Toward a theory of superconductivity in two-dimensional systems with arbitrary carrier densities in an external magnetic field

V. P. Gusynin, V. P. Loktev, and I. A. Shovkovyi

N. N. Bogolyubov Institute of Theoretical Physics, National Academy of Sciences of Ukraine, 252143, Kiev, Ukraine
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We derive self-consistent equations that describe the behavior of the superconducting gap (order parameter) and chemical potential of metallic two-dimensional (2D) systems as a function of external magnetic field H, temperature T, and carrier density. For systems with low carrier densities, the pairs are localized below T_c; we show that in such systems the derivative $dH_{c2}(T)/dT$ at the point $T=T_c$ is considerably smaller than it is for a system in which Cooper pairs form. We find that in external fields that satisfy the criterion for the quantum limit, these systems are characterized by the appearance of a nontrivial nonuniform order parameter at sufficiently high temperatures. © *1995 American Institute of Physics.*

1. INTRODUCTION

Despite the exertion of much effort both in the experimental and theoretical directions, the nature of high-temperature superconductivity remains unknown. However, this effort should not be considered fruitless. At the present time we can confidently say that a consensus exists with regard to the factors that typify high-temperature superconductivity in the copper oxides[1–3]: a normal-phase conductivity whose behavior is quasi-two-dimensional, and, as a rule, unusual from the point of view of the theory of Landau Fermi liquids; a density of mobile carriers (primarily associated with oxygen) that is rather small (in any case, much smaller than in normal metals); and an extraordinarily strong interaction of the spins of the latter with primarily localized spins of the copper ions, making it necessary to develop a representation for the mutual coupling of superconductivity and magnetism. A simultaneous and first-principles description of these features within the framework of a unified theoretical approach is hardly possible; therefore, it is more common to investigate their roles separately, in order to ensure that the resulting conclusions are as reliable as possible. However, the development of approaches and methods that allow a more multifaceted study of the nature of high-temperature superconductivity remains an important and current problem of theory.

In this connection it is worth mentioning some recent and successful approaches to the modeling of certain features of high-temperature superconductivity in these materials, among them the 2D character of their conductivity and the relatively small[1)] (in general arbitrarily small) densities of fermions $n_f \equiv N_f/V$ (where N_f is the total number of fermions and V is the volume) present in them that feel a locally attractive potential.[5,6] It has turned out that a very simple but at the same general model can be used to derive, with some rigor, an effective low-energy Lagrangian for the superconducting phase (which can be endowed with an anisotropic order parameter[7]) and to study all the physically interesting characteristics of this phase. In particular, it has been shown that the equation for the chemical potential μ, which ordinarily is ignored in the standard theory of superconductivity, is no longer given by the trivial relation $\mu=\varepsilon_F$ (where ε_F is the Fermi energy); rather, it is necessary to derive and solve self-consistent equations for the gap Δ (or, which is the same thing, the order parameter) and the chemical potential, which can be not only positive but also negative, depending on T (including values of $T<T_c$, where T_c is the superconducting transition temperature). The positive range of values corresponds to overlapping pairs of bound fermions (Cooper-type pairing), while the negative range corresponds to pairs that are nonoverlapping in real space (localized pairing). These systems can support both types of pairing; which type occurs is primarily determined either by the value of the density n_f for a given T, or (clearly the more interesting situation) by the value of $T(<T_c)$ for a given value of n_f. In this latter case, we can induce a crossover from one type of pairing to another by varying T [more precisely, $\mu(T)$], which to a certain extent (since $\mu(T_c)=0$) explains how the superconductivity of high-T_c systems can resemble a superfluid condensate of local pairs which are nevertheless not seen above T_c.

In this work, our goal is to study the behavior of these metallic 2D systems, in particular how their superconducting phase forms in an external magnetic field H perpendicular to the plane of motion of the fermions. The interest in problems of this kind has been extraordinarily high recently due to the assertion made in[8,9] (see, also, the review[10]) that a fundamentally new nontrivial phase (with a finite order parameter) can exist in the range of finite T and ultra-strong fields that induce true Landau quantization, implying new relations between the superconducting order parameter and the quantity **H**. However, because the authors of these papers limited themselves to the BCS approximation, they did not discuss the equation for μ; thus, they missed the fundamental (in our view) fact that the chemical potential of a 2D metal depends significantly not only on T but also, as we will show below, on **H**.

In this paper we will derive a self-consistent set of equations for calculating both Δ and μ, and analyze its solutions in the range of both large and small values of n_f and **H** (the

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criteria for largeness and smallness will be stated below). We will then discuss how the effect of the field on the superconducting condensate depends on the carrier density, which leads us to an expression[2)] for the critical curve $H_{c2}(T)$ (where H_{c2} is the second critical field of the superconductor), starting from the assumption that the phase transition from the normal state to the superconducting state is a transition of second order. Finally, we derive the value of the derivative $dH_{c2}(T)/dT$ for $T=T_c$, and clarify the role of two-particle bound states for the ranges of large and small values of H. From this discussion it will become clear how important it is to take into account the local constituent bosons, which manifest themselves as fluctuations in the bare complex Fermi field.

2. MODEL AND BASIC EQUATIONS

As a starting point, let us take the Hamiltonian H_f for the simplest 2D Fermi system with a local attraction between particles, which we investigated in detail in,[5] and additional terms that take into account the presence of the constant magnetic field $\hbar=1$:

$$H_f=-\int d\mathbf{r}\,\mathscr{H}_f(\mathbf{r}),\quad \mathscr{H}_f(\mathbf{r})=-\psi_\sigma^+\left[\frac{1}{2m}\left(\partial_j-i\frac{e}{c}A_j\right)^2+\mu\right]\psi_\sigma-\frac{1}{2}g\psi_\sigma^+\psi_{\bar\sigma}^+\psi_{\bar\sigma}\psi_\sigma,$$
$$(\psi\equiv\psi(x),\ \mathbf{A}\equiv\mathbf{A}(x),\ x\equiv\mathbf{r},t,\ \mathbf{r}=(X,Y),\ \bar\sigma\equiv-\sigma=\uparrow,\downarrow). \tag{2.1}$$

Here A is the vector potential of the field $\mathbf{H}\|\hat z$, e and m are the charge and effective mass of a carrier (electron or hole), and $g>0$ is the fermion–fermion interaction constant corresponding to attraction. It is clear that, by specifying that the external magnetic field is given, we are neglecting terms in H_f caused by inhomogeneity of this field induced by the spatial distribution of the condensate. We do this because, these terms turn out to be small,[11] at least for values of the field that are not too far from the critical curve.

Using the Nambu representation[12] $\Psi=(\psi_\uparrow^+,\psi_\downarrow)$, we transform the Hamiltonian (2.1) to the form

$$H_f=-\int d\mathbf{r}\left[\Psi^+\tau_z\left(\frac{\mathbf{D}^2}{2m}+\mu\right)\times\Psi+g\Psi^+\tau_+\Psi\Psi^+\tau_-\Psi\right],\quad \mathbf{D}\equiv\partial-i\frac{e}{c}\tau_z\mathbf{A}, \tag{2.2}$$

(where τ are the Pauli matrices), which is convenient for calculating the partition function Z_f of the Fermi particles. Ths function is associated with the thermodynamic potential Ω_f of the particles through the standard relation ($k_B=1$):

$$Z_f=\exp(-\Omega_f/T)=\mathrm{Tr}\,\exp(-H_f/T). \tag{2.3}$$

Since in this case $\Omega_f=\Omega(V,T,\mu)$ and we are interested in the dependence of the physical quantities on the concentration of bare Fermi particles, let us write here the relation between μ and n_f that we will need for further calculations:

$$n_f=-V^{-1}\partial\Omega_f/\partial_\mu. \tag{2.4}$$

Using the Hamiltonian (2.2), Eq. (2.3) can be identically represented by a functional integral over the Grassmann variables[13] Ψ and Ψ^+:

$$Z_f=\int d\Psi d\Psi^+\exp\left\{-\int_0^\beta d\tau\int d\mathbf{r}[\Psi^+\partial_\tau\Psi+\mathscr{H}_f(\mathbf{r})]\right\},\quad \beta\equiv1/T, \tag{2.5}$$

these variables satisfy the antiperiodic boundary condition $\Psi(\tau,\mathbf{r})=-\Psi(\tau+\beta;\mathbf{r})$. As usual (see, e.g., Ref. 14), we now introduce an auxiliary complex Bose field $\Phi\equiv\Phi(\tau;\mathbf{r})$ (the Hubbard–Stratonovich method) with the periodic boundary condition $\Phi(\tau;\mathbf{r})=\Phi(\tau+\beta;\mathbf{r})$ to convert (2.5) to the form

$$Z_f=\int\int d\Psi d\Psi^+ d\Phi d\Phi^*\exp\left\{-\int_0^\beta d\tau\int d\mathbf{r}\left\{\Psi^+\left[\partial_\tau-\tau_z\left(\frac{\mathbf{D}^2}{2m}+\mu\right)+\tau_-\Phi+\tau_+\Phi^*\right]\Psi+g^{-1}|\Phi|^2\right\}\right\}, \tag{2.6}$$

which allows us to carry out the integration over the Fermi variables. Finally, as in Ref. 5 we are led to an expression for the partition function in terms of an effective action $S_{\mathrm{eff}}\equiv S_{\mathrm{eff}}(\Phi,\Phi^+)$:

$$Z_f=\int d\Phi d\Phi^* e^{-S_{\mathrm{eff}}},\quad S_{\mathrm{eff}}=-\mathrm{Tr\,Ln}\,G^{-1}+g^{-1}\int_0^\beta d\tau\int d\mathbf{r}|\Phi|^2, \tag{2.7}$$

where $G\equiv G(\tau_1,\tau_2;\mathbf{r}_1,\mathbf{r}_2)$, i.e., the Green's function of the fermions, is a functional of the field Φ. Its explicit form is determined by the equation

$$\left[-\partial_{\tau_1}+\tau_z\left(\frac{\mathbf{D}^2}{2m}+\mu\right)-\tau_-\Phi-\tau_+\Phi^*\right]G=\delta(\tau_1-\tau_2)\times\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{2.8}$$

the boundary conditions for which are supplied by the equation $G(\tau_1,\tau_2;\mathbf{r}_1,\mathbf{r}_2)=-G(\tau_1+\beta,\tau_2;\mathbf{r}_1,\mathbf{r}_2)$.

If the solution to Eq. (2.8) were known, it could be substituted into (2.7), allowing us to find a formally exact expression for the partition function and thereby reproduce any physical quantity of interest. However, in practice the quantity S_{eff}, and therefore Z_f as well, is impossible to compute exactly, so that approximations are necessary. The first, and most widely used, approximation is to use the method of steepest descent to compute the integral (1.7) near extrema Φ of the action S_{eff}; this corresponds to the mean-field (self-consistent field) approximation. In this case the saddle point is determined from the condition

$$\frac{\delta S_{\mathrm{eff}}(\Phi,\Phi^*)}{\delta\Phi^*(\tau;\mathbf{r})}=\mathrm{tr}\,G(\tau,\tau;\mathbf{r},\mathbf{r})\tau_+ +g^{-1}\bar\Phi=0, \tag{2.9}$$

which is a direct consequence of our explicit expression for S_{eff}. Another assumption is that the system is in "steady

state," i.e., that the solution $\bar{\Phi}\equiv\Phi(\mathbf{r})$, from which we derive the system order parameter according to the definition $\Delta\equiv|\bar{\Phi}|$, is independent of the "time" τ (in general this solution can be nonuniform and depend on T). In this case, the Green's function defined by (2.8) depends only on the difference $\tau_1-\tau_2$, and admits the following Fourier expansion:

$$G(\tau_1-\tau_2;\mathbf{r}_1,\mathbf{r}_2)=T\sum_{n=-\infty}^{\infty}G_n(\mathbf{r}_1,\mathbf{r}_2)\exp[-i\omega_n(\tau_1-\tau_2)],$$

$$\omega_n=\pi T(2n+1). \tag{2.10}$$

Substituting (2.10) into (2.8), we obtain the following equation for the coefficients $G_n(\mathbf{r}_1,\mathbf{r}_2)$:

$$\left[i\omega_n+\tau_z\left(\frac{\mathbf{D}_1^2}{2m}+\mu\right)-\tau_-\Phi-\tau_+\Phi^*\right]G_n(\mathbf{r}_1,\mathbf{r}_2)=\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{2.11}$$

By introducing $\bar{\Phi}$, we can find the partition function (2.7) in explicit form. This calculation consists of identifying those parts of (2.7) that depend only on $\bar{\Phi}$ and are, therefore, nonfluctuating, and including the fluctuation-induced corrections within the Gaussian approximation, i.e.,:

$$Z_f=e^{-\bar{S}_{\text{eff}}}\int d\Delta\Phi d\Delta\Phi^*\exp\Bigg[-\int_0^\beta\int_0^\beta d\tau_1\times d\tau_2\int\int d\mathbf{r}_1 d\mathbf{r}_2\Delta\Phi^*(\tau_1;\mathbf{r}_1)\Gamma^{-1}\times(\tau_1-\tau_2;\mathbf{r}_1,\mathbf{r}_2)\Delta\Phi(\tau_2;\mathbf{r}_2)\Bigg], \tag{2.12}$$

Here $\bar{S}_{\text{eff}}\equiv S_{\text{eff}}(\bar{\Phi},\bar{\Phi}^*)$ is the effective action in the self-consistent field approximation, $\Delta\Phi=\Phi-\bar{\Phi}$ is the fluctuating component of the constituent Bose field, and Γ is the propagator of the latter, which is introduced through the definition

$$\Gamma^{-1}(\tau_1-\tau_2;\mathbf{r}_1,\mathbf{r}_2)=g^{-1}\delta(\tau_1-\tau_2)\delta(\mathbf{r}_1-\mathbf{r}_2)+\text{tr}\,G(\tau_1-\tau_2;\mathbf{r}_1\mathbf{r}_2)\tau_-G(\tau_2-\tau_1;\mathbf{r}_2,\mathbf{r}_1)\tau_+|_{\Phi=\bar{\Phi}}. \tag{2.13}$$

The analytic structure of the function Γ consists of isolated poles that correspond to bound two-fermion states (in other words, bosons) and branch cuts corresponding to the continuum of two-particle Fermi excitations. Note that in writing (2.12) we have omitted fluctuations of the field $\mathbf{H}$ near its value (weakly nonuniform in general). The role of these fluctuations is primarily to mediate a possible conversion of the second-order phase transition between a normal metallic and a superconducting phase into a first-order phase transition.[15]

Direct integration of expression (2.12) over the fluctuations of the auxiliary field gives

$$Z_f=\exp(-\bar{S}_{\text{eff}}-\text{Tr Ln}\,\Gamma^{-1}),$$

from which we eventually find the required thermodynamic potential [see (2.3)]:

$$\beta\Omega_f=g^{-1}\int_0^\beta d\tau\int d\mathbf{r}|\bar{\Phi}|^2-\text{Tr Ln}\,G^{-1}+\text{Tr Ln}\,\Gamma^{-1}. \tag{2.14}$$

Finally, by substituting (2.14) into (2.4) we can write the concentration of Fermi particles in the system as a sum:

$$n_f=T\sum_n\text{tr}\,\tau_zG_n(\mathbf{r},\mathbf{r})-TV^{-1}\partial(\text{Tr Ln}\,\Gamma^{-1})/\partial\mu, \tag{2.15}$$

whose terms formally correspond to contributions from particles described by the various types of statistics. The first term, which we obtain from the representation (2.10) and the definition of the Green's function [given by Eq. (2.8)], describes the usual fermion density in the mean-field approximation. The second term is more interesting and, to some extent, unusual; its presence in (2.15) is exclusively due to the inclusion of fluctuations $\Delta\Phi$ and $\Delta\Phi^*$ of the constituent field in the functional integral (2.7). The presence of this term in the expression for n_f shows unambiguously that the interaction results in dynamic processes that separate the original system into two parts (even, generally speaking, in the normal phase)—a purely fermionic subsystem and a subsystem described by boson degrees of freedom. In this case, the partial concentrations of these particles—fermions and bosons—in the two subsystems is entirely controlled by the terms (2.15), while their total number is determined by the density n_f.

In concluding this section, we note that the formalism we used above to derive the system of self-consistent equations (2.9) and (2.15) directly reproduces several previously-known results derived by Gor'kov concerning the behavior of a superconductor in a magnetic field.[16] These results are obtained here even in the zeroth (mean-field) approximation if we impose the condition $\mu=\varepsilon_F$, i.e., without taking into account the self-consistent relation between μ and n_f. In addition, it is worth noting that the importance of this latter relation was first pointed out by Leggett ,[17] and that fluctuations in the boson subsystem were takem into account in the papers.[18–20] However, the authors of these papers discussed only 3D systems in the absence of an external magnetic field.

3. EQUATIONS FOR THE CRITICAL CURVE

Thus, Eqs. (2.9) and (2.15) completely describe the behavior of the order parameter and chemical potential of a 2D system as a function of the "external" parameters T, $\mathbf{H}$, and n_f. In this case, the most interesting quantities are the critical points along the curve $H_{c2}(T)$, which separates the normal and superconducting phases (especially near the point $T=T_c$). Because our equations are explicit in form, it follows that the primary ingredient we need to calculate is the Green's function given by Eq. (2.11). Let us rewrite the latter in an integral form that is more convenient for subsequent analysis:

$$G_n=G_n^{(0)}+G_n^{(0)}\hat{\Phi}G_n,\quad \hat{\Phi}\equiv\tau_-\bar{\Phi}+\tau_+\bar{\Phi}^*. \tag{3.1}$$

Here we have used operator notation for brevity, for which $G_n^{(0)}\equiv G_n^{(0)}(\mathbf{r},\mathbf{r})$ is the Green's function of the system in the normal state, which can be found in the Appendix. Equation

(3.1) possesses the remarkable property of allowing a series solution, at least for cases where the order parameter (i.e., the value of the energy gap) happens to be small in the immediate neighborhood of the critical curve.

Since the integral Eq. (3.1) has a Dyson structure, the most natural method of finding its solution is by iteration. In fact, it is not difficult to see that the usual expansion holds:

$$G_n=G_n^{(0)}+G_n^{(0)}\hat{\Phi}G_n^{(0)}+G_n^{(0)}\hat{\Phi}G_n^{(0)}\hat{\Phi}G_n^{(0)}+\dots . \quad (3.2)$$

By substituting this expansion into (1.9) along with (1.10), we are led to the equation

$$\bar{\Phi}(\mathbf{r})=\int d\mathbf{r}_1\bar{\Phi}(\mathbf{r}_1)\Big[K_1(\mathbf{r},\mathbf{r}_1)+\int\int d\mathbf{r}_2 d\mathbf{r}_3 K_2 \times(\mathbf{r},\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\bar{\Phi}(\mathbf{r}_2)\bar{\Phi}(\mathbf{r}_3)\Big], \quad (3.3)$$

with the kernel

$$K_1(\mathbf{r},\mathbf{r}_1)=-gT\sum_{n=-\infty}^{\infty}\operatorname{tr} G_n^{(0)}(\mathbf{r},\mathbf{r}_1)\tau_- G_n^{(0)}(\mathbf{r}_1,\mathbf{r})\tau_+,$$

$$K_2(\mathbf{r},\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=-gT\sum_{n=-\infty}^{\infty}\operatorname{tr} G_n^{(0)}(\mathbf{r},\mathbf{r}_1)\tau_- G_n^{(0)} \times(\mathbf{r}_1,\mathbf{r}_2)\tau_+ G_n^{(0)}(\mathbf{r}_2,\mathbf{r}_3)\tau_- G_n^{(0)} \times(\mathbf{r}_3,\mathbf{r}_1)\tau_+ \quad (3.4)$$

which allows us to describe not only the phase transition curve but also the value of the order parameter near it. This latter fact is because we have kept the lowest nonlinear term in (3.3), which, of course, limits the usefulness of this equation to the range of rather small values of $\bar{\Phi}$. This includes the critical curve itself, on one side of which $\bar{\Phi}\neq 0$, while on the other side $\bar{\Phi}=0$. If we start from the assumption mentioned above, i.e., that the curve $H_{c2}(T)$ of interest represents a second-order phase transition (i.e., the order parameter changes continuously across it [for all n_f] as the parameters T and H vary), we can assume that the value $\bar{\Phi}=0$ also characterizes points on the curve itself. This latter assertion implies that the critical parameters are none other than those for which the equation

$$\bar{\Phi}(\mathbf{r})=\int d\mathbf{r}_1 K_1(\mathbf{r},\mathbf{r}_1)\bar{\Phi}(\mathbf{r}_1) \quad (3.5)$$

has a nonzero solution.

In order to find this solution in explicit form, we note that the kernel (3.4) of this equation is completely determined by the free Green's functions given in the Appendix. These expressions allow us to rewrite Eq. (3.5) in the form [see (A7)]

$$\bar{\Phi}(\mathbf{r})=\int d\mathbf{r}_1 K_1^{(\mathrm{hom})}(\mathbf{r}-\mathbf{r}_1)\exp\left(\frac{i}{l^2}[\mathbf{r}\mathbf{r}_1]_z\right)\bar{\Phi}(\mathbf{r}_1), \quad (3.6)$$

where l [see (A6)] is a magnetic length proportional to the period of the hexagonal Abrikosov lattice of the superconductor in the magnetic field[10]; the kernel

$$K_1^{(\mathrm{hom})}(\mathbf{r})=-gT\sum_{n=-\infty}^{\infty}\operatorname{tr} G_n^{(\mathrm{hom})}(\mathbf{r})\tau_- G_n^{(\mathrm{hom})}(\mathbf{r})\tau_+ \quad (3.7)$$

is the spatially uniform part of the full kernel $K_1(\mathbf{r},\mathbf{r}_1)$ specified by the Green's function (A8). It was shown in Ref. 21 that Eq. (3.6) has an exact and nontrivially nonuniform solution

$$\bar{\Phi}(\mathbf{r})=\Delta\exp(-r^2/2l^2) \quad (3.8)$$

whose "amplitude" $\Delta\equiv\Delta(T)$ gives the maximum value of the system order parameter (corresponding to $\mathbf{H}=0$) for $T\leqslant T_c$. We may say that this expression describes a superconducting vortex because, in contrast to a magnetic vortex, it describes an order parameter that is nonzero in a certain region and decays rapidly towards the periphery of the latter.[3)]

The final form of the first of two equations we need to determine the critical curve is obtained from (2.6) by substituting Eq. (3.8) into it and going to the momentum representation; doing so, we find after integrating over $\mathbf{r}$

$$1-\int d\mathbf{r}K_1^{(\mathrm{hom})}(\mathbf{r})\exp(-\mathbf{r}^2/2l^2)=1+(gTl^2/(2\pi)^3) \times\sum_n\int\int d\mathscr{K}\,d\mathbf{k}\exp(-\mathscr{K}^2l^2/2)\operatorname{tr} G_n^{(\mathrm{hom})}(\mathscr{K}/2+\mathbf{k})\tau_- G_n(-\mathscr{K}/2+\mathbf{k})\tau_+=0. \quad (3.9)$$

Here it is clear that nonconservation of the total momentum of a pair is wholly due to the external magnetic field.

The quantity μ enters into Eq. (A.11) by way of the Green's function. Its connection with n_f on the critical curve follows from Eq. (2.15) if we also use the expansion (2.2) in the latter, which eventually gives

$$n_f=\frac{T}{(2\pi)^2}\sum_n\int d\mathbf{k}\operatorname{tr} G_n^{(\mathrm{hom})}(\mathbf{k})\tau_z - TV^{-1}\frac{\partial}{\partial\sim\mu}\operatorname{Tr}\ln\Gamma_0^{-1}, \quad (3.10)$$

where Γ_0^{-1} is defined by Eq. (2.13). The second term on the right side of this equation contains the free Green's function $G^{(0)}(\tau_1-\tau_2;\mathbf{r}_1,\mathbf{r}_2)$, which is subject to Eq. (A1).

We now can obtain an explicit expression for $\operatorname{Tr}\ln\Gamma_0^{-1}$, and from it the corresponding contribution Δn_f to (3.10). Although these calculations apply to the weak-field range, in the opposite case the correction Δn_f is not very important.

First of all, let us separate out the coordinate-independent portion of the propagator Γ_0^{-1}, using the definition (2.13) and the explicit form (A7) for the free Green's functions that appears in it. Then

$$\Gamma_0^{-1}(\tau_1-\tau_2;\mathbf{r}_1,\mathbf{r}_2)=\exp\left(\frac{i}{l^2}[\mathbf{r}_1\mathbf{r}_2]_z\right)\Gamma_{\mathrm{hom}}^{-1}(\tau_1-\tau_2;\mathbf{r}_1-\mathbf{r}_2), \quad (3.11)$$

where obviously [compare (2.13)],

$$\Gamma_{\mathrm{hom}}^{-1}(\tau_1-\tau_2;\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{g}\delta(\tau_1-\tau_2)\delta(\mathbf{r}_1-\mathbf{r}_2)+\operatorname{tr} G^{(\mathrm{hom})}(\tau_1$$

$$-\tau_2;\mathbf{r}_1-\mathbf{r}_2)\tau_- G^{(\mathrm{hom})}(\tau_2-\tau_1;\mathbf{r}_2 -\mathbf{r}_1)\tau_+|_{\dot{\Phi}=0}. \quad (3.12)$$

If the Fourier series (2.10) is used to expand (3.11) and (3.12) in terms of harmonics with the boson frequencies $\Omega_n=2\pi Tn$, the following expressions for the corresponding Fourier coefficients result:

$$\Gamma_n^{-1}(\mathbf{r}_1,\mathbf{r}_2)=\exp\left(\frac{i}{l^2}[\mathbf{r}_1\mathbf{r}_2]_z\right)\Gamma_{\mathrm{hom}}^{-1}(n;\mathbf{r}_1-\mathbf{r}_2),$$

$$\Gamma_{\mathrm{hom}}^{-1}(n;\mathbf{r})=\frac{1}{g}\,\delta(\mathbf{r})+T\sum_{n_1}\mathrm{tr}\,G_{n_1}^{(\mathrm{hom})}(\mathbf{r})\tau_- G_{n_1-n}^{(\mathrm{hom})}(\mathbf{r})\tau_+|_{\dot{\Phi}=0}, \quad (3.13)$$

the last of which can also be derived in the momentum representation by analogy with (A10):

$$\Gamma_{\mathrm{hom}}^{-1}(n;\mathscr{K})=\int d\mathbf{r}\,\exp(-i\mathscr{K}\,\mathbf{r})\Gamma_{\mathrm{hom}}^{-1}(n;\mathbf{r})=\frac{1}{g}+T \times\sum_{n_1}\frac{1}{(2\pi)^2}\int d\mathbf{k}\,\mathrm{tr}\,G_{n_1}^{(\mathrm{hom})}\left(\frac{\mathscr{K}}{2}+\mathbf{k}\right) \times\tau_- G_{n_1-n}^{(\mathrm{hom})}\left(-\frac{\mathscr{K}}{2}+\mathbf{k}\right)\tau_+|_{\dot{\Phi}=0}. \quad (3.14)$$

Note that the vector $\mathscr{K}$, in contrast to the vector $\mathbf{k}$ of the fermion Green's function, here corresponds to the total momentum of a pair of particles.

Since the propagator Γ enters into Eqs. (2.14) and (3.10) through the operation

$$\mathrm{Tr\,Ln}\,\Gamma^{-1}=T\sum_n\sum_{\mathscr{K}}\langle\mathscr{K}|\ln\Gamma_n^{-1}|\mathscr{K}\rangle, \quad (3.15)$$

it is clear that its computation requires the calculation of matrix elements of the complete inverse propagator, not just its spatially uniform part. In this case, when $\mathbf{H}=0$ we clearly have

$$\langle\mathscr{K}_1|\Gamma_n^{-1}|\mathscr{K}_2\rangle=\delta(\mathscr{K}_1-\mathscr{K}_2)\Gamma_{\mathrm{hom}}^{-1}(n,\mathscr{K}_1), \quad (3.16)$$

from which it follows directly that the following relations hold in the absence of a field:

$$\langle\mathscr{K}|\mathrm{Ln}\,\Gamma_n^{-1}|\mathscr{K}\rangle=\ln\Gamma_{\mathrm{hom}}^{-1}(n,\mathscr{K}),$$
$$\mathrm{Tr\,Ln}\,\Gamma^{-1}=VT\sum_n\frac{1}{(2\pi)^2}\int d\mathscr{K}\,\ln\Gamma_{\mathrm{hom}}^{-1}(n;\mathscr{K}). \quad (3.17)$$

Inclusion of the external field leads to the appearance of spatial nonuniformity and, consequently, to violation of the conservation of momentum law expressed by (3.16) and (3.17). Thus, a first-principles calculation requires knowledge of all the matrix elements, including the nondiagonal ones. However, when the field is small ($l\to\infty$) the expansion (3.17) is correct up to terms of order $O(\mathbf{H}^2)$.[23]

In other words, since ours is a theory that describes weak-field effects and includes terms no higher than quadratic, we may limit ourselves to the uniform part of the two-particle propagator Γ in the matrix elements (3.15). Since an expansion of the function $\Gamma_{\mathrm{hom}}^{-1}(n;\mathscr{K})$ itself in powers of l^{-1} does not contain any terms $\propto l^{-2}$[see the definition (3.12), and also (A13)], the representation (3.14) will be accurate to this order if we substitute the Green's function of the system in the absence of a field into it, i.e.,

$$G_n^{(\mathrm{hom})}(k)=\frac{1}{i\omega_n-\tau_z\xi(\mathbf{k})}. \quad (3.18)$$

Standard summation of the series (3.14) (see, e.g., Ref. 24) over the fermion frequencies using the Green's function (3.18) gives

$$\Gamma_{\mathrm{hom}}^{-1}(n;\mathscr{K})=\frac{1}{g}-\frac{1}{(2\pi)^2}\int d\mathbf{k} \times\frac{1-n_F(\mathscr{K}/2+\mathbf{k})-n_F(\mathscr{K}/2-\mathbf{k})}{\xi(\mathscr{K}/2+\mathbf{k})+\xi(\mathscr{K}/2-\mathbf{k})-i\Omega_n}, \quad (3.19)$$

where $n_F(\mathbf{k})=[\exp\xi(\mathbf{k})/T+1]^{-1}$ is the Fermi distribution function.

Formally, the integral in (3.19) diverges if the width of the conduction band of the 2D metal $W\to\infty$. However, it is easy to avoid this divergence if we anticipate Sec. 3 and use the regularization $W\to\infty, g\to 0$, which enters into the theory of the energy of the two-particle bound state,[5,6] which equals

$$\varepsilon_b=-2W\,\exp(-4\pi/gm) \quad (3.20)$$

It is clear from the derivation given above that this quantity is independent of the magnitude of the magnetic field to order $\propto l^{-2}$.

Thus, for weak external fields the propagator of the constituent bosons can be described by the following expression:

$$\Gamma_{\mathrm{hom}}^{-1}(n;\mathscr{K})=\lim_{W\to\infty}\frac{m}{4\pi}\left[\ln\frac{2W}{|\varepsilon_b|}-\int_0^{2W}du\left(u+\frac{\mathscr{K}^2}{4m}-2\mu-i\Omega_n\right)^{-1}\right]+\frac{1}{(2\pi)^2}\int d\mathbf{k}\left[n_F\left(\frac{\mathscr{K}}{2}+\mathbf{k}\right)+n_F\left(\frac{\mathscr{K}}{2}-\mathbf{k}\right)\right]\left(\frac{\mathscr{K}^2+4\mathbf{k}^2}{4m}-2\mu-i\Omega_n\right)^{-1}, \quad (3.21)$$

which can be used in concrete calculations.

We will find Δn_f from the corresponding correction to the thermodynamic potential (2.14):

$$\Delta\Omega_f=VT\sum_n\frac{1}{(2\pi)^2}\int d\mathscr{K}\,\ln\Gamma_{\mathrm{hom}}^{-1}(n;\mathscr{K}),$$

which takes the following form when we pass from summation to integration in the complex plane of the variable ω:

$$\Delta\Omega_f=-i\frac{V}{(2\pi)^3}\int d\mathscr{K}\int d\omega\,n_B(\omega)\ln\Gamma_{\mathrm{hom}}^{-1}(\omega;\mathscr{K}), \quad (3.22)$$

here $n_B(\omega)=(\exp\omega/T-1)^{-1}$ is the Bose distribution function. The contour C for integration with respect to ω is shown in Fig. 1. Let us deform this contour into the contour C_1 (see Fig. 1), and, following Ref. 18 define the phase

$$\delta(\omega;\mathscr{K})=-\mathrm{Arg}\,\Gamma_{\mathrm{hom}}^{-1}(\omega+i0;\mathscr{K}). \quad (3.23)$$

This lets us write the expression under the logarithm sign in (3.23) in the form $|\Gamma_{\mathrm{hom}}^{-1}|\exp(-i\delta)$. Then the integral contain-

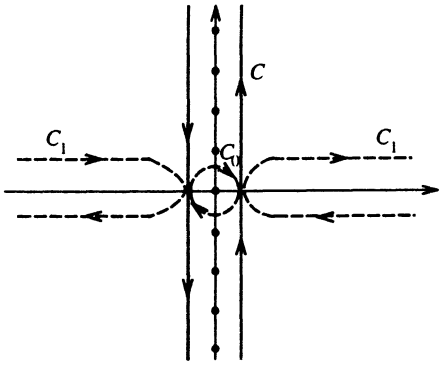

FIG. 1.

ing Γ_{hom}^{-1} turns out to equal zero because the integrations over the upper and lower branches of the contour C_1 along the real axis mutually compensate, which converts the fluctuation part of Ω_f to the form

$$\Delta\Omega_f = -\frac{V}{(2\pi)^3}\int d\mathscr{K}\int_{C_1} d\omega\, n_B(\omega)\delta(\omega;\mathscr{K})$$

$$= -\frac{V}{(2\pi)^3}\int d\mathscr{K}\int_{-\infty}^{\infty} d\omega\, n_B(\omega)\delta(\omega;\mathscr{K}), \quad (3.24)$$

the last equation is derived by deforming the integration contour C_1 into an integration over the real ω axis. This can be done by adding the contour C_0 (see Fig. 1) around the point $\omega=0$ to the contour C_1 and then subtracting the contribution arising from the quantity $\delta(0,\mathscr{K})$. However, this latter quantity is annihilated by the condition that the ground state of the system as a whole be stable, which takes the form[18]

$$\text{Re}\,\Gamma_{\text{hom}}^{-1}(\omega;\mathscr{K})>0, \quad \text{Im}\,\Gamma_{\text{hom}}^{-1}(0;\mathscr{K})=0,$$

which necessarily implies that $\delta(0,\mathscr{K})=0$.

In accordance with (2.15), (3.24) leads to the required fluctuation correction to the particle concentration in the system

$$\Delta n_f = \frac{1}{4\pi^3}\int d\mathscr{K}\int_{-\infty}^{\infty} d\omega\, n_B(\omega)\frac{\partial}{\partial\mu}\delta(\omega;\mathscr{K}). \quad (3.25)$$

In particular, we can restrict the discussion to values of this quantity on the critical curve for $|\mu_c|\gg T_c$, i.e., the region of most interest to us. Then in this region, where pair formation is most advantageous, we can obtain for $\Gamma_{\text{hom}}^{-1}(\omega;\mathscr{K})$ [see (3.21)] the expression

$$\text{Re}\,\Gamma_{\text{hom}}^{-1}(\omega;\mathscr{K}) = \frac{m}{4\pi}\ln\left|\frac{\mathscr{K}/4m-2\mu-\omega}{\varepsilon_b}\right| + O\times\left[\exp\left(-\frac{|\mu_c|}{T_c}\right)\right], \quad (3.26)$$

which shows that $\Gamma_{\text{hom}}^{-1}(\mathscr{K}^2/4m-2\mu+\varepsilon_b;\mathscr{K})=0$. In this case, the phase (3.23) develops a discontinuity of the form

$$\delta(\omega;\mathscr{K}) = \pi\theta\left(\omega-\frac{\mathscr{K}^2}{4m}+2\mu-\varepsilon_b\right).$$

By substituting this function into (3.25), we eventually find the correction:

$$\Delta n_f = \frac{1}{2\pi^2}\int d\mathscr{K} n_B\left(\frac{\mathscr{K}^2}{4m}-2\mu+\varepsilon_b\right) \quad (3.27)$$

to the fermion number caused by fluctuations of the auxiliary field. In this limit, the correction is expressed entirely in terms of a Bose distribution function of particles with mass $2m$ and chemical potential 2μ, as should be the case in principle; the energy origin used here is the quantity ε_b from (3.20). Consequently, as we said in Sec. 1, the constituent field and its fluctuations actually are responsible for the formation of a subsystem of new and relatively stable particles in the original system of attracting Fermi particles—bosons made up of bound states of pairs of fermions in real space. Nevertheless, we can show that the next correction to (3.27) [which has the same order of smallness as in (3.26)] turns out to be negative, i.e., it indicates that the constituent bosons have a finite lifetime due to decay processes.

If, however, $\mu\gg T_c$, then an analysis analogous to that given above for the case $\mu<0$ is not feasible. However, physical considerations clearly indicate that the positiveness of μ is a direct consequence of the small role played by stable localized pairs. Consequently, the inclusion of their fluctuations is not very important along the critical curve, implying that $\Delta n_f=0$ (for 3D and quasi-2D systems at least).

Thus, the system (3.9)–(3.10) including the contribution (3.27) from boson fluctuations describes the critical parameters $\mu(T), H_{c2}(T)$ self-consistently for $|T-T_c|\gg T_c$ and a given value of n_f. The zero-field limit in this case corresponds to $l=\infty$; then, taking into account that

$$\lim_{l\to\infty}(2\pi)^{-1}\exp(-\mathscr{K}^2l^2/2)\to\delta(\mathscr{K}),$$

it is easy to use (3.9) to obtain relation (45) of Ref. 6 where the model (2.1) was discussed for $\mathbf{H}=0$. In what follows we will treat the cases of small and large magnetic fields separately.

4. WEAK MAGNETIC FIELD

This case is realized if the cyclotron frequency $\omega_{\mathbf{H}}\ll T_c$ [see (A6)] and if it is possible to expand Eq. (2.9) with respect to l^{-1}, using the explicit form of the Green's function (A13). After making the necessary transformations, we obtain the equation

$$\frac{4\pi}{gm} = \int_0^W \frac{du}{u-\mu}\,\text{th}\,\frac{u-\mu}{2T} - \frac{4T}{ml^2}\sum_{n=0}^{\infty}\int_{-\mu}^{W-\mu}\frac{du}{u}\frac{d}{du}\frac{u^2(u+\mu)}{\omega_n^2+u^2}. \quad (4.1)$$

The second term in the right-hand side of this equation is entirely due to the presence of the field; W as above, is the width of the conduction band.

It is not difficult to see that the value of T_c for $\mathbf{H}=0$ is given by the relation

$$\frac{4\pi}{gm}=\int_0^W \frac{du}{u-\mu_c}\,\mathrm{th}\,\frac{u-\mu_c}{2T_c},\quad \mu_c\equiv\mu(T_c), \tag{4.2}$$

which was analyzed in Ref. 6. Assuming that the external field causes only rather small changes in the parameters required for the critical curve, let us substitute (4.2) into Eq. (4.1). After a regularization [5] of the form $W\to\infty$, $g\to 0$, (4.1) acquires the form

$$\int_0^\infty du\left[\frac{\mathrm{th}(u-\mu)/2T}{u-\mu}-\frac{\mathrm{th}\,(u-\mu_c)/2T_c}{u-\mu_c}\right]$$
$$=\frac{2T}{ml^2}\sum_{n=0}^{\infty}\left[\frac{1}{\omega_n^2}+\frac{\pi\mu}{2\omega_n^3}+\frac{\mu\ \mathrm{arctg}(\mu/\omega_n)}{\omega_n^3}\right], \tag{4.3}$$

with a left side that is is convenient for computation in the range of small $T-T_c$. The summation over frequencies in the right side of (4.3) can also be carried out if we make use of the expressions[25]

$$\sum_{n=0}^{\infty}(2n+1)^{-\nu}=(1-2^{-\nu})\zeta(\nu)=\begin{cases}\pi^2/8, & \nu=2,\\ 7\zeta(3)/8, & \nu=3,\end{cases}$$

$$\left(\frac{2}{\pi}\right)^3\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\,\mathrm{arctg}\,\frac{a}{2n+1}=\frac{1}{2}\int_0^a\frac{du}{u^3}\,(u-\mathrm{th}\,u)$$
$$=\frac{7\zeta(3)}{2\pi^2}-\frac{1}{2}\int_a^\infty\frac{du}{u^3}(u-\mathrm{th}\,u),\quad a>0,$$

(where $\zeta(\nu)$ is the Riemann zeta function). This allows us to quickly find the general expression for the slope of the critical curve in the vicinity of T_c:

$$\frac{e}{4mc}\left(\frac{dH_{c2}}{dT}\right)_{T_c}=\frac{1+\mathrm{th}(\mu_c/2T)-(T_c/\mu_c)(\partial\mu_c/\partial T_c)\mathrm{th}(\mu_c/2T_c)}{1+[7\zeta(3)/2\pi^2](\mu_c/T_c)+(|\mu_c|/2T_c)\int_0^{|\mu_c|/2T_c}(du/u^3)(u-\mathrm{th}\,u)}. \tag{4.4}$$

A number of experimental facts suggest that the value of this derivative in high-temperature superconductors is very small (or even zero).[26,27] However, when we attempt to discuss the shape of the $H-T$ phase diagram theoretically in these compounds, we find that deviations from the predictions of the BCS theory appear only if it is assumed that the superconductivity of these compounds is superfluidlike in character, with the role of the bosons played, e.g., by heavy small-radius bipolarons.[28]

As we pointed out above, although the slope in question is still given by Eq. (4.4), this is no longer a "closed form" expression in our case because it includes the unknown value μ_c, whose connection with n_f is given by a separate equation. Furthermore, this equation is also needed to determine the value of $\partial\mu_c/\partial T_c\equiv(\partial\mu/\partial T)_{T_c}$, which also contributes to the slope of the critical curve. It turns out that numerical methods are needed to find the slope over the entire range of variation of μ; however, as we are interested only in the qualitative properties of the model, we will limit ourselves to an analysis of limiting cases.

Localized pairs. These states correspond to the condition $\mu<0$ or $\varepsilon_F\ll|\varepsilon_b|$[where ε_B is defined in (2.20)].[5,6,20] If, in this case, we assume that $|\mu_c|\gg T_c$, then Eq. (3.27) can be used for (3.10). After integrating it, we find

$$n_f=\frac{mT}{\pi}\ln\left[1+\exp\left(-\frac{|\mu|}{T}\right)\right]-\frac{2mT}{\pi}\ln\left[1-\exp\left(\frac{2\mu-\varepsilon_b}{T}\right)\right]+O\left[\exp\left(-\frac{|\mu|}{T}\right)\right]$$
$$\approx-\frac{2mT}{\pi}\ln\left[1-\exp\left(\frac{2\mu-\varepsilon_b}{T}\right)\right], \tag{4.5}$$

in which we have omitted exponentially small terms, so that the second equation we need is simple in form.

In principle, the system (4.4) and (4.5) completely describes a 2D metal and its critical curve near T_c if we assume that the critical temperature introduced in (4.2) is itself finite for any value of n_f. However, it is easy to establish that Eqs. (4.2) and (4.5) are mutually consistent only at the point $T_c=0$. This we find to be in complete agreement with general and well-known conclusions about the ability of fluctuations to destroy order parameters in isotropic spaces with dimensions smaller than three.

It is necessary to ensure that T_c is stable against fluctuations and nonzero, which must be the case if the quantity (4.4) is to be meaningful. One way to do this is to take into account 3D (or quasi-2D) effects, which change things in a radical way. In this case, Eq. (4.5) is modified so that it leads to the condition [19] $\mu_c=\varepsilon_b/2$, which imposes no limitation on T_c. The expression for the critical value of the concentration $n_c^{3D}\equiv n_f^{3D}(T)$ has the form

$$n_c^{3D}=\frac{1}{4\pi^3}\int d\mathcal{K}\,d\mathcal{K}_z\left[\exp\left(\frac{\mathcal{K}^2}{4m}+\frac{\mathcal{K}_z^2}{4M}\right)T_c^{-1}-1\right]^{-1}$$
$$=\frac{2}{\pi^{3/2}}\zeta\left(\frac{3}{2}\right)M^{1/2}mT_c^{3/2}, \tag{4.6}$$

where $M(\gg m)$ is the fermion mass in the z direction. In a certain sense, we can introduce quasi-two-dimensionality into a strictly 2D system if, following Part 1 of Ref. 29 we expand the phase-space volume of the latter by allowing the particles to displace perpendicular to the plane with an average energy given by the condition $\mathcal{K}_z^2/4M\approx T$. Then the 2D concentration at finite T (including T_c) can be defined as

$n_f = n_f^{3D}\mathscr{K}_z^{-1}$. If we then take into account (4.6) and the presumed relation between T_c and $\mathscr{K}_z$, we are led to the relation

$$n_c \equiv n_f(T_c) = \pi^{-3/2}\zeta(3/2)mT_c, \tag{4.7}$$

which correctly describes the relation between the particle density and temperature in a 2D system. More importantly, it predicts a finite value of T_c at finite values of n_f. Here it is appropriate to note that the use of the equation $\varepsilon_F = \pi n_f/m$ in (4.7), which is standard for 2D conductors, gives $T_c \approx 0.68\varepsilon_F$, which is close to the value obtained for localized pairs in 3D space.[19]

These discussions allow us to transform Eq. (4.6) into a form that is convenient for treating the 2D case with arbitrary μ and T by writing

$$n_f = \frac{2mT}{\pi^2}\int_0^\infty du\, u^{1/2}\left\{\exp\left[u + \frac{\varepsilon_b - 2\mu}{T}\right] - 1\right\}^{-1},$$

from which it follows that $\partial\mu_c/\partial T_c = 0$. From this we eventually find:

$$-\frac{e}{4mc}\left(\frac{dH_{c2}}{dT}\right)_{T_c} \approx \left(\frac{\mu_c}{T_c}\right)^2 \exp\left(-\frac{|\mu_c|}{2T_c}\right),$$

$$|\mu_c| = \frac{|\varepsilon_b|}{2} \gg T_c \sim \varepsilon_F. \tag{4.8}$$

This expression indicates that the slope of the critical curve at the point T_c for the constituent bosons is exponentially small; this conclusion corresponds qualitatively, but is not identical to, the case of Bose condensation, where the critical curve has power-law behavior ($\propto (T - T_c)^{3/2}$).[14]

Cooper pairs. These form if $\mu > 0$ and $\mu_c \gg T_c$.[5,20] Furthermore, in this case fluctuations of the auxiliary field are suppressed, and we can neglect all but the first term in (3.10) [compare (4.5)]:

$$n_f = \frac{mT}{\pi}\ln\left[1 + \exp\left(\frac{\mu}{T}\right)\right] \approx \frac{\mu_c m}{\pi},$$

which immediately gives the derivative

$$\partial\mu_c/\partial T_c \approx -(\mu_c/T_c)\exp(-\mu_c/T_c) \approx 0.$$

Substituting it into (4.5), we find the required slope:

$$-\frac{e}{4mc}\left(\frac{dH_{c2}}{dT}\right)_{T_c} = 2\pi^2[7\zeta(3)]^{-1}\frac{T_c}{\mu_c},$$

$$\mu_c = \varepsilon_F \gg T_c = \frac{\gamma}{\pi}(2|\varepsilon_b|\varepsilon_F)^{1/2}, \tag{4.9}$$

(where γ is the Euler constant), which has only algebraic smallness. Recall that (4.9) practically coincides with the well known expression for 3D metals (up to a numerical factor; see part 5 of Ref. 30).

Thus, the results we have obtained show that the behavior of the phase diagram of a metal, and in particular the slope of its critical curve, depend significantly on the original carrier density. If the latter is such that $\varepsilon_F \ll |\varepsilon_b|$ (the regime of localized pairs), then this derivative turns out to be much smaller than for the case $\varepsilon_F \gg |\varepsilon_b|$ (Cooper pairs). By varying the number of carriers, we can track the dependence of $dH_{c2}(T)/dT$ on this quantity. In the BCS regime [see (4.9)] the slope of the critical curve is $\propto n_f^{-1/2}$, i.e., it can even increase as n_f decreases. However, once it has attained its maximum value, it begins to decrease (probably as the number of constituent local bosons and the importance of their role increase) down to practically zero [see (4.8)]. In this case, neither the slope of the critical curve nor the H–T phase diagram as a whole can be described by a single equation supplemented by the condition $\mu = \varepsilon_F$.

5. STRONG MAGNETIC FIELDS

In order to estimate the limit of strong magnetic fields, it is convenient to write the expression for the critical curve using the representation (A12) for the Green's function in the form of a sum over Landau levels[31]

$$\frac{4\pi}{gm} = \omega_H \sum_{n_1,n_2}^{\infty} \frac{(n_1+n_2)!}{n_1!n_2!2^{n_1+n_2}} \frac{1 - n_F(\varepsilon_{n_1}) - n_F(\varepsilon_{n_2})}{\varepsilon_{n_1} + \varepsilon_{n_2}},$$

$$\varepsilon_n \equiv \omega_H\left(n + \frac{1}{2}\right) - \mu \tag{5.1}$$

[for n_F see (3.19)]. In this case, the second equation, which relates μ to n_F, takes the form

$$\frac{\varepsilon_F}{\omega_H} = \sum_{n=0}^{\infty} n_F(\varepsilon_n). \tag{5.2}$$

We will limit ourselves to the mean-field approximation here, since, on the one hand, no self-consistent description of a 2D system with arbitrary density n_f in a strong magnetic field $\mathbf{H}$ is available, while on the other hand, the considerable nondiagonal character of the propagator Γ in both the coordinate and momentum space representations makes the problem of including fluctuation-induced corrections intractable even in the most natural basis to represent them, i.e., the Landau states.

In the limit $\varepsilon_F/\omega_H = \pi c n_f/eH \ll 1$, we find from (5.2)

$$\frac{1}{T}\left(\frac{\omega_H}{2} - \mu\right) = \ln\frac{\omega_H}{\varepsilon_F},$$

consequently, it is sufficient to limit ourselves to the asymptotic expression $n_F(\varepsilon_n) \approx \exp(-\varepsilon_n/T)$ in Eq. (5.1). If we now renormalize the constant g in the same way as we did in (3.20) for the bound states, we reduce Eq. (5.1) to the form

$$\psi\left(1 - \frac{2\mu}{\omega_H}\right) + \ln\frac{\omega_H}{|\varepsilon_b|} = -2\exp\left(-\frac{\omega_H - 2\mu}{4T}\right) \times \Phi\left(\frac{1+\exp(-\omega_H/2T)}{2}, 1, 1 - \frac{2\mu}{\omega_H}\right), \tag{5.3}$$

where

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz}, \quad \Phi(z,s,v) = \sum_{n=0}^{\infty}\frac{z^n}{(n+v)^s}.$$

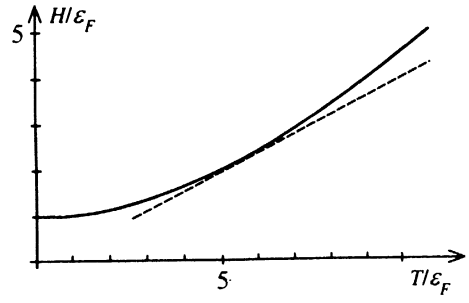

FIG. 2. Form of the critical curve in the strong-field limit. For comparison we show the form of the critical curve taken from Ref. 10 with a dashed curve.

In the region $1-2\mu/\omega_H \gg 1$, we can use the asymptotic forms of the function $\psi(z)_{z\to\infty} \propto \ln z-(2z^2)^{-1}$ in (5.3) and $\Phi(z,1,v)_{v\to\infty}\to v^{-1}$ to find the solution (under the condition $\varepsilon_F \ll \omega_H \lesssim |\varepsilon_b|$)

$$T=\frac{|\varepsilon_b|}{2}\ln^{-1}\frac{\omega_H}{\varepsilon_F} \tag{5.4}$$

(compare with the critical temperature $T_c=(|\varepsilon_b|/2)\ln^{-1}(|\varepsilon_b|/2\varepsilon_F)$ in the mean-field approximation[6]).

In the other limit $1-2\mu/\omega_H \ll 1$ (where we must use $\psi(z)_{z\to 0} \propto -z^{-1}$, $\Phi(z,1,v)_{v\to 0}\sim v^{-1}$), we are led to a curve on the phase diagram given by the equation

$$T=\frac{\omega_H}{2}\ln^{-1}\frac{\omega_H}{|\varepsilon_b|}\ln^{-1}\frac{\omega_H}{\varepsilon_F}=\frac{|\varepsilon_b(\mathbf{H})|}{2}\ln^{-1}\frac{\omega_H}{\varepsilon_F}, \tag{5.5}$$

which is correct for $\varepsilon_F, |\varepsilon_b| \ll \omega_H$ and arbitrary ratios of ε_F to $|\varepsilon_b|$ (the binding energy $\varepsilon_b(\mathbf{H})$ for two charged fermions in the presence on a magnetic field was calculated in[23]:

$$\delta_b(\mathbf{H})=-\omega_H\ln^{-1}(\omega_H/|\varepsilon_b|).$$

Thus, at sufficiently high fields ($\omega_H \gg \varepsilon_F$) and temperatures, a 2D metal actually possesses a nontrivial ($\tilde{\Phi} \neq 0$) phase, with a nonuniform order parameter, which appears under conditions of complete penetration of the field into the sample. We will not attempt to discuss the nature of this new phase (to do so we must calculate the value of the current, which is an independent problem); we will, however, note that, in contrast to,[10] the critical curve will be a complicated function that remains nonlinear even in the large-field limit (see Fig. 2).

6. CONCLUSIONS

To summarize our investigation, we have obtained a completely self-consistent system of equations for a 2D superconductor in an external magnetic field. We have shown that when the density of bare carriers is low, one of the equations, specifically the one that relates the carrier density to the chemical potential of the system, gives a nontrivial contribution that determines the behavior of the critical curve. In the opposite case of high densities, this equation is almost trivial, and in principle the system can be described by the BCS approximation. The dependence of the behavior of the critical curve on concentration mentioned above in the region of small fields should be verifiable experimentally for high-T_c superconductor systems, where the carrier density is an "external" parameter that is relatively easy to control.

Finally, we may perhaps have a new "high-field" ($H>(\pi c/e)n_f$) ordered phase to study in these systems, if we also choose high-T_c superconducting compounds with a sufficiently small carrier density that the required values of field become realistically attainable, as we discussed above. However, in view of the extreme simplicity of the model we have chosen to investigate these questions, we did not attempt to compare these functions with the available experiments, which up to now are not sufficiently diverse.

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APPENDIX

The Green's Function for Free Fermions in an External Magnetic Field

The equation for the Green's function has the form (1.7), in which $\Phi=\Phi^*=0$:

$$\left[-\partial_{\tau_1}+\tau_z\left(\frac{\mathbf{D}_1^2}{2m}+\mu\right)\right]G^{(0)}(\tau_1-\tau_2;\mathbf{r}_1,\mathbf{r}_2)=\delta(\tau_1-\tau_2)\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{A1}$$

where $\mathbf{D}$ is defined in (2.2) with the symmetric gauge $A=(1/2)(-Y,X)$. The boundary condition for this equation is given in (2.8). Accordingly, the equations of type (A1) for the coefficients of the Fourier expansion of $G_n^{(0)}(\mathbf{r}_1,\mathbf{r}_2)$, i.e., (2.10), have the form (2.11) for the free Green's function, where we have also set $\Phi=\Phi^*=0$. Abstracting from any concrete representation, we can write this expression in operator form

$$(i\omega_n-\mathscr{H}_f^{(0)})G_n^{(0)}=\mathbf{I}, \tag{A2}$$

where $\mathscr{H}_f^{(0)}$ corresponds to the Hamiltonian density (2.2) in the Nambu representation with $g=0$, and $\mathbf{I}$ is a unit matrix. Following Schwinger's convention,[32] let us rewrite (A2) using an integration over "proper" time such that

$$G_n^{(0)}=(i\omega_n-\mathscr{H}_f^{(0)})^{-1}=i\theta(-\omega_n)\int_0^\infty d\tau\exp[i(\mathscr{H}_f^{(0)}+i|\omega_n|)\tau]-i\theta(\omega_n)\int_0^\infty d\tau\exp[-i(\mathscr{H}_f^{(0)}-i|\omega_n|)\tau],$$

from which we obtain

$$G_n^{(0)}(\mathbf{r}_1,\mathbf{r}_2)=i\int_0^\infty d\tau\,\exp(-|\omega_n|\tau)\{\langle\mathbf{r}_1|[\theta \times(-\omega_n)\exp(i\,\mathscr{H}_f^{(0)}\tau)-\theta(\omega_n) \times\exp(-i\mathscr{H}_f^{(0)}\tau)]|\mathbf{r}_2\rangle\}. \tag{A3}$$

The matrix elements that enter into (A3) are easily rewritten in terms of path integrals:

$$\langle\mathbf{r}_1|\exp(\pm i\mathscr{H}_f^{(0)}\tau)|\mathbf{r}_2\rangle=A\int d\mathbf{r}(t)d\mathbf{p}(t) \times\exp\left\{i\int_0^\tau dt[\mathbf{p}(t)\mathbf{r}(t)\pm\mathscr{H}_f^{(0)}]\right\}, \tag{A4}$$

where A is the normalization constant determined from the equation

$$\langle\mathbf{r}_1|\exp(\pm i\,\mathscr{H}_f^{(0)}\tau)|\mathbf{r}_2\rangle|_{\tau=0}=\delta(\mathbf{r}_1-\mathbf{r}_2).$$

In the vector potential gauge we have chosen, the integral (A4) turns out to be Gaussian; after identifying a completed square in its exponential, it can be calculated to the end. In particular, the integration over momentum leads to the expression

$$\langle\mathbf{r}_1|\exp(\pm\mathscr{H}_f^{(0)}\tau)|\mathbf{r}_2\rangle =A\int d\mathbf{r}(t)\exp\left\{i\int_0^\tau dt\,\tau_z\left[\mp\frac{m}{2}\dot{\mathbf{r}}^2(t) +\frac{1}{2l^2}[\mathbf{r}(t)\dot{\mathbf{r}}(t)]_z\mp\mu\right]\right\}, \tag{A5}$$

in which the constant A is renormalized, while

$$\mathbf{r}(0)=\mathbf{r}_2,\mathbf{r}(\tau)=\mathbf{r}_1$$

can be used as boundary conditions.[4]

Integrating (A5), we eventually find that

$$\langle\mathbf{r}_1|\exp(\pm i\mathscr{H}_f^{(0)}\tau)|\mathbf{r}_2\rangle =\pm\frac{im\omega_\mathbf{H}}{4\pi\sin(\omega_\mathbf{H}\tau/2)}\tau_z\exp\frac{i}{2}\tau_z\left\{-\frac{1}{2l^2}[\mathbf{r}_1\mathbf{r}_2]_z \mp\frac{1}{4l^2}(\mathbf{r}_1-\mathbf{r}_2)^2\mathrm{ctg}(\omega_\mathbf{H}\tau/2)\mp\mu\tau\right\}, \tag{A6}$$

where, as in (A5), the notation $l\equiv(c/eH)^{1/2}$ has been used for the magnetic length, and $\omega_\mathbf{H}=eH/cm$ for the cyclotron frequency. Eventually, after substituting (A6) into (A3), we are led to the final expression for the Green's function:

$$G_n^{(0)}(\mathbf{r}_1,\mathbf{r}_2)=\exp\left(-\frac{i}{2l^2}\tau_z[\mathbf{r}_1\mathbf{r}_2]_z\right)G_n^{(\mathrm{hom})}(\mathbf{r}_1-\mathbf{r}_2), \tag{A7}$$

in which the spatially uniform portion is identified to be:

$$G_n^{(\mathrm{hom})}(\mathbf{r})=-\frac{m\omega_\mathbf{H}}{4\pi}\int_0^\infty d\tau\frac{1}{\sin(\omega_\mathbf{H}\tau/2)}\exp(-|\omega_n|\tau) \times\left\{\theta(-\omega_n)\exp\left\{-i\tau_z\left[\mu\tau +\frac{\mathbf{r}^2}{4l^2}\mathrm{ctg}\left(\frac{\omega_\mathbf{H}\tau}{2}\right)\right]\right\}+\theta(\omega_n)\exp\left\{i\tau_z\left[\mu\tau +\frac{\mathbf{r}^2}{4l^2}\mathrm{ctg}\left(\frac{\omega_\mathbf{H}\tau}{2}\right)\right]\right\}\right\}. \tag{A8}$$

At first glance, it may seem that the representation (A8) is not entirely convenient, because the denominator of the expression under the integral sign reduces to zero at points $\tau=2\pi j\omega_\mathbf{H}^{-1}$ (where j is a non-negative integer). However, in fact the strong oscillations of the exponentials in the curly brackets regulate the behavior of the integral in the vicinity of these poles. We can demonstrate this by writing (A8) in another form, using the identity

$$i\,\mathrm{ctg}\,u=1-2\,\exp(2iu)/[\exp(2iu)-1],$$

and also the generating function for the Laguerre polynomials[25]

$$(1-z)^{-1-\alpha}\exp[uz/(z-1)]=\sum_{j=0}^\infty L_j^\alpha(u)z^j,\quad |z|<1,$$

These expressions allow us to express the Green's function (A8) in the form

$$G_n^{(\mathrm{hom})}(\mathbf{r})=\frac{1}{2\pi l^2}\exp\left(-\frac{\mathbf{r}^2}{4l^2}\right)\sum_{j=0}^\infty L_j\left(\frac{\mathbf{r}^2}{2l^2}\right)\left\{i\omega_n -\tau_z\left[\omega_\mathbf{H}\left(j+\frac{1}{2}\right)-\mu\right]\right\}^{-1}. \tag{A9}$$

After the analytic continuation $i\omega_n\to\omega$, the function $G_n^{(\mathrm{hom})}(\mathbf{r})$ becomes the Green's function $G^{(\mathrm{hom})}(\omega;\mathbf{r})$ for free fermions in a magnetic field when $T=0$. The poles of the latter correspond to the Landau equation for values of the frequency $\omega_j=\tau_z[\omega_\mathbf{H}(j+1/2)-\mu]$, as it should be.

The coordinate dependence of the Green's function $G_n^{(\mathrm{hom})}(\mathbf{r})$ also shows that it is most natural and convenient to write it in the momentum representation:

$$G_n^{(\mathrm{hom})}(\mathbf{k})=\int d\mathbf{r}\,\exp(-i\mathbf{k}\mathbf{r})G_n^{(\mathrm{hom})}(\mathbf{r}). \tag{A10}$$

Substituting (A8) and (A9) into (A10) give respectively

$$G_n^{(\mathrm{hom})}(\mathbf{k})=i\int_0^\infty d\tau\frac{\exp(-\omega_n\tau)}{\cos(\omega_\mathbf{H}\tau/2)}\{\theta(-\omega_n)\exp\{-i\tau_z[\mu\tau -k^2l^2\mathrm{tg}(\omega_\mathbf{H}\tau/2)]\}-\theta(\omega_n) \times\exp\{i\tau_z[\mu\tau-k^2l^2\,\mathrm{tg}\,(\omega_\mathbf{H}\tau/2)]\}\}, \tag{A11}$$

$$G_n^{(\mathrm{hom})}(\mathbf{k})=2\,\exp(-k^2l^2)\sum_{j=0}^\infty(-1)^jL_j(2k^2l^2)\left\{i\omega_n -\tau_z\left[\omega_\mathbf{H}\left(j+\frac{1}{2}\right)-\mu\right]\right\}^{-1}. \tag{A12}$$

These representations are equivalent; however, the first is more convenient to use in the range of small magnetic fields, while the second is better for large magnetic fields. In particular, from (A11) we can show that if $\mathbf{H}\to0$ (i.e., $l\to\infty$), to within terms of order l^{-4} the Green's function has the form

$$G_n^{(\mathrm{hom})}(\mathbf{k})=\frac{1}{i\omega_n-\tau_z\xi(\mathbf{k})}\left\{1-\frac{(4m^2l^4)^{-1}}{[i\omega_n-\tau_z\xi(\mathbf{k})]^2}-\tau_z\frac{(4m^3l^4)^{-1}\mathbf{k}^2}{[i\omega_n-\tau_z\xi(\mathbf{k})]^3}\right\}, \tag{A13}$$

in which $\xi(\mathbf{k})=k^2/2m-\mu$.

[1)]The problem of the transition from an insulating state to a metallic state due to doping of the originally insulating high-T_c superconducting compounds, and the minimal carrier concentration corresponding to this transition, below which they turn out to entirely localized, in thlis type of problem has not been addressed. The distinctive features of such a 2D metal–insulator transition were studied, e.g., in Ref. 4.

[2)]As we are interested only in the transition between the superconducting and normal phases, we will not discuss the other critical curve, namely $H_{c1}(T)$.

[3)]The authors of the recent paper[22] have noted that the general solution to this equation is given by the same expression (3.8) with the replacement $\Delta\to f(Z)$ with $Z=X+iY$, where $f(Z)$ is the derivative of a holomorphic function. The zeroes of this function coincide with the positions of the centers of Abrikosov vortices, whose mutual spacings are not fixed by the linearized equation (3.5) and must be found by solving Eq. (3.3).

[4)]As a check, we write the "classical" trajectory that arises in computing (A5): $X(t)=a\cos\omega t+b\sin\omega t-\bar{X}-\Delta Y\ \mathrm{ctg}\ (\omega_H\pi/2)$, $Y(t)=a\sin\omega t-b\cos\omega t+Y+\Delta X\ \mathrm{ctg}\ (\omega_H\pi/2)$, $a\equiv-\Delta X+\Delta Y\ \mathrm{ctg}\ \omega_H\pi/2)$, $b\equiv\Delta X\ \mathrm{ctg}\ (\omega_H\pi/2)+\Delta Y$, $\bar{\mathbf{r}}\equiv(\mathbf{r}_1+\mathbf{r}_2)/2, \Delta\mathbf{r}=(\mathbf{r}_1-\mathbf{r}_2)/2$.

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