Kinetic theory of a relativistic gas with internal degrees of freedom in the presence of weak sources

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(Submitted 15 December 1994)
Zh. Éksp. Teor. Fiz. **107**, 1877–1894 (June 1995)

Dissipative relativistic hydrodynamic equations are derived from the linearized kinetic equation for a gas with internal parameters. The method does not depend on any a priori assumptions about the relative size of dissipative terms, spatial gradients, or the collision integral. It is shown that the transition from the kinetic to the hydrodynamic description is reversible. The hydrodynamic model is nonlocal. This makes the dissipation compatible with causality. For a multicomponent mixture without reactions it is shown how this theory helps to describe viscosity, thermal conductivity, and diffusion. Relations between the nonlocal kernels and the classic transport coefficients are obtained. © *1995 American Institute of Physics.*

1. INTRODUCTION

A consistent description of dissipative processes (such as viscosity, thermal conductivity, diffusion, etc.) in a relativistic theory involves considerable complications. Specifically, in nonrelativistic mechanics these phenomena are described by parabolic equations, which give rise to an infinite signal velocity. This is not permitted in a systematic relativistic theory.

In the first relativistic model of a nonideal gas (Eckart[1] and Landau and Lifshitz[2]) a covariant theory was derived including dissipativity. Then it was shown[3] that both these models are special cases of a broader class of models derived using the Chapman–Enskog method from the relativistic kinetic equation and differing in the definition of the local 4-velocity of the medium. All these models suffer from a shortcoming inherent in the parabolic Navier–Stokes–Fourier model of a nonrelativistic viscous thermally conductive gas: the signal velocity in the comoving inertial coordinate frame is infinite.

In Ref. 4 it was shown that the finite signal velocity is a result of taking into account the dispersion in the transport coefficients, that is, in a model with legacy. For a relativistic gas consisting of structureless massive particles a method was proposed for going from the kinetic equation to the equations of nonlocal hydrodynamics.[5] The hydrodynamic model consistently combines the properties of dissipativity and causality. The hydrodynamic kernels are expressed in terms of collision operators. The nonrelativistic version of the method has been described in Ref. 6.

In the present work the results obtained in Ref. 5 are extended to the case of a relativistic gas with internal degrees of freedom. It is shown that for a certain class of sources the kinetic equation is equivalent to a system of nonlocal hydrodynamic equations. The present technique for deriving the equations of dissipative relativistic hydrodynamics differs from the methods of Chapman and Enskog, Grad, and Hilbert.[7–10] It is assumed that the gas can contain particles with internal structure (i.e., with vibrational and rotational degrees of freedom), and it can also be a mixture of different components. This latter assumption allows us to consider the phenomenon of diffusion.

The technique used here requires no *a priori* assumptions about the magnitude of the dissipative terms, the spatial gradients, or the collision integral. It also requires no information about the explicit functional form of the collision integral. The method is based on introducing hydrodynamic sources into the kinetic equation and projecting the distribution functions onto the space of hydrodynamic variables. Previously similar projection operations were used for equations without sources in Refs. 11 and 12. However, in the systematic passage from kinetic theory to nonlocal hydrodynamics the use of sources is essential. In fact, operations with homogeneous equations implicitly presuppose the imposition of a Cauchy problem, which is dubious in a theory with temporal nonlocality.

Section 2 gives a systematic account of the properties of the relativistic equation needed in the subsequent exposition. In Sec. 3 we present a derivation of the equations of nonlocal hydrodynamics and show that the hydrodynamic model is equivalent to the original linearized kinetic equation. In Sec. 4 the properties of the nonlocal kernels arising in the constitutive relations of the hydrodynamic model are analyzed. In particular, we derive relations associated with dissipativity and time reversibility at the microscopic level (analogous to the Onsager relations). Section 5 treats a multicomponent relativistic gas and shows that the phenomena of viscosity, thermal conductivity, and diffusion can be described without violating causality. In Sec. 6 the results of the work are discussed.

We use a system of units in which Planck's constant $\hbar$, the velocity of light in vacuum c, and the Boltzmann constant k are equal to unity. Greek subscripts run through the values 0,1,2,3 corresponding to some inertial space-time coordinate system x^α, where x^0 is time; roman subscripts a,b,c,d run through the values 1,2,3, corresponding to the spatial Cartesian coordinates x^a; and we write $\partial_\alpha=\partial/\partial x^\alpha$. The space-time subscripts are raised and lowered using the Minkowsky metric $\eta_{\alpha\beta}=\mathrm{diag}(1,-1,-1,-1)$. The roman subscripts A,B,C run through the values $0,\ldots,\ N-1$, where N is the number of quantities given in Eq. (1.3) below which are

1063-7761/95/061041-08$10.00

conserved in particle collisions. Summation over repeated indices is assumed unless otherwise specified.

2. PROPERTIES OF THE RELATIVISTIC KINETIC EQUATION

In formulating the relativistic kinetic theory we will on the whole follow the approach and notation of Refs. 3 and 5. Consider a gas whose molecules are characterized by the coordinates x^α and 4-momenta p^β and by some collective parameter r associated with the other degrees of freedom. The parameter r can run through discrete or continuous values. Thus, r parameters can include the number of components in the case of a mixture of gases, the rotational angular momentum of a molecule, the label of the vibrational mode of a polyatomic molecule, etc.

A state of the system is described by the one-particle distribution function $f=f(x^\alpha,p^\beta,r)$. For a given value of the parameter r the 4-momentum p^β satisfies the condition

$$\eta_{\alpha\beta}p^\alpha p^\beta=m^2(r), \quad p^0\geqslant 0. \tag{1.1}$$

Here $m(r)$ is the rest mass of the gas particles characterized by the parameter r. The relation (1.1) is an equation on the mass surface for particles with positive mass and on the future light cone when the mass is equal to zero. The Lorentz-invariant measure on the surface (1.1) is

$$d\nu_r=(p^0)^{-1}dp^1dp^2dp^3.$$

Let $d\mu(r)$ be the measure in the space of the parameters r. In the state space of a gas particle, i.e., in the space of the parameter p^α and r associated with the condition (1.1), we define the measure

$$d\zeta=d\nu_r d\mu(r).$$

The distribution function satisfies the dynamic equation

$$p^\alpha\partial_\alpha f=\mathrm{St}[f]+S. \tag{1.2}$$

Here St[f] is the so-called collision integral; in the general case it is a nonlinear operator in the space of functions of p^α and r. The function $S=S(x^\alpha,p^\beta,r)$ describes the interchange of material, heat, etc., between the gas and an external medium. The use of the kinetic equation (1.2) without a source term implies that the macroscopic fields produced by the gas (electromagnetic, gravitational, etc.) are negligibly small. The introduction of a source S in the kinetic equation enables us to rigorously treat nonequilibrium distributions defined globally in time.

We assume that the particle collision dynamics admits N integrals

$$J_A=J_A(p^\alpha,r). \tag{1.3}$$

Then the operator St[f] satisfies N relations

$$\int J_A(p^\alpha,r)\mathrm{St}[f](x^\beta,p^\alpha,r)d\zeta=0 \tag{1.4}$$

for an arbitrary distribution function f. We will assume that the set of constants of motion (1.3) is complete in the sense that in the absence of sources the equilibrium state is fully characterized by the values $\langle J_A\rangle$. The distribution function corresponding to equilibrium takes the form

$$f_e=f_e(p^\alpha,r)=(2\pi)^{-3}\exp(F^AJ_A), \tag{1.5}$$

where F^A are constants. For an arbitrary distribution function (1.5) we have identically

$$\mathrm{St}[f_e]=0. \tag{1.6}$$

Relations (1.2) and (1.4) yield the relativistic hydrodynamic equations with sources,

$$\partial_\alpha Q_A^\alpha=s_A, \tag{1.7}$$

where we have used the following definitions for the 4-currents Q_A^α and sources s_A:

$$Q_A^\alpha=Q_A^\alpha(x^\beta)=\int p^\alpha J_A(p^\beta,r)f(x^\alpha,p^\beta,r)d\zeta,$$

$$s_A=s_A(x^\alpha)=\int J_A(p^\beta,r)S(x^\alpha,p^\beta,r)d\zeta. \tag{1.8}$$

We now investigate the dynamics of linear perturbations about some equilibrium distribution f_0 of the form (1.5) induced by weak sources S. Let F_0^A be parameters corresponding to f_0 in expression (1.5). We will use the customary form of the distribution function subjected to linearization:[3] $f=f_0(1+\varphi)$. Then Eq. (1.2) is transformed into

$$(p^\alpha\partial_\alpha-L)\varphi=s, \quad s=f_0^{-1}S, \tag{1.9}$$

where L is a linear operator defined through functional differentiation of the collision integral: $L\varphi=f_0^{-1}D\,\mathrm{St}[f_0](f_0\varphi)$. The functions φ, regarded from the standpoint of their dependence on the arguments p^α and r, belong to a space H which can conveniently be taken to be a Hilbert space with the scalar product

$$(\varphi_1,\varphi_2)=\int f_0\varphi_1^*\varphi_2 d\zeta.$$

By H_h we denote the subspace spanned by the family of vectors J_A, and by H_a the orthogonal complement of H_h: $H=H_h\oplus H_a$. In the subspace H_h the metric tensor $\gamma_{AB}=(J_A,J_B)$ is defined, by means of which we can raise and lower the subscripts A,B,C. In particular, a family of vectors J^A is defined such that $\delta_A^B=(J_A,J^B)$.

The sources s_A can be regarded as weak if for the characteristic time scale T_* of the problem in question (e.g., for a wave period) we have the set of inequalities

$$|T_*\gamma^{AB}s_A|\ll|F_0^A|.$$

Without specifying the form of the collision integral we can assert that the operator L satisfies a number of conditions. Thus, by differentiating Eq. (1.6) with respect to F^A we find the relations

$$LJ_A=0. \tag{1.10}$$

Then we have relations associated with reality,

$$(L\varphi)^*=L\varphi^*, \tag{1.11}$$

dissipativity,

$$L+L^+\leqslant 0, \tag{1.12}$$

and the conservation laws (1.4),

$$L^{+}J_A=0. \tag{1.13}$$

In the subspace H_a the inequality (1.12) becomes strict.

We assume that the interaction of the gas particles is time-reversible on the microscopic level (T-invariance in quantum theory[13]). In the function space H we define the time inversion operation I. Thus, the operator I changes the sign of the spatial components of the 4-momenta, but leaves the helicity unchanged if it enters into the set of parameters r. The operator I is self-adjoint and satisfies $I^2=1$. The integrals (1.3) can be chosen to be eigenfunctions of the operator I with eigenvalues ± 1:

$$IJ_A=\varepsilon_A J_A, \quad \varepsilon_A=\pm 1. \tag{1.14}$$

Let the perturbed equilibrium distribution be invariant under time reversal,

$$If_0=f_0. \tag{1.15}$$

Then reversibility on the microscopic level imposes a restriction on L:

$$L^{+}=ILI. \tag{1.16}$$

In many situations the operator L commutes with the operator I, and so in place of (1.16) a stronger condition $L^{+}=L$ holds.

We will use the following notation: for the function $g=g(x^{\alpha})$ the Fourier transform $g_F(k_\alpha)$ is defined by

$$g_F=g_F(k_\alpha)=\int \exp(-ik_\alpha x^{\alpha})g(x^{\alpha})dx^{\alpha}.$$

Fourier-transforming Eq. (1.9), we find the linear operator equation

$$G\varphi_F=s_F,$$

$$G=ik_\alpha p^{\alpha}-L. \tag{1.17}$$

By using Eq. (1.17) we can investigate the rate at which a signal propagates from the point source $S=f_0 h_{*}\delta(x^{\alpha})$, $h_{*}\in H$. From (1.17) we find the equation

$$G\varphi_F=h_{*}. \tag{1.18}$$

Consider Eq. (1.18) for complex values k_α, such that the imaginary parts of the wave 4-vector $K_\alpha=\operatorname{Im} k_\alpha$ satisfy the conditions

$$K^{\alpha}K_\alpha>0, \quad K^0<0. \tag{1.19}$$

Then using (1.12) we obtain the inequality

$$G+G^{+}\geqslant -2K_\alpha p^{\alpha}>0. \tag{1.20}$$

Together with the conditions (1.1) this inequality ensures that the vector function $\varphi_F=G^{-1}h_{*}$ is defined and holomorphic in the complex tube (1.19). By following Ref. 14 we can use this together with some readily verified asymptotic expressions to show that the function φ in real space-time vanishes outside the future light cone,

$$x^0\geqslant(=\eta_{ab}x^{a}x^{b})^{1/2}. \tag{1.21}$$

Hence the kinetic equation (1.2) satisfies the causality principle of the special theory of relativity.

3. DERIVATION OF THE EQUATIONS OF LINEAR RELATIVISTIC DISSIPATIVE HYDRODYNAMICS

We introduce some notation. Let $P_h:H\rightarrow H_h$, $P_a:H\rightarrow H_a$ be projection operators and let $I_h:H_h\rightarrow H$, $I_a:H_a\rightarrow H$ be injection operators, $G_{hh}=P_hGI_h$, $G_{ah}=P_aGI_h$, $G_{ha}=P_hGI_a$, $G_{aa}=P_aGI_a$. Note that according to Eqs. (1.10) and (1.13) we have $LH_h=0$, $LH_a\subseteq H_a$. We split the distribution function φ into two parts, $h=P_h\varphi$, $a=P_a\varphi=\varphi-h$.

We define the hydrodynamic 4-currents as linear perturbations of the 4-currents Q_A^{α} [cf. Eq. (1.18)],

$$g_{\mathrm{A}}^{\alpha}=(J_{A}p^{\alpha},\varphi).$$

From Eq. (1.9) together with (1.13) we find the hydrodynamic equations with sources,

$$\partial_\alpha g_{\mathrm{A}}^{\alpha}=s_{\mathrm{A}}, \quad s_{\mathrm{A}}=(J_{\mathrm{A}},s). \tag{2.1}$$

In order to solve these equations with given right-hand sides, we need to know the constitutive relations, i.e., expressions for the components g_{A}^{α} of the 4-currents in terms of the given set of hydrodynamic variables. Specification of the constitutive relations together with the dynamic equations (2.1) closes the hydrodynamic model.

In Ref. 5 we follow approach (A), in which the 4-currents g_A^{α} are expressed in terms of the components $h^{A}=(J^{A},\varphi)=(J^{A},h)$. An explicit physical meaning can be ascribed to the quantities h^{A}. To see this we find the equilibrium distribution (1.5) with parameters $F^{A}=F_0^{A}+\Delta F^{A}$ such that at a given point x^{α} it coincides with the distribution $f_0(1+h)$. Then we find the relation

$$h^{A}(x^{\alpha})=\Delta F^{A}. \tag{2.2}$$

A second approach (B) is possible, in which the spatial components g_A^{a} are expressed in terms of g_A^{0}. Below we consider both approaches.

In order to find the constitutive relations we assume that the source term in Eq. (1.9) as a function of p^{α} and r belongs to the space H_h. Then the "nonhydrodynamic" part of the distribution function a can be expressed according to (1.17) in terms of h,

$$a_F=-G_{aa}^{-1}G_{ah}h_F. \tag{2.3}$$

We express the hydrodynamic 4-currents in terms of the components $h^{A}=(h,J^{A})$. If we use relation (2.3) it is not difficult to calculate the constitutive relations in form A:

$$g_{AF}^{\alpha}=U_{AB}^{\alpha}h_F^{B},$$

$$U_{AB}^{\alpha}=Z_{AB}^{\alpha}-{}^{0}A_{AB}^{\alpha}, \quad Z_{AB}^{\alpha}=(p^{\alpha}J_A,J_B),$$

$${}^{0}A_{AB}^{\alpha}=(p^{\alpha}J_A,G_{aa}^{-1}G_{ah}J_B). \tag{2.4}$$

By virtue of relations (1.11) for real k_β we have

$${}^{0}A_{AB}^{\alpha}(k_\beta)^{*}={}^{0}A_{AB}^{\alpha}(-k_\beta). \tag{2.5}$$

We write

$$(W^{AB})=(U_{AB}^{0})^{-1}, \quad C_A^{a\mathrm{B}}=U_{AC}^{a}W^{CB}, \tag{2.6}$$

Then from (2.4) we find the constitutive relations in the form B:

$$g^{a}_{AF}=C^{aB}_{A}g^{0}_{BF}. \tag{2.7}$$

In analogy to (2.5) we have

$$C^{aB}_{A}(k_{\alpha})^{*}=C^{a\mathrm{B}}_{A}(-k_{\alpha}). \tag{2.8}$$

It is clear that the possibility of rigorously specifying the constitutive relations in the form B is directly related to the determinability of the matrix W^{AB} in Eq. (2.6). Let us study this question in more detail. We define the matrices

$$L^{\alpha}_{AB}=(p^{\alpha}J_{A},G^{-1}J_{B}),(\Delta^{AB})=(L^{0}_{AB})^{-1}.$$

It is easy to see that there is a representation

$$g^{\alpha}_{AF}=L^{\alpha}_{AB}s^{B}_{F},$$

which yields an alternative representation for the coefficients in relation (2.7),

$$C^{aB}_{A}=L^{a}_{AC}\Delta^{CB}.$$

Thus, the question reduces to the determinability of the matrix Δ^{AB}. It turns out that for this a sufficient condition is that the inequalities that supplement the dissipativity condition (1.12) hold:

$$Lp^{0}+p^{0}L^{+}\leqslant 0, \tag{2.9}$$

where the inequality becomes strict on H_{a}. Then the reversibility of the matrix L^{0}_{AB} follows from the chain of conditions

$$L^{0}_{AB}+L^{0*}_{BA}=-(G^{\pm 1}J_{A},(Lp^{0}+p^{0}L^{+})G^{-1}J_{B})\geqslant 0. \tag{2.10}$$

We will further assume that when we treat the constitutive relations in the form B given by Eq. (2.7), condition (2.9) is satisfied. It is noteworthy that in nonrelativistic kinetic theory in the analogous derivation for the determinability of the constitutive relations in form B the usual dissipativity condition (1.12) is sufficient.

If we go over to real space-time, the constitutive relations in the form A of Eq. (2.4) assume the form of a sum of convolutions with real kernels characterizing the nonlocality of the theory. By virtue of (1.20) the functions U^{α}_{AB} are analytic in the complex tube (1.19) and can only have algebraic singularities at its boundary. From this it follows that in real space-time the hydrodynamic nonlocality kernels vanish outside the future light cone.[14] Thus, the constitutive relations in form A are causal.

In similar fashion we can prove the same assertion for the constitutive relations in form B as well.

4. PROPERTIES OF THE HYDRODYNAMIC KERNELS

We obtain restrictions on the coefficient functions in the constitutive relations, derived from covariance, dissipativity, and reversibility on the microscopic level. Where feasible we will omit the subscripts $A,B,\ldots$, associated with the labels of the conserved quantities (1.3) and use the matrix formalism.

We begin with the constitutive relations in the form A of Eq. (2.4). The nonlocal nature of the theory is connected with the matrices ${}^{0}A^{\alpha}_{AB}$.

Let the equilibrium state f_0 be invariant under the spatial rotation group $SO(3)$. The requirement of invariance under the action of this group imposes restrictions on the functional form of the matrices ${}^{0}A^{\alpha}$:

$$(R_{0}(g){}^{0}A)_{AB}(k_{0},gk_{b})={}^{0}A^{0}_{AB}(k_{0},k_{b}), \tag{3.1}$$

$$(R_{1}(g){}^{0}A)^{a}_{AB}(k_{0},gk_{b})={}^{0}A^{a}_{AB}(k_{0},k_{b}) \tag{3.2}$$

for arbitrary $g\in SO(3)$. Here R_0 is a representation of the rotation group in the linear space Φ_0 of quantities of the form ${}^{0}A^{0}_{AB}$, and R_1 is a representation of the rotation group in the linear space Φ_1 of quantities of the form ${}^{0}A^{a}_{AB}$. We recall that the subscripts A,B can also be transformed under the group $SO(3)$. In the spaces Φ_0, Φ_1 we identify a maximal set of linearly independent invariants with respect to the subgroup of rotations that preserve the vector k_a: $I^{n}_{0} = I^{n}_{0AB}(k_b)$, $I^{n}_{1} = I^{na}_{1AB}(k_b)$. These invariants can be chosen in the form of polynomials of k_a such that

$$(I^{na}_{mAB}(k_{b}))^{*}=I^{na}_{mAB}(-k_{b}).$$

Then we can display the most general functional form of the coefficient matrices ${}^{0}A^{\alpha}_{AB}$ satisfying relations (3.1) and (3.2):

$${}^{0}A^{0}_{AB}=I^{n}_{0AB}\,{}^{0}X_{0n},{}^{0}A^{a}_{AB}=I^{na}_{1AB}\,{}^{0}X_{1n}. \tag{3.3}$$

Here ${}^{0}X_{mn}={}^{0}X_{mn}(k_{\alpha})$ are scalar quantities which are invariant under the action of the rotation group. In consequence of relations (2.5) these functions satisfy the relation

$$({}^{0}X_{mn}(k_{\alpha}))^{*}={}^{0}X_{mn}(-k_{\alpha}).$$

This means that the functions ${}^{0}X_{mn}$ are Fourier transforms of real kernels ${}^{0}Y_{mn}={}^{0}Y_{mn}(x_{\alpha})$:${}^{0}Y_{mnF}={}^{0}X_{mn}$. Since the theory is causal, the supports of these kernels lie in the future cone (1.21).

We define the matrix

$${}^{0}B_{AB}=-ik^{0}_{\alpha}A^{\alpha}_{AB}=(G_{ah}J_{A},G^{-1}_{aa}G_{ah}J_{B}).$$

From the dissipativity condition (1.12) we have

$$G^{-1}_{aa}+G^{-1+}_{aa}=-G^{-1+}_{aa}(L+L^{+})G^{-1}_{aa}\geqslant 0.$$

From this we find the matrix inequality corresponding to the dissipativity condition (1.12) in the hydrodynamic model:

$${}^{0}B+{}^{0}B^{+}\geqslant 0. \tag{3.4}$$

Now let us find the implications of reversibility (1.14)–(1.16) on the microscopic level. Substituting (1.16) into the expression for ${}^{0}B$ and using (2.5) we find

$${}^{0}B_{AB}(k_{0},k_{a})=\varepsilon_{A}\varepsilon^{0}_{B}B_{BA}(k_{0},-k_{a}). \tag{3.5}$$

The substitution of expressions (3.3) in (3.4) or (3.5) allows us to find the restrictions on the kernels ${}^{0}X_{mn}$ implied by dissipativity and T-invariance respectively.

Although the functions J_A are linearly independent, the functions $p^{\alpha}J_A$ and J_B may be linearly dependent. Then from each relation of the form

$$\Lambda^{A}_{\alpha}p^{\alpha}J_{A}+\lambda^{A}J_{A}=0$$

we find a condition on the coefficient functions,

$$\Lambda^{A}_{\alpha}\,{}^{0}A^{\alpha}_{AB}=0, \tag{3.6}$$

which can be interpreted as degeneracy of the system.

Now we proceed to analyze the constitutive relations in the form B (2.7). It is easy to see that the identity

$$ik_{\alpha}L^{\alpha}_{AB}(k_{\beta})=\gamma_{AB}$$

holds. From this it follows that the coefficients in the constitutive relations (2.7) are related by

$$ik_{a}C^{a}=\gamma\Delta-ik_{0}. \tag{3.7}$$

Let us write ${}^{1}\mathrm{A}^{a}=C^{a}\gamma$, ${}^{1}B=ik_{a}{}^{1}A^{a}$. It turns out to be easier technically to investigate the coefficients that enter into the matrices ${}^{1}A^{a}$. These matrices allow the 3-currents g^{a}_{A} to be expressed in terms of the quantities $\xi^{A}=\gamma^{AB}g^{0}_{B}$. The reversible transformation from the matrices ${}^{1}A^{a}$ to the matrices C^{a} can easily be established using the metric γ.

The requirement of invariance with respect to the group $SO(3)$ imposes the following restrictions on the functional form of the matrices ${}^{1}A^{\alpha}$:

$$(R_{1}(g)\,{}^{1}A)^{a}_{AB}(k_{0},gk_{b})={}^{1}A^{a}_{AB}(k_{0},k_{b}). \tag{3.8}$$

The most general functional form of the coefficient matrices ${}^{1}A^{a}_{AB}$ satisfying relations (3.8), is

$${}^{1}A^{a}_{AB}=I^{na}_{1AB}\;{}^{1}X_{1n}. \tag{3.9}$$

Here ${}^{1}X_{mn}={}^{1}X_{mn}(k_{\alpha})$ are scalar quantities invariant under the action of the rotation group. As a result of Eqs. (2.8) these functions satisfy the relation

$$({}^{1}X_{mn}(k_{\alpha}))^{*}={}^{1}X_{mn}(-k_{\alpha}).$$

This means that the functions ${}^{1}X_{mn}$ are Fourier transforms of some real kernels ${}^{1}Y_{mn}={}^{1}Y_{mn}(x^{\alpha})\!:{}^{1}Y_{mnF}={}^{1}X_{mn}$. Since the theory is causal, the supports of these functions lie in the future cone (1.21).

From (3.7) we have

$${}^{1}B=\gamma\Delta\gamma-ik_{0}\gamma.$$

Hence using (2.10) we find the inequality

$${}^{1}B+{}^{1}B^{+}=\gamma\Delta^{+}(L^{0+}+L^{0})\quad\Delta\gamma\geqslant 0. \tag{3.10}$$

This inequality is the dissipativity condition in the form (2.9) in the hydrodynamic model.

If we assume the reversibility conditions (1.14)–(1.16), then it is easy to derive the reciprocity relations analogous to (3.5):

$${}^{1}B_{AB}(k_{0},k_{a})=\varepsilon_{A}\varepsilon^{1}_{B}B_{BA}(k_{0},-k_{a}). \tag{3.11}$$

As before, the substitution of expressions (3.9) in (3.10) or (3.11) allows us to obtain restrictions on the kernels ${}^{1}X_{mn}$ resulting from dissipativity or reversibility respectively.

Conditions (3.10) and (3.11) have the standard form of the dissipativity and reversibility conditions which obtain for mechanical systems with legacy.[15] It is for this reason that the constitutive relations in the form B are preferable to the ones in form A.

5. HYDRODYNAMICS OF A MULTICOMPONENT RELATIVISTIC GAS

Let us consider a relativistic K-component gas of particles which may have internal structure. In this case the parameter r has the form of a set (i,r'), where i runs through the values $1,\ldots,K$ corresponding to the labels of the components, and the parameter r' is related to the internal degrees of freedom (e.g., the vibrational degrees). Integration with respect to the measure $d\mu(r)$ reduces to summation over the parameter i and integration with respect to some measure $d\zeta'(r')$ associated with the parameter r'. We further assume that the indices i,j run through the values $1,\ldots,K$, while the indices I, J run through the values $(3+i)$, $i=1,\ldots,K$. If the indices i,j and the indices I, J are used in a single formula their values are related by $I=i+3$, $J=j+3$. Assume that there are no reactions between particles and that the following is a complete set of $K+4$ conserved quantities:

$$J_{\alpha}(p^{\beta},r)=p_{\alpha},\quad J_{J}(p^{\beta},r)=\delta_{ji}.$$

According to Ref. 3 the parameters F^{A} in the equilibrium distribution (1.5) have the following form:

$$F^{\alpha}=-\beta u^{\alpha},\quad F^{I}=\beta\mu_{i}. \tag{4.1}$$

Here β is the inverse temperature, μ_{i} are the chemical potentials of the components, and u^{α} is a 4-vector satisfying the conditions

$$u^{\alpha}u_{\alpha}=1,\quad u^{0}>0, \tag{4.2}$$

which can be interpreted as the 4-velocity of the medium in the equilibrium state.

Using the notation (1.8) we define the energy-momentum tensor $T^{\alpha\beta}$ of the medium and the flux components j^{α}_{i}:

$$T^{\beta}_{\alpha}=Q^{\beta}_{\alpha},\quad j^{\alpha}_{i}\equiv Q^{\alpha}_{i+3}.$$

Equations (1.7) in this case assume the form

$$\partial_{\beta}T^{\alpha\beta}=f^{\alpha},\quad \partial_{\alpha}j^{\alpha}_{i}=\nu_{i}.$$

Here $f^{\alpha}=\eta^{\alpha\beta}s_{\beta}$ is the density of the external 4-forces and ν_{i} are the external particle sources. We define the total particle flux as

$$j^{\alpha}=\sum_{i} j^{\alpha}_{i}.$$

In terms of the currents j^{α}_{i} and j^{α} we can define the partial densities n_{i} and the 4-velocities u^{α}_{i} of the component species, as well as the total density n and 4-velocity u^{α} of the gas:

$$j^{\alpha}_{i}=n_{i}u^{\alpha}_{i},\quad j^{\alpha}=nu^{\alpha}. \tag{4.3}$$

The definition (4.3) of the velocity of the medium corresponds to the Eckart approach, and also to the definition of the 4-velocity u^{α} of Eqs. (4.1) and (4.2), assumed previously. We define the diffusive fluxes using the formula (no summation over the subscript i!)

$$d^{\alpha}_{i}=n_{i}(u^{\alpha}_{i}-u^{\alpha}). \tag{4.4}$$

Next, the energy-momentum tensor is represented in the form

$$T^{\alpha\beta}=\varepsilon u^{\alpha}u^{\beta}+q^{\alpha}u^{\beta}+u^{\alpha}q^{\beta}+\pi^{\alpha\beta}, \tag{4.5}$$

where the heat-flux vector q^{α} and the stress tensor $\pi^{\alpha\beta}$ are subject to the restrictions

$$q^{\alpha}u_{\alpha}=0, \quad \pi^{\alpha\beta}u_{\alpha}=0. \tag{4.6}$$

Relation (4.5) together with (4.6) can be regarded as a definition of the internal energy ε, as well as q^{α} and $\pi^{\alpha\beta}$:

$$\varepsilon=T^{\alpha\beta}u_{\alpha}u_{\beta}, \tag{4.7}$$

$$q^{\alpha}=T^{\alpha\beta}u_{\beta}-\varepsilon u^{\alpha}, \tag{4.8}$$

$$\pi^{\alpha\beta}=T^{\alpha\beta}-(\varepsilon u^{\alpha}u^{\beta}+q^{\alpha}u^{\beta}+u^{\alpha}q^{\beta}). \tag{4.9}$$

The viscous stress tensor

$$\tau^{\alpha\beta}=p\Gamma^{\alpha\beta}-\pi^{\alpha\beta}, \tag{4.10}$$

is defined in terms of the stress tensor; here we have written $\Gamma^{\alpha\beta}=u^{\alpha}u^{\beta}-\eta^{\alpha\beta}$ and p is the hydrostatic pressure, given by the function $p=p(\varepsilon,n_i)$, which is evaluated in terms of equilibrium distributions.

Now we go on to consider the dynamics of small perturbations superposed on a rest state. For an arbitrary physical quantity A we will use the expansion $A = \overset{0}{A} + \overset{1}{A}$, where $\overset{0}{A}$ is the value in the rest state and $\overset{1}{A}$ is the perturbed value. In particular, from (4.2) we have

$$\overset{0}{u}{}^{\alpha}=\delta_0^{\alpha}, \quad \overset{1}{u}{}^{0}=0, \quad \overset{1}{u}{}^{a}=u^{a}. \tag{4.11}$$

Let us investigate the algebraic structure of the constitutive relations in the form A. In place of the coefficient functions Z^{α}_{AB}, ${}^0A^{\alpha}_{AB}$ it is convenient to consider the functions $Z_{\alpha AB}=\eta_{\alpha\beta}Z^{\beta}_{AB}$, ${}^0A_{\alpha AB}=\eta_{\alpha\beta}{}^0A^{\beta}_{AB}$. We have the obvious symmetry relations

$$Z_{\alpha AB}=Z_{\alpha BA}, \quad Z_{\alpha\beta A}=Z_{\beta\alpha A}, \quad {}^0A_{\alpha\beta A}={}^0A_{\beta\alpha A}. \tag{4.12}$$

In addition relations of the form (3.6) hold:

$$\sum_I {}^0A_{\alpha IB}=0, \quad \sum_I {}^0A_{\alpha AI}=0. \tag{4.13}$$

When the symmetry relations (4.12) are taken into account, the following coefficients are nontrivial:

I. $Z_{000}, Z_{0ab}, Z_{00I}, Z_{0IJ}, Z_{abI}$,

II. ${}^0A_{000}, {}^0A_{00a}, {}^0A_{0a0}, {}^0A0ab, {}^0A_{00I}, {}^0A_{0aI}$,

${}^0A_{0I0}, {}^0A_{0Ia}, {}^0A_{0IJ}$;

III. ${}^0A_{ab0}, {}^0A_{abc}, {}^0A_{abI}, {}^0A_{aI0}, {}^0A_{aIb}, {}^0A_{aIJ}$.

Set I is simply the set of numbers calculated according to the technique described in Ref. 3. The values of these coefficients are given in Sec. 1 of the Appendix.

Next, sets II and III are broken into sums according to Eqs. (3.3). The corresponding expressions are given in Sec. 2 of the Appendix.

We write $\lambda=-\eta_{ab}k^a k^b$. Using the representation of the coefficients ${}^0A_{ABC}$ in terms of the scalar kernels, we can find corresponding expressions for the matrices ${}^0B_{AB}$, and also write down the Onsager relations (3.5) (cf. Sec. 3 of the Appendix).

We write down some consequences of the dissipativity conditions (3.4):

$$\mathrm{Re}(-ik^{00}X_{00}+\lambda^0X_{02})\geqslant 0, \tag{4.14}$$

$$\mathrm{Re}(-ik^{00}X_{03}-\lambda^0X_{14})\geqslant 0, \tag{4.15}$$

$$\mathrm{Re}(-ik^{00}X_{03}-2\lambda^0X_{14}-\lambda^0X_{12}+ik^0\lambda^0X_{04} +\lambda^{20}X_{13})\geqslant 0, \tag{4.16}$$

$$({}^0B_{IJ})+({}^0B_{IJ})^{+}\geqslant 0. \tag{4.17}$$

Now we will show that this nonlocal hydrodynamic model describes the phenomena of diffusion, thermal conduction, and viscosity. Equation (4.4) can be used to express the diffusive flux in terms of the parameters h^A, which from (2.2) and (4.1) have a clear physical meaning:

$$d_i^a=n_i(u_i^a-u^a)=Q^a_{i+3}+\overset{0}{n}_i\beta_0^{-1}h^a.$$

Going over to Fourier transforms and assuming $I=i+3$, we find

$$d^a_{iF}=Q^a_{IF}+\overset{0}{n}_i\beta_0^{-1}h^a_F=(Z^a_{Ib}+\overset{0}{n}_i\beta_0^{-1}\delta^a_b)h^b_F-{}^0A^a_{IA}h^A_F. \tag{4.18}$$

Substituting the expression calculated previously for Z^a_{Ib}, we can easily show that the first term in Eq. (4.18) (the nondissipative part) vanishes. Thus, the diffusive flux is given by the nonlocal expression

$$d^a_{iF}=-{}^0A^a_{IA}h^A_F, \tag{4.19}$$

which includes the effects of baro- and thermodiffusion. We recall that in the classical interpretation diffusion is a flux of material caused by gradients in the chemical potentials. In order to distinguish this classical component we must set $k^0=0$ in expression (4.19), and then restrict ourselves to the linear approximation in k^a. As a result we find the symmetric matrix of diffusion coefficients

$$D_{ij}=-\overset{0}{\beta}{}^0X_{1\ \ 12\ I\ J}|_{k_\alpha=0}.$$

This matrix satisfies the relations

$$\sum_i D_{ij}=0, \quad \sum_j D_{ij}=0$$

and the dissipative inequality [cf. Eq. (4.17)]

$$(D_{ij})\geqslant 0.$$

Similarly, from Eqs. (4.7)–(4.11) (see also Ref. 5) it is not difficult to derive expressions for the thermal flux and the viscous stress tensor:

$$q^a_F=-{}^0A^a_{0A}h^A_F, \quad \tau_{abF}=-{}^0A_{abA}h^A_F.$$

Setting $k^0=0$ in these equations and then restricting ourselves to first order in k^a, we can distinguish the terms in these equations related to thermal conduction and viscosity.

If κ, η_V, η_S are the classical thermal conductivity, bulk viscosity, and shear viscosity coefficients, respectively, then we have the relations

$$\kappa=(\overset{0}{\beta})^{2\,0}X_{02}|_{k_\alpha=0}, \tag{4.20}$$

$$\eta_V+\frac{4}{3}\,\eta_S=(-2\,{}^0X_{14}-{}^0X_{12})|_{k_\alpha=0}, \tag{4.21}$$

$$\eta_S=-{}^0X_{14}|_{k_\alpha=0}. \tag{4.22}$$

The dissipative inequalities (4.14)–(4.16) ensure that the right-hand sides of these expressions are nonnegative.

Thus, nonlocal hydrodynamics enables us to consistently combine the description of diffusion, thermal conduction, and viscosity processes with causality.

6. CONCLUSION

To conclude this work we make a number of comments about the various stages of the above arguments.

In Sec. 5 it was shown that the classical dissipative phenomena (viscosity, thermal conduction, and diffusion) can be described in a relativistic theory by means of constitutive relations in the form A without losing causality, by using the dissipativity condition (1.12) corresponding to the familiar H-theorem.[3,9,10] Consequently, the significance of the constitutive relations in form B, and hence the dissipativity condition (2.9), in relativistic hydrodynamics deserves further study. It is of interest to note that since nonrelativistic kinetic theory uses the operator $(\partial_0+v^a\partial_a)$ in place of the differential operator $(p^\alpha\partial_\alpha)$ in the basic equation, the constitutive relations in form A and in form B are the same to within a constant matrix.

One application of the degeneracy condition (3.6) of the coefficient matrix ${}^0A^{\alpha}_{AB}$ is the case in which the particles of a multicomponent gas have no internal structure. Then from (1.1) we have

$$\eta_{\alpha\beta}p^\alpha p^\beta=m_i^2J_I.$$

From this, in accordance with (3.6), we find the identity

$${}^0A^{\alpha}_{\alpha B}=0. \tag{5.1}$$

Substituting expressions for the coefficients ${}^0A^{\alpha}_{AB}$ with $B=a$ in (5.1) we easily find

$$(3\,{}^0X_{14}+2\,{}^0X_{12})|_{k_\alpha=0}=0,$$

which in accordance with Eqs. (4.20)–(4.22) yields the familiar result[10] $\eta_V=0$.

In the expressions for the coefficients ${}^0A^{\alpha}_{AB}$ the appearance of terms with ε_{abc}, which are not invariant under spatial inversion, is associated with the possible violation of P-invariance in the particle collisions. The latter is possible when weak interactions are considered.

For practical applications of nonlocal hydrodynamics it is necessary to have expressions for the invariant kernels ${}^0X_{nm}$. It is found that their general functional form can be specified when $L^+=L$, $k_a=0$, or $k_0=0$ holds.

It is sufficient to consider the case of the function

$$\Psi(k_\alpha)=(a_1,G^{-1}_{aa}a_2),\quad a_1,a_2\in H_a.$$

We write $\Theta=(-P_aLI_a)^{1/2}$, $\Lambda=\Theta^{-1}P_ap^0I_a\Theta^{-1}$ (these are operators in H_a) and $dE(\tau)$, the spectral measure for the operator Λ (Ref. 16). Then we find the representation

$$\Psi_1(k_0)=\Psi(k_\alpha)|_{k_a=0}=(\Theta^{-1}a_1,(ik_0\Lambda+1)^{-1}\Theta^{-1}a_2)$$

$$=\int(ik_0\tau+1)^{-1}d(\Theta^{-1}a_1,E(\tau)\Theta^{-1}a_2). \tag{5.2}$$

The representation (5.2) is the usual expression for the relaxation kernel in the form of the integral over the spectrum of internal relaxation times.

Arguing similarly, we can derive the expression

$$\Psi_2(k_a)=\Psi(k_\alpha)|_{k_0=0}=P(\lambda)\int_{R\geqslant 0}(\lambda R^2+1)^{-1}dA_2(R). \tag{5.3}$$

Here $P(\lambda)$ is a polynomial in the square of the wave number. The representation (5.3) is interpreted naturally as an expansion of the spatial nonlocality kernel into an integral over the spectrum of correlation radii.

Of course, the practical application of Eqs. (5.2) and (5.3) is dependent on the possibility of evaluating the weight functions appearing in the integrals on the corresponding spectrum. For this we can use various numerical techniques based on an explicitly defined operator L.

APPENDIX

1. In the following expressions we use the notation $z=\beta m(r)$, where $K_n(\zeta)$ is the modified Bessel function of the second kind of order n.

$$Z_{000}=(2\pi)^{-2}\beta^{-5}\int d\mu(r)\exp(\beta\mu_i)z^3[75K_3(z)$$

$$+9zK_2(z)],$$

$$Z_{0ab}=\sigma^1\delta_{ab},$$

$$\sigma^1=(2\pi)^{-2}25\beta^{-5}\int d\mu(r)\exp(\beta\mu_i)z^3K_3(z),$$

$$Z_{00I}=2(2\pi)^{-2}\beta^{-4}\exp(\beta\mu_i)\int d\zeta'(r')z^2[3K_2(z)$$

$$+zK_1(z)],$$

$$Z_{0IJ}=\sigma^2\delta_{IJ},$$

$$\sigma^2=9(2\pi)^{-2}\beta^{-3}\exp(\beta\mu_i)\int d\zeta'(r')z^2K_2(z),$$

$$Z_{abI}=\sigma^3_I\delta_{ab},$$

$$\sigma^3_I=2(2\pi)^{-2}\beta^{-4}\exp(\beta\mu_i)\int d\zeta'(r')z^2K_2(z).$$

2. The representation (3.3) in this model assumes the following form:

$${}^0A_{000}={}^0X_{00},\quad {}^0A_{00a}=ik_a{}^0X_{01},\quad {}^0A_{0a0}=ik_a{}^0X_{02},$$

$${}^0A_{0ab}=\delta_{ab}{}^0X_{03}+ik_aik_b{}^0X_{04}+\varepsilon_{abc}ik_c{}^0X_{05},$$

$$
\begin{aligned}
&{}^0A_{00I}={}^0X_{06I}, \quad {}^0A_{0aI}=ik_a{}^0X_{07I}, \quad {}^0A_{0I0}={}^0X_{08I},\\
&{}^0A_{0Ia}=ik_a{}^0X_{09I}, \quad {}^0A_{0IJ}={}^0X_{010IJ},\\
&{}^0A_{ab0}=\delta_{ab}{}^0X_{10}+ik_aik_b{}^0X_{11},\\
&{}^0A_{abc}=\delta_{ab}ik_c{}^0X_{12}+ik_aik_bik_c{}^0X_{13}+(ik_a\delta_{bc}\\
&\qquad +ik_b\delta_{ac}){}^0X_{14}+(ik_aik_d\varepsilon_{bcd}+ik_bik_d\varepsilon_{acd}){}^0X_{15},\\
&{}^0A_{abI}=\delta_{ab}{}^0X_{16I}+ik_aik_b{}^0X_{17I},\\
&{}^0A_{aI0}=ik_a{}^0X_{18I},\\
&{}^0A_{aIb}=\delta_{ab}{}^0X_{19I}+ik_aik_b{}^0X_{110I}+\varepsilon_{abc}ik_c{}^0X_{111I},\\
&{}^0A_{aIJ}=ik_a{}^0X_{112IJ}.
\end{aligned}
$$

According to (4.13) the following conditions on the scalar kernels should hold:

$$\sum_I {}^0X_{0nI}=0, \quad n=6,7,8,9,$$

$$\sum_I {}^0X_{010IJ}=0, \quad \sum_J {}^0X_{0\ 10\ I\ J}=0,$$

$$\sum_I {}^0X_{1nI}=0, \quad n=6,7,8,9,10,11,$$

$$\sum_I {}^0X_{012IJ}=0, \quad \sum_J {}^0X_{012IJ}=0.$$

3. Using the equations of the previous section of the Appendix we find expressions for the matrix ${}^0B_{AB}$ (recall that $\lambda=-\eta_{ab}k^ak^b$):

$$
\begin{aligned}
&{}^0B_{00}=-ik^{00}X_{00}+\lambda{}^0X_{02},\\
&{}^0B_{0b}=ik_b(-ik^{00}X_{01}+{}^0X_{03}-\lambda{}^0X_{04}),\\
&{}^0B_{0I}=-ik^{00}X_{06I}-\lambda{}^0X_{07I},\\
&{}^0B_{a0}=ik_a(-ik^{00}X_{02}+{}^0X_{10}-\lambda{}^0X_{11}),\\
&{}^0B_{ab}=\delta_{ab}(-ik^{00}X_{03}-\lambda{}^0X_{14}),\\
&\qquad +ik_aik_b(-ik^{00}X_{04}+{}^0X_{12}-\lambda{}^0X_{13}+{}^0X_{14}),\\
&\qquad +\varepsilon_{abc}ik_c(-ik^{00}X_{05}-\lambda{}^0X_{15}),\\
&{}^0B_{aI}=ik_a(-ik^{00}X_{07I}+{}^0X_{16I}-\lambda{}^0X_{17I}),\\
&{}^0B_{I0}=-ik^{00}X_{08I}-\lambda{}^0X_{18I},\\
&{}^0B_{Ia}=ik_a(-ik^{00}X_{09I}+{}^0X_{19I}-\lambda{}^0X_{110I}),\\
&{}^0B_{IJ}=-ik^{00}X_{010IJ}-\lambda{}^0X_{112IJ}.
\end{aligned}
$$

By assuming $\varepsilon_0=\varepsilon_I=1$, $\varepsilon_a=-1$ we can easily write down Onsager relations (3.5) for the scalar kernels:

$$
\begin{aligned}
&-ik^{00}X_{01}+{}^0X_{03}-\lambda{}^0X_{04}+ik^{00}X_{02}-{}^0X_{10}+\lambda{}^0X_{11}=0,\\
&ik^{00}X_{06I}+\lambda{}^0X_{07I}-ik^{00}X_{08I}-\lambda{}^0X_{18I}=0,\\
&ik^{00}X_{07I}-{}^0X_{16I}+\lambda{}^0X_{17I}-ik^{00}X_{09I}\\
&+{}^0X_{19I}=\lambda{}^0X_{110I}=0,\\
&ik^{00}X_{010IJ}+\lambda{}^0X_{112IJ}=ik^{00}X_{010JI}+\lambda{}^0X_{112JI}.
\end{aligned}
$$

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Translated by David L. Book