T}{N_en^2}.$$

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