Dynamics of a periodic wave in a model including quadratic and cubic nonlinearities

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The integrable system of equations arising in different models of the coherent interaction of light in a two-level medium is considered. In particular, the models describe single-photon lasing, Raman scattering of light, four-wave mixing, etc. In two- and four-wave processes it is assumed that one of the fields is time-independent. Interaction processes are treated in a medium with an effective quadratic nonlinearity, taking into account the third-order nonlinearity associated with frequency modulation. The model is studied for the first time using the periodic form of the inverse-scattering method. Equations are derived which reduce the derivation of the general periodic single-phase solution to calculated integrals. For this solution Whitham equations are derived, describing the slow spreading and modulation of a compact soliton packet. Special account is taken of pumping of the upper energy level of the transition in the medium. The situation is investigated in which the nonlinear two-level active medium is in a ring cavity. Multistable behavior of the variables of a periodic wave is described for the first time as a function of their initial values and the magnitude of the gain in this system. It is shown that qualitatively different lasing regimes develop, depending on the sign of the coupling constant characterizing the contribution of the cubic nonlinearity and the resonance conditions. © *1995 American Institute of Physics.*

1. INTRODUCTION

The evolution of high-power laser pulses in nonlinear media is one of the most intriguing and complicated problems of theoretical physics. Recently developed mathematical techniques enable one to solve the Cauchy problem for a number of models describing such phenomena as Raman scattering, two-photon absorption, and three- and four-wave mixing, which are important for nonlinear optics (see, e.g., Ref. 1). At present the inverse scattering method (ISM)[2] remains the analytical tool which permits one to obtain the most complete description of the evolution of short pulses in a nonlinear medium and to predict new nonlinear optical effects. Besides the study of the dynamics of fields in passive media, the ISM has been applied to study fields associated with the generation of pulses in one-pass lasers,[3] in which a propagating sink field extracts energy stored in a previously inverted medium. Note, however, that this lasing scheme has been implemented experimentally much less often than that employed in dye lasers and ion and some solid-state lasers, where the upper level of the energy transition is pumped continuously (see, e.g., Ref. 4 and work cited therein).

Burtsev *et al.*[5] have shown that the pumping of the upper level in a two-level system can be included in the Maxwell–Bloch model while retaining the integrability of the inverse scattering method. A number of other well-known integrable models of nonlinear optical processes occurring under conditions of resonance with the intrinsic energy transition of the medium also admit the use of the ISM when a generalized analog of this pumping is included.[6] In the present work it is proposed to describe a laser with continuous pumping of the upper level by means of a generalization of the integrable model first proposed and studied in connection with the ISM in our previous work.[7] This model describes the interaction of three fields in a quadratic medium, taking into account the correction to the frequency of the cubic nonlinearity. In the present work this integrable model is generalized for the first time to the case in which the energy transition is pumped. It is shown that this pumping and the nonlinear modulation of the frequency give rise to qualitatively new effects, such as multistable dependence of the soliton variables on their initial values and on the pump variables. The physical applications of the model are described in more detail in the following section.

The phenomenon of the modulation instability has been extensively studied in a number of integrable and nonintegrable models.[8] In particular, it has been shown that the solution of the nonlinear Schrödinger equation in the form of a harmonic with constant amplitude evolves into a nonlinear (in general, N-phase) quasiperiodic solution.[9] Very similar development is observed in the numerical study of the growth of instabilities in a two-level laser with a pump.[10] Based on these facts, in media where the transverse relaxation time is sufficiently long one should expect quasiperiodic waves to be generated if the seed field consists of a pulse which is sufficiently long and powerful and if the relaxation time is long in comparison with the scale of the individual solitons (breathers, etc.) which make up the train of nonlinear pulses. We note the recent work in which a similar phenomenon was observed in connection with the generation of a light field in solid-state pumped lasers.[4] Duncan *et al.*[11] discussed the experimental prospects for producing periodic waves with Raman scattering in a two-level me-

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dium for relaxation times much longer than the growth times of the nonlinear process. This problem is also of interest because, as shown by Menyuk,[12] the soliton regime of Raman scattering with strong pump depletion is possible, in contrast to the periodic regime, only over short times. In the present work it is assumed that the envelope of the generated light field has the form of a quasiperiodic nonlinear wave. We assume that the process by which the field evolves is completely determined by the variation of its parameters, i.e., the adiabatic approximation is used. Thus, the first step consists of finding the general periodic single-phase solution[2] of the model with periodic boundary conditions.

In Sec. 3, for the first time, the single-phase solution of this model is constructed by means of the periodic version of the ISM.[13] The shape of the envelope of the generated field is determined by a polynomial function. In the same section it is shown that the applicability of the ISM when pumping of the upper level is included implies that the roots of this polynomial depend on the effective length of the medium. When pumping is included exact periodic solutions of the model cannot be found, so we employ the quasiclassical approximation. It is assumed that the typical scale on which the fields change due to the action of the pump and that on which the soliton-solition interaction occurs are sufficiently different that these processes can be separated approximately in the analytical description. The problem is divided into two parts: 1) the quasiclassical description of the soliton dynamics in a compact packet, and 2) the study of the time dependence of the variables due to the action of the pump.

In Sec. 4, we introduce the Whitham modulation equations[14] and solve them for the case of a compact soliton packet. These equations are used in Sec. 5 to describe the regimes in which a modulated periodic wave propagates. It is shown that when a periodic wave is injected into the nonlinear system a "sparse" train of solitons can be transformed into a "highly compact" packet and vice versa. We also treat a ring cavity with an active nonlinear medium. It is shown that the feedback supplied by the cavity and nonlinear frequency modulation give rise to multistability of the variables of the periodic wave, a special case of which is soliton multistability. The latter was studied in Refs. 15–17. The present case differs with respect to the physical mechanism of multistability and in the inclusion of level pumping. In particular, it is shown for the first time that the dependence of the soliton variables on the magnitude of the pump can be multistable.

2. FUNDAMENTAL EQUATIONS AND THE MODELS OF COHERENT THREE-WAVE INTERACTION WHICH THEY DESCRIBE

The basic system of equations which we studied takes the form

$$\partial_x F_+ = i[vF_+ - \zeta_2|R_+|^2 F_+/2 + \epsilon F_3 R_+],$$
$$\partial_x F_3 = i/2[F_+ R_- - F_- R_+] + W,$$
$$\partial_y R_+ = 2i[\zeta_1 F_3 R_+ + F_+] - \epsilon \zeta_1^2 W R_+, \qquad (1)$$

where $\epsilon = \pm 1$.

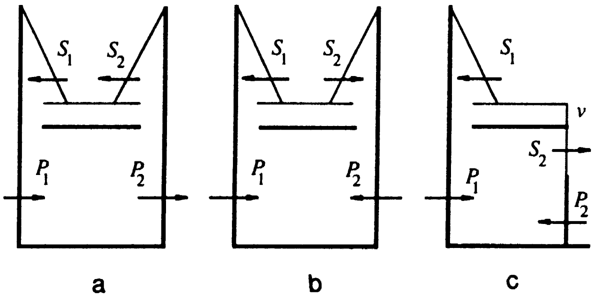

FIG. 1. Schematics of four-wave mixing in a two-level medium (model 1 of Sec. 2). The horizontal lines represent energy levels and mismatches. The heavy lines correspond to the envelope P_i, the carrier frequency is Ω_i, the carrier wave number is k_i, and the phase velocity is $\pm V_i$ (the upper sign corresponds to propagation from left to right and vice versa, as shown by the arrows). The light lines correspond respectively to the quantities S_i, ω_i, l_i, and $\pm U_i$. The subscript $i=1$ ($i=2$) labels the right (left) pair. Schematic (a) also describes two oppositely propagating fields having polarization components P_i and S_i respectively.

Equations (1) arise in many nonlinear optics problems. We briefly summarize the main nonlinear optical processes which can be described by the solutions of Eq. (1).

1. Four-wave mixing

The initial electric field takes the form

$$\bar{E}(z,t) = \sum_{j=1}^{2} \{P_j \exp[i(q_j z - \Omega_j t)] + S_j \exp[i(l_j z - \omega_j t)]\} + \text{c.c.}, \qquad (2)$$

where P_j and S_j are slowly varying envelopes (see, e.g., Refs. 1 and 18), Ω_j and ω_j are the carrier frequencies, and q_j and l_j are the wave vectors, respectively. It is assumed that the two pairs of fields propagate in opposite directions, so that the two-photon resonance conditions $\Omega_j \mp \omega_j = \omega_0 + \nu$ hold; here the frequency mismatch satisfies $\nu \ll \omega_0, \omega_j, \Omega_j$, $j=1,2$ (Fig. 1). The resonance conditions not only allow the nonlinear mixing effect to be enhanced considerably (by orders of magnitude), but also permit us to drop the terms in the equations which describe the cubic self-interaction of the fields. This is also what allows the ISM to be employed in this model. The process of nonlinear mixing of the fields is determined by the two-photon-induced Kerr nonlinearity.[19] Additional conditions for the applicability of the ISM in the case of four time-dependent waves are the following: the model is one-dimensional, the phase velocities of the fields propagating in a given direction are equal to one another, and the wave mismatch is equal to zero. Different combinations of the resonant conditions and choices of the directions of propagation for the fields give rise to more than ten different four-wave mixing schemes. The analysis carried out in Ref. 20 shows that all integrable versions reduce to three mathematically distinct models. In the majority of experiments on the observation of four-wave mixing which are familiar to us one of the fields can be taken to be constant to a good approximation.[18] This condition also substantially reduces the difficulty in synchronizing the field pulses, required for the effect to be observable. The constancy of one of the fields is ensured, e.g., in the limit $|P_1|\gg|P_2|$. It is important to note that the latter condition allows us to extend the range of physical variables in which the ISM can be applied. The important requirement that there be no wave mismatch and that the waves with envelopes $P_{1,2}$ have equal phase velocities is thereby removed. Expansion with respect to the ratio $|P_2|/|P_1|$ reduced the time-dependent equations for four-wave mixing to the system (1).

For the interaction scheme (a) illustrated in Fig. 1 we have the following values of the parameters $\Delta=q_2-l_2-q_1+l_1$, where Δ is the wave mismatch and $V_{1,2}$ and $U_{1,2}$ are the phase velocities of the fields $P_{1,2}$ and $S_{1,2}$, respectively:

$$\alpha_{11}=\frac{2\pi\Omega_1^2 n_0 N_0}{k_1\hbar\nu}|\kappa(\Omega_1)|^2,$$

$$\alpha_{21}=\frac{2\pi\omega_1^2 n_0 N_0}{l_1\hbar\nu}\kappa^*(\Omega_2)\kappa(\Omega_1),$$

$$\alpha_{12}=\frac{2\pi\Omega_1^2 n_0 N_0}{k_1\hbar\nu}\kappa^*(\Omega_1)\kappa(\Omega_2),$$

$$\alpha_{22}=\frac{2\pi\omega_1^2 n_0 N_0}{l_1\hbar\nu}|\kappa(\Omega_2)|^2. \tag{3}$$

Here N_0 is the atomic density and n_0 is the constant difference in the populations of the states associated with the transition. We assumed that the scattering tensor $\kappa_{ij}=\delta_{ij}\kappa$ is real. Similar expressions for β_{ij} can be derived from Eqs. (3) by means of the interchanges $\Omega_1\leftrightarrow\Omega_2$, $\omega_1\leftrightarrow\omega_2$, $q_1\leftrightarrow q_2$, $l_1\leftrightarrow l_2$. The notation used in Eqs. (1) has the following interpretation for the scheme of Fig. 1a:

$$F_-=F_+^*,\quad R_-=R_+^*,\quad \partial_\eta=\partial_z+V^{-1}\partial_t,$$

$$\partial_\xi=\partial_z-V^{-1}\partial_t,$$

$$m^2=\alpha_{11}\alpha_{22}(\beta_{11}\beta_{22})^{-1},\quad u=\kappa(\Omega_1)/\kappa(\Omega_2),$$

$$V=V_{1,2}=U_{1,2},\quad \epsilon=1,$$

$$s_1=S_1,\quad p_1=P_1,\quad s_2=\mp S_2\sqrt{\frac{\beta_{11}}{\alpha_{11}}},$$

$$p_2=\mp P_2\sqrt{\frac{\beta_{22}}{\alpha_{22}}},$$

$$\bar{\zeta}_1=-\frac{u^2+m^2}{2mu},\quad \zeta_2=-\frac{1+u^2m^2}{2mu},$$

$$\bar{y}=\alpha_{11}\int_{-\infty}^{\eta}I_1(\eta)d\eta,\quad x=\alpha_{22}\int_{-\infty}^{\xi}I_2(\xi)d\xi,$$

$$I_1(\eta)=\frac{u}{m}\left|s_1\right|^2+\frac{m}{u}\left|s_2\right|^2,$$

$$I_2(\xi)=mu|p_1|^2+\frac{1}{mu}|p_2|^2,$$

$$F_3=\frac{1}{I_1}\left(\frac{m}{u}\left|s_2\right|^2-\frac{u}{m}\left|s_1\right|^2\right),$$

$$R_3=\frac{1}{I_2}\left(u/m\left|p_2\right|^2-\frac{1}{um}\left|p_1\right|^2\right),$$

$$\left.\begin{matrix}F_+\\R_+\end{matrix}\right\}=\left.\begin{matrix}2s_1s_2^*/I_1\\2p_1p_2^*/I_2\end{matrix}\right\}\exp\left\{i\frac{x(u^2-m^2)/2+\bar{y}(1-u^2m^2)}{mu}\right\}.$$

In this approximation (R_3=const) we have

$$v=\Delta\alpha_{22}^{-1}I_2^{-1}+\zeta_2R_3,\quad y=\bar{y}R_3/2,\quad \zeta_1=\bar{\zeta}_1R_3^{-1}.$$

The normalization has been chosen so that the pseudospin $\vec{F}$ satisfies the relation $\mathbf{F}=(F_3,F_\pm)$

$$\epsilon|F_+|^2+F_3^2=1. \tag{4}$$

To save space we omit the corresponding expressions for schemes (b) and (c) of Fig. 1 (see Ref. 7). Note that for those schemes $\epsilon=-1$ holds.

The special case of four-wave mixing in this model, $\zeta_{1,2}=\epsilon=-1$, corresponds to degenerate four-wave mixing used in wave-front conjugation.[21]

2. Interaction of two polarized waves in a medium with cubic nonlinearity

Two wave packets propagating in opposite directions in a medium with characteristic frequency close to the sum or difference of the carrier frequencies has been actively studied both theoretically and experimentally.[21] In Ref. 22 a Hamiltonian is used to describe this process which yields equations formally identical with the model of an anisotropic chiral field over the group $O(3)$. For the case of uniaxial anisotropy this problem is formally equivalent to the model treated above under the condition $V_1=V_2$, $U_1=U_2$, $\zeta_1=\zeta_2$, $\Delta=0$, $\epsilon=1$. Here the quantities $P_{1,2}$ represent the polarization components of one wave and $S_{1,2}$ those of the other. In the limit $|P_1|\gg|P_2|$ the equations of this model also reduce to Eqs. (1). In Ref. 22 it is noted that this model can be used to describe the propagation of laser fields in a plasma under the condition that the plasma frequency is equal to the difference of the carrier frequencies of these waves. For the models presented above the requirement that the pair of waves propagate in opposite directions is introduced only to simplify the description. After minor changes all results can be used as well for pairs of waves propagating at slightly different angles; this is used to increase the effective interaction length. The approximation used here enables us to include the wave mismatch that results without losing the integrability of the model.

3. Raman scattering and two-photon propagation of fields in a two-level medium

Two-photon processes in a two-level medium have been investigated in numerous theoretical and experimental treatments (see, e.g., Refs. 1 and 19). A number of interesting analytical results have been obtained by studying several integrable models. The most interesting of these was described in Ref. 23, where the change in occupation numbers was taken into account under resonance conditions and it was assumed that the phase velocities of the pump field and the Stokes field are equal. These two restrictions, which are important experimentally, can be removed if we assume that the pump is not depleted or the difference in the occupation numbers of the two states of the transition is constant. In particular, the latter assumption is applicable in experiments on cooperative Raman scattering.[24] In many experiments on induced Raman scattering the constancy of the difference in occupation numbers holds to high accuracy (10^{-4}–10^{-7}; Refs. 11 and 25). But at the same time, nonlinear frequency modulation effects, which in our model are described by terms with constant coupling $\zeta_{1,2}$, play an important role. In Ref. 11 a model of Raman scattering corresponding to Eqs. (1) with $\zeta_2\neq 0$, $\epsilon=1$, $\zeta_1=0$ was used for numerical analysis of the field conversion process. Good agreement was found with experimental observations for pulses of length less than the relaxation time of the medium by factors of a few times 10. Thus, the use of a model of the form (1) to describe two-photon processes is not only facilitated by the conditions of a majority of the familiar experiments but also greatly extends the range of physical parameters and also permits the inclusion of continuous pumping of the levels while retaining the applicability of the ISM.

When the difference in the occupation numbers of the levels associated with the transition is constant, the Maxwell–Bloch equations can be written in the form (1), where

$$\bar{\zeta}_1=\zeta_2=\frac{b_1-b_2}{\kappa_0},\quad x=\kappa_0 N_0 z,$$

$$N_0^2=r_3^2+r_+r_-,\quad \bar{y}=\kappa_0\int_{-\infty}^{\tau} I_1(\eta)d\eta,$$

$$\tau=t-\frac{z}{c},\quad I_1(\eta)=|S_1|^2+\epsilon|S_2|^2,\quad F_3=\frac{|S_1|^2-\epsilon|S_2|^2}{I_1},$$

$$R_3=\frac{r_3}{N_0},\quad R_+=\frac{r_+}{N_0}\exp\left[i(b_1+b_2)\int_{-\infty}^{\tau} I_1(\eta)d\eta\right],$$

$$F_+=2\exp\left[i(b_1+b_2)\int_{-\infty}^{\tau} I_1(\eta)d\eta\right]\times\begin{cases} S_1S_2^*/I_1, & \epsilon=1\\ S_1^*S_2^*/I_1, & \epsilon=-1\end{cases},$$

here r_3 is the difference in the populations of the levels associated with the transition, N_0 is the number of atoms, r_+ is the polarizability of the medium, t and z are the temporal and spatial variables respectively, $S_{1,2}$ are the slowly varying envelopes of the fields with carrier frequencies $\Omega_{1,2}$ such that $\Omega_1-\epsilon\Omega_2=\omega_0+\nu_0$, holds, where ω_0 is the transition frequency and ν_0 is the mismatch. The values of the constants $b_{1,2}$ and κ_0 are given by Steudel.[23] In this approximation the transition to the variables of Eqs. (1) is given by the relations $v=\nu_0(\kappa_0N_0)^{-1}+\zeta_2R_3$, $y=\bar{y}R_3/2$, $\zeta_1=\bar{\zeta}_1R_3^{-1}$. The values given above correspond to levels with constant occupation numbers. The case in which the pump is not depleted is derived by the formal interchange $R_+\leftrightarrow F_+$, $R_3\leftrightarrow F_3$, $x\leftrightarrow 2y$. Raman scattering (two-photon absorption) corresponds to the value $\epsilon=1(-1)$.

The ordered models of three-wave interactions, which have been used, e.g., to study induced Brillouin scattering, excitation of exciton transitions, etc.,[21] also reduce to Eqs. (1) under appropriate restrictions. An important limitation on the applicability of the ISM for these models is the condition that the phase velocities of the two fields be equal. If the difference in the carrier frequencies of the two "fast" waves is comparable with the time scale on which the third "slow" field varies, it is necessary to take into account the time derivatives of the nonlinear polarizability of the medium in deriving the model equations. It is easy to show that retaining the first time derivative of the product of the envelopes of the fast fields leads to a system of equations similar to (1) (Ref. 7).

4. Effective two-level model; model of a Raman laser

The three-level scheme for the interaction of molecules with laser radiation and the Stokes field can also be reduced to an effective two-level system when the mismatch is sufficiently large and depletion of the pump is neglected. This procedure has been carried out in recent treatments.[26] It can be shown that when relaxation is neglected these equations are identical with (1) except for notation when pumping of the upper level is omitted ($W=0$).

5. Single-photon interaction with a two-level medium, including the quadratic Stark effect

The Maxwell–Bloch equations of this model take the form

$$\left(\partial_z+\frac{1}{c_0}\partial_t\right)E(z,t)=\frac{2\pi\omega_0N_0}{c_0}(dR+i\kappa EN),$$

$$\partial_tR=-i(\nu_0+\kappa|E|^2)R+\frac{d}{\hbar}EN,$$

$$\partial_tN=-\frac{d}{\hbar}(E^*R+R^*E)+w. \tag{5}$$

Here E is the slowly-varying envelope of the field with frequency $\omega=\omega_0+\nu_0$; ω_0 and d are its transition frequency and dipole moment; $\kappa=\kappa_1-\kappa_2$, where $\kappa_{1,2}$ are the polarizabilities of the levels; R is the polarizability of the transition; and N is the difference in the level populations. Equations (5) are the same as (1) after the substitutions

$$\zeta_1=2\zeta_2=\frac{\kappa\hbar\sqrt{\Omega}}{d^{-2}},\quad \Omega=\frac{2\pi N_0\omega_0d^2}{\hbar},\quad 2y=z\frac{\sqrt{\Omega}}{c},$$

$$F_3=\frac{N}{N_0},\quad W=\frac{w}{\sqrt{\Omega}},\quad x=\left(t-\frac{z}{c}\right)\sqrt{\Omega},\quad v=-\frac{\nu_0}{\sqrt{\Omega}},$$

$$R_+=\frac{i\sqrt{2}dE}{\hbar\Omega},\quad F_+=R\sqrt{2},\quad \epsilon=1.$$

The last model with $\kappa=0$ is used most often of those treated above to describe coherent effects occurring in a two-level medium (including one with gain); see, e.g., Ref. 1. In particular solutions it has been shown that the nonlinear frequency shift results in substantial modification of the soliton shape. Recent work[27] has revealed that the optical Stark effect has a major effect on the gain of radiation pulses in high-power laser systems. The two-level model of a single-pass laser amplifier (neglecting pumping, $W=0$) has been studied using the ISM in Ref. 3, and including the quadratic Stark effect in our work.[7] Burtsev *et al.* have shown that the apparatus of the ISM can be applied to a model corresponding to the special case $\epsilon=1$, $\zeta_{1,2}=0$ of the model (1), where pumping of the upper level is included.[5] The list of nonlinear optics models which are integrable by means of the ISM, taking into account an analog of the pump, was substantially extended by Burtsev and Gabitov.[6] The model investigated in the present work, however, is absent from this list.

3. CONSTRUCTION OF SINGLE-PHASE SOLUTIONS OF THE MODEL

There exist different versions of the ISM apparatus, which enable one to construct periodic (in general, N-phase) solutions of systems of equations with periodic boundary conditions. More convenient for our purposes is the approach presented in Refs. 13, which is what we will use here.

As already pointed out, the periodic version of the ISM apparatus is being applied to the present integrable system for the first time. The presence of a "pump" makes the spectral parameters depend on position. We will assume that this dependence is slow in comparison with the nonlinear period of the wave, i.e., the separation between solitons which make up the train. We will find an exact periodic solution for $W=0$, $\zeta_{1,2}\neq 0$. The next stage will consist of treating the effect of changes in the spectral parameters of the periodic solution due to the pump ($W\neq 0$, $\zeta_2=0$, $\zeta_1\neq 0$) on its shape.

The Lax representation for the integrable system (1) neglecting the pump was found in Ref. 7. In the presence of a pump it retains its form when the dependence of the spectral parameter λ on y is taken into account.

Thus, Eqs. (1) can be represented in the form of a compatibility condition for the following two linear systems of equations:

$$\partial_x\Phi=\begin{pmatrix} i(\lambda-\bar{v}-\zeta_2|R_+|^2/4) & \sqrt{c^2+a^2\lambda}R_+ \\ \sqrt{c^2+a^2\lambda}R_- & -i(\lambda-\bar{v}-\zeta_2|R_+|^2/4)\end{pmatrix}\Phi, \tag{6}$$

$$\partial_y\Phi=\frac{1}{\lambda}\begin{pmatrix} i(2\zeta_1\lambda+\epsilon)\bar{F}_3/2 & -\sqrt{c^2+a^2\lambda}F_+ \\ -\sqrt{c^2+a^2\lambda}F_- & -i(2\zeta_1\lambda+\epsilon)\bar{F}_3/2\end{pmatrix}\Phi, \tag{7}$$

where Φ is a two-component function and we have written $c^2=-\epsilon/4$, $\epsilon=\pm 1$, $a^2=-(\zeta_1+\zeta_2)/2$. The Lax representation (6), (7) is valid for $W=0$, $\zeta_{1,2}\neq 0$, where $\bar{F}_3$ with a bar is the same as F_3 in (1) without a bar. The condition for the applicability of the Lax representation for $W\neq 0$, $\zeta_1\neq 0$, $\zeta_2=0$ (the term containing ζ_2 is eliminated by a simple transformation of the fields R_+, F_+, which renormalizes ζ_1) requires that the spectral parameters satisfy the condition

$$\partial_y\lambda=W\,\frac{2\zeta_1\lambda+\epsilon}{2\lambda} \tag{8}$$

and $\bar{F}_3=F_3+i\epsilon\zeta_1 W/2$.

Following Ref. 13 we introduce the following quadratic characteristic functions:

$$f=1/2(\phi_1\psi_2+\phi_2\psi_1),\quad g=\phi_1\psi_1,\quad h=\phi_2\psi_2, \tag{9}$$

where $\phi_{1,2}$ and $\psi_{1,2}$ denote different solutions of the system (6), (7). These functions satisfy the following system:

$$\begin{aligned}
&\partial_x f=\sqrt{c^2+a^2\lambda}(R_+h+R_-g),\\
&\partial_y f=-\frac{\sqrt{c^2+a^2\lambda}}{\lambda}(F_+h+F_-g),\\
&\partial_x g=2i(\lambda^2-\bar{v}-\zeta_2|R_+|^2/4)g+\sqrt{c^2+a^2\lambda}R_+f,\\
&\partial_y g=\frac{i}{\lambda}[(2\zeta_1\lambda+\epsilon)F_3g-\sqrt{c^2+a^2\lambda}F_+f],\\
&\partial_x h=-2i(\lambda^2-\bar{v}-\zeta_2|R_-|^2/4)h+\sqrt{c^2+a^2\lambda}R_+f,\\
&\partial_y h=-\frac{i}{\lambda}[(2\zeta_1\lambda+\epsilon)F_3h-\sqrt{c^2+a^2\lambda}F_-f].
\end{aligned}\tag{10}$$

It follows from (10) that the function $P(\lambda)=f^2+gh$ is independent of both variables, i.e., $\partial_y P(\lambda)=0$, $\partial_x P(\lambda)=0$. The explicit form of the periodic solution is determined by the dependence of P on the spectral parameter λ. The single-phase solution is given by the polynomial

$$P(\lambda)=\prod_{k=1}^{4}(\lambda-\lambda_k)=\sum_{j=0}^{4}P_j\lambda^j, \tag{11}$$

where λ_k are the roots of the polynomial, fixed by the initial conditions. It can be shown that the corresponding quadratic eigenfunctions satisfying the system (6), (7) can be found in the form

$$f=\sum_{k=0}^{2}f_k\lambda^k,\quad g=\sqrt{c^2+a^2\lambda}(g_0+g_1\lambda),$$
$$h=\sqrt{c^2+a^2\lambda}(h_0+h_1\lambda). \tag{12}$$

Substituting these expressions in Eqs. (10) we find

$$\begin{aligned}
&R_+h_1=-R_-g_1,\quad g_1=iR_+f_2,\quad h_1=-iR_-f_2,\\
&R_+h_0=-R_-g_0,\\
&F_3g_0=-2i\epsilon F_+f_0,\quad F_3h_0=2i\epsilon F_-f_0,\quad \partial_y f_2=0.
\end{aligned}\tag{13}$$

From (10)–(13), using expressions for the zeroth and fourth powers of λ, we find

$$\begin{aligned}
&g_1=iR_+,\quad g_0=2i\sqrt{P_0}F_+,\quad h_1=-iR_-,\\
&h_0=\sqrt{P_0}R_-,\quad f_0=-\epsilon\sqrt{P_0}F_3,\quad f_2=1.
\end{aligned}\tag{14}$$

Here the coefficient P_4 has been set equal to unity without loss of generality. The above relations show that $\mu(\lambda,x,y)=-2\sqrt{P_0}F_+/R_+$ is a zero of the function $g(\lambda)$, i.e., the latter can be represented in the form

$$g=i\sqrt{c^2+a^2\lambda}(\lambda-\mu)R_+. \tag{15}$$

Substituting this expression in (10) and using (14 and $f^2(\lambda=\mu=P(\mu)$ we find

$$\partial_x g(\lambda=\mu)=2i\sqrt{c^2+a^2\lambda}F_+\sqrt{P(\mu)}$$

and a similar expression for $\partial_y g$. Hence

$$\partial_x\mu=2i\sqrt{P(\mu)},\quad \partial_y\mu=i\sqrt{P(\mu)/P_0}.$$

From the last two equations it follows that μ depends on a single variable θ [the corresponding solution of the original system of equations (1) is called a single-phase solution]

$$\partial_\theta\mu=i\sqrt{P(\mu)},\quad \theta=2x+\frac{y}{\sqrt{P_0}}. \tag{16}$$

From (10) and (16) it follows that the functions F_3 and $|R_+|$ depend only on the similarity variable θ. This property permits one of the relations following from (1) to be integrated:

$$|R_+|^2=8\sqrt{P_0}F_3-8C, \tag{17}$$

where C is a real constant.

The conditions under which P can be written in the form of a finite polynomial, generally speaking, impose a number of restrictions on the values of the spectral parameters and hence the form of the solution. For an N-phase solution the number of such restrictions increases with N. For the single-phase solution they can be found directly from the expansion (11) and (17). Thus, for the first to third powers of λ we find

$$2\,f_0f_1-c^2|R_+|^2(\mu+\mu^*)-a^2|R_+|^2|\mu|^2=P_1,$$
$$2\,f_0f_2-c^2|R_+|^2-a^2|R_+|^2(\mu+\mu^*)+f_1^2=P_2,$$
$$2\,f_2f_1-a^2|R_+|^2=P_3. \tag{18}$$

Combining the equations in (18) pairwise we find a number of relations among the functions. A similar relation follows from (17):

$$|R_+|^4+16|R_+|^2(C+|\mu|^2\epsilon)+64(C^2-P_0)=0. \tag{19}$$

Comparing (18) and (19) we find that the coefficients or roots of the polynomial and the constant C should satisfy the following relation:

$$64a^4C^2+P_3^2=64a^4P_0+16\epsilon a^2P_1+4P_2+\epsilon 8C$$
$$+16a^2CP_3. \tag{20}$$

Equation (20) can be regarded as a definition for the constant C. In this case the roots of the polynomial P can be arbitrary. In the general case the constant C is determined by the propagation regime of the "seed" fields. For example, for two-photon propagation with field intensities such that $F_3=0$ holds it is natural to assume that the polarizability of the medium vanishes, i.e., $C=0$. In the case of a multipass system the constant C can be found from the boundary conditions. The condition (20) imposes a restriction on the roots of the polynomial P. Note that the existence of such conditions is due to the symmetry of the model. Thus, in the case of the chiral field model based on the group $O(3)$ or the model describing four-wave mixing of fields in a two-layer medium, there are two such conditions.[28] The reduction of these models by virtue of the assumption that one of the fields is constant decreases the number of conditions of the form (20) to a single one and gives rise to (1). This kind of restriction does not occur, e.g., when the ISM apparatus described above is applied to weakly nonlinear models such as the nonlinear Schrödinger equation[13] or its variant containing a differentiated nonlinearity.[29] The spectral problem (6) also arises in the case of the modified nonlinear Schrödinger equation.[29] However, comparison of the relations found above with the analogous ones in Ref. 29 shows that besides condition (20) the periodic solutions of these models in general are different.

The expressions given above also enable us to find a more general single-phase solution of the Maxwell–Bloch equations in a two-level medium for both $\epsilon=1$ and $\zeta_{1,2}=W=0$ than that which is familiar from the literature.[30] No explicit expressions for the roots satisfying (20) can be found in the general case. We will give an example of a polynomial whose roots satisfy relation (20) for arbitrary complex $a\neq0$, $\lambda_{1,2}$ and $C=0$, $\epsilon=1$:

$$P_{ex}(\mu)=\left[\mu-\frac{1}{2}\left(\lambda_1+\frac{1}{2a}\right)-\lambda_2\right]\left[\mu-\frac{1}{2}\left(\lambda_1+\frac{1}{2a}\right)+\lambda_2\right]$$
$$\times\left[\mu-\frac{1}{(2a)^2}\right](\mu-\lambda_1^2).$$

The functions F_3,, $|R_+|$, $|F_+|$ can be expressed in terms of C, $\mu(\theta)$ by means of the relations (4) and (17) given above. Thus, e.g.,

$$|R_+|^2=8[\pm\sqrt{(\epsilon|\mu|^2-C)^2+P_0-C^2}-(\epsilon|\mu|^2$$
$$-C)],\quad |F_+|^2=\frac{|R_+|^2|\mu|^2}{4P_0}. \tag{21}$$

The integral

$$\int_{\mu_0}^{\mu}\frac{d\mu}{\sqrt{P(\mu)}}=i(\theta-\theta_0) \tag{22}$$

can be expressed in terms of Jacobi elliptic functions. We give two solutions for which the roots σ_n, $n=1-4$ of the polynomial P are imaginary: $\sigma_n=i\lambda_n$, $\operatorname{Im}\lambda_n=0$, $\lambda_1>\lambda_2>\lambda_3>\lambda_4$. For $\lambda_1>\chi_1\geqslant\lambda_2$ we have[31]

$$\mu_1=i\chi_1=i\,\frac{\lambda_1(\lambda_2-\lambda_4)+\lambda_4(\lambda_1-\lambda_2)\mathrm{sn}^2(i\Theta,k_1)}{\lambda_2-\lambda_4+(\lambda_1-\lambda_2)\mathrm{sn}^2(i\Theta,k_1)}, \tag{23}$$

where

$$2\Theta=(\theta-\theta_0)\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)},$$

$$k_1^2=\frac{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}.$$

For $\lambda_2>\chi_2\geqslant\lambda_3$ we have

$$\mu_2=i\chi_2=i\,\frac{\lambda_2(\lambda_1-\lambda_3)-\lambda_1(\lambda_2-\lambda_3)\mathrm{sn}^2(\Theta,k_2)}{(\lambda_1-\lambda_3)-(\lambda_2-\lambda_3)\mathrm{sn}^2(\Theta,k_2)}, \tag{24}$$

where Θ is the same as in (23) and we have written

$$k_2^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}.$$

4. THE WHITHAM EQUATIONS

The existence of a representation of Eqs. (1) in the form of a compatibility condition for the two linear systems of equations (6) and (7) including a nonzero pump $W \neq 0$ implies that the spectral parameter depends on the position y. For a one-soliton solution this dependence means that the shape and velocity depend on y and the solution of Eq. (8) is completely determined. For the one-phase solution the time dependence of the roots of the polynomial P satisfies a complicated system of nonlinear equations, which can be found by studying the evolution of the integrals of the model. These equations cannot be solved using the ISM. Consequently, for $W=0$ we use the quasiclassical approximation first employed by Whitham.[14] In this approach the integrals of motion are averaged over the fast nonlinear oscillations or over the period between solitons formed by the periodic wave. We study processes that take place on characteristic scales of the variables that are much larger than this period, i.e., the slow modulation of a large packet of "closely spaced" nonlinear pulses. Averaging over the period yields the set of Whitham equations for the spectral parameters, the roots of the polynomial P.

In Refs. 32 and 33 the technique developed using the ISM is presented, permitting the Whitham equations to be found directly in diagonal form. We will use the method of Ref. 33, which allows us to reduce a complicated (generally speaking) problem to a general procedure.

Using the Lax representation (6), (7) we can easily find the following relation:

$$\partial_y\left[\frac{\sqrt{c^2+a^2\lambda}}{g}R_+\right]=\partial_x\left[\frac{\sqrt{c^2+a^2\lambda}}{\lambda g}F_+\right]. \tag{25}$$

We introduce a new normalization for the functions f,g,h: $f^2+hg=1$. Using (15) and (25) we find

$$\partial_y\left[\frac{\sqrt{P(\lambda)}}{\lambda-\mu}\right]=\partial_x\left[\frac{\sqrt{P(\lambda)}}{2\sqrt{P_0}}\left(\frac{1}{\lambda}-\frac{1}{\lambda-\mu}\right)\right]. \tag{26}$$

The oscillation period T is determined by the following integral:

$$T=\int d\theta\int\frac{d\mu}{\sqrt{-P(\mu)}}=\frac{4K(k)}{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}, \tag{27}$$

where $K(k)$ is the complete elliptic integral of the first kind with modulus k: $k^2=[(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)]/[(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)]$, and the λ_k are the roots of the polynomial P arranged so that $\lambda_1>\lambda_2>\lambda_3>\lambda_4$. The integral in (27) is carried out around the cut between λ_1 and λ_2 or between λ_3 and λ_4. The average over the period T is performed using the obvious formulas

$$\left\langle\frac{1}{\lambda-\mu}\right\rangle=\frac{1}{T}\int\frac{1}{\lambda-\mu}\,d\theta=\frac{1}{T}\int\frac{1}{\lambda-\mu}\frac{1}{\sqrt{-P(\mu)}}d\mu. \tag{28}$$

Assuming successively $\lambda=\lambda_n$, $n=1-4$ we find from (26) and (27)

$$\lim_{\lambda\to\lambda_n}\left\langle\frac{1}{\lambda-\mu}\right\rangle=-2\partial_{\lambda_n}(\ln T). \tag{29}$$

Expression (26) has a singularity for $\lambda=\lambda_n$ resulting from the derivatives

$$\partial_x\sqrt{P(\lambda)},\quad \partial_y\sqrt{P(\lambda)}.$$

In Ref. 33 it is shown that the condition that the coefficients of these two derivatives vanish yields the desired Whitham equations. In our case these equations take the form

$$\partial_y\lambda_n+\frac{1}{V_n}\partial_x\lambda_n=0,\quad \frac{V_0}{V_n}=1-\left[\lambda_n\left\langle\frac{1}{\lambda_n-\mu}\right\rangle\right]^{-1}. \tag{30}$$

Here

$$\left\langle\frac{1}{\lambda_1-\mu}\right\rangle=\frac{(\lambda_2-\lambda_4)E(k)-(\lambda_1-\lambda_4)K(k)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_4)K(k)},$$

$$\left\langle\frac{1}{\lambda_2-\mu}\right\rangle=\frac{(\lambda_1-\lambda_3)E(k)-(\lambda_2-\lambda_3)K(k)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)K(k)},$$

$$\left\langle\frac{1}{\lambda_3-\mu}\right\rangle=\frac{(\lambda_2-\lambda_4)E(k)-(\lambda_2-\lambda_3)K(k)}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)K(k)},$$

$$\left\langle\frac{1}{\lambda_4-\mu}\right\rangle=\frac{(\lambda_1-\lambda_3)E(k)-(\lambda_1-\lambda_4)K(k)}{(\lambda_1-\lambda_4)(\lambda_3-\lambda_4)K(k)},$$

where $V_0 = -2\sqrt{P_0} = -2\sqrt{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$. The expression $E(k)$ refers to the complete elliptic integral of the second kind with the same modulus k as in Eq. (27). Recall that only three of the λ_n are independent if the constant C is given in advance.

The diagonal form of Eqs. (30), which results when the above approach is employed, gives rise to very important simplifications in the solution of these equations (cf. Ref. 34). It is obvious that the evolution of the roots λ_k is determined by the initial values of the fields. To be specific let us consider the case in which a step function is propagating, i.e., the field R_+ has the form $R_+(x,y)=\text{const}$, $x=0$, $y\leq 0$ and $R_+(0,y)=0$, $y>0$. Let us choose the solution (23). We restrict ourselves primarily to a detailed study of this case, since the purpose of the present work is to describe the qualitative behavior of a new effect. Note that these properties are preserved when more general solutions are considered, including those with the restriction (20).

Consider the behavior of the solution near the leading edge of the step. Numerical results show that when powerful steplike pulses propagate a packet of pulses develops in the neighborhood of this front, whose shape approaches a soliton asymptotically (see, e.g., Refs. 2 and 34). This limit corresponds to $\lambda_2\to\lambda_3$, $k_1\to 1$. Assuming $\lambda_{2,3}=\zeta\pm\varepsilon/2$, $\varepsilon\ll\lambda_{2,3}$, we find

$$k^2\approx 1-\varepsilon\frac{(\lambda_1-\lambda_4)}{(\lambda_1-\zeta)(\zeta-\lambda_4)}+O(\varepsilon^2). \tag{31}$$

From Eq. (30) we find

$$\frac{V_0}{V_{1,2}}\approx 1\pm 2\varepsilon\ln(\varepsilon)+O(\varepsilon). \tag{32}$$

The self-similar solution of (30) such that λ_i depends on the single variable x/y follows from the representation of the Whitham equations in the form

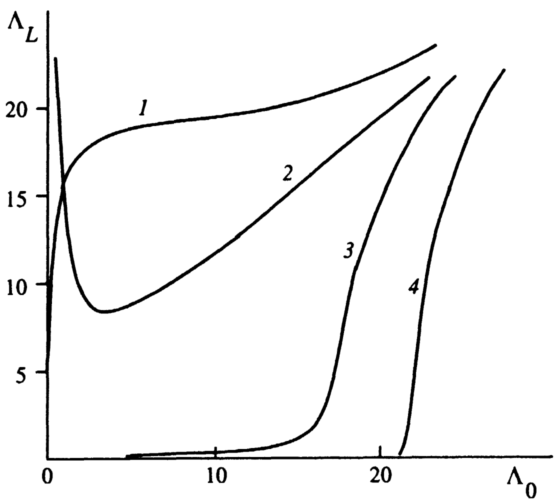

FIG. 2. Dependence of $\Lambda_L=\lambda(L)\zeta_1-\epsilon$ on $\Lambda_0=\lambda(0)\zeta_1-\epsilon$: *1)* $\epsilon=1$, $g=W\zeta_1^2L=10$; *2)* $\epsilon=-1$, $g=10$; *3)* $\epsilon=1$, $g=-10$; *4)* $\epsilon=-1$, $g=-10$, respectively.

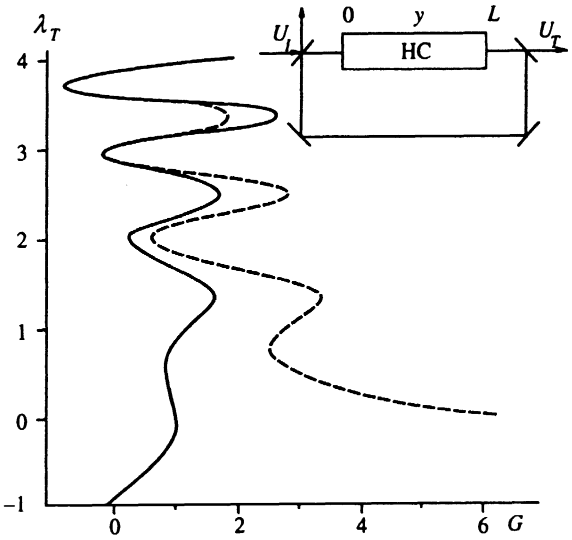

FIG. 3. Dependence of λ_T (the spectral data of a transmitted wave) on $G=LW$ (the gain); $\zeta_1=1$, $T_r=0.1$. The solid trace corresponds to $\epsilon=1$ and the broken trace to $\epsilon=-1$. The insert depicts a schematic of the ring cavity with the nonlinear medium (NM). Here $\lambda_I=5\sqrt{T_r}$ is the parameter of the field U_I injected into the cavity and $U_T=\sqrt{T_r}R_+(L,x)$ is the transmitted field.

$$\left(V_i-\frac{x}{y}\right)\partial_y\lambda_i=0. \tag{33}$$

From (32) and (33) it follows that

$$\varrho=\frac{V_0y}{x}-1=\frac{1}{\zeta}\frac{4(\zeta-\lambda_1+\varepsilon/2)\varepsilon K(k)}{(\lambda_1-\zeta)E(K)-\varepsilon K(k)}\approx\frac{4\varepsilon}{\zeta}\ln\left(\frac{Q}{\varepsilon}\right)+O(\varepsilon), \tag{34}$$

where $Q=4(\lambda_1-\zeta)(\zeta-\lambda_4)(\lambda_1-\lambda_4)^{-1}$. In the logarithmic approximation the solution to (34) takes the form

$$\varepsilon\approx\varrho\zeta\left[4\ln\left(\frac{Q}{\varrho}\right)\right]^{-1}. \tag{35}$$

The separation L_0 between the peaks of the solitons increases logarithmically slowly:

$$L_0\approx V_0T\approx 4V_0\ln\left[\frac{4Q}{\varrho\zeta}\ln\left(\frac{Q}{\varrho}\right)\right]\frac{1}{\sqrt{(\lambda_1-\zeta)(\zeta-\lambda_4)}}. \tag{36}$$

The second limiting case $k_1\to 0$ corresponds to quasiharmonic oscillations at the trailing edge of the wave packet. Note that similar solutions of the Whitham equations occur in other models.[30,35] The details of the model are specified by the restrictions on the set of roots of the polynomial P.

5. MULTISTABILITY OF A PERIODIC WAVE

Let $m_w, m_{ob}=W^{-1}$ be the characteristic scales on which the packet envelope varies due to the interaction of the solitons in the packet according to the Whitham equations, $m_w\ll m_{ob}$ and $m_w\gg m_{ob}$ the reciprocal of the amplification length, respectively. In the first case the gain occurs adiabatically slowly in comparison with the intersoliton interaction. The solution of Eq. (8)

$$\Lambda(0)-\Lambda(L)-\epsilon\ln[\Lambda(0)/\Lambda(L)]=WL\zeta_1^2,$$
$$\Lambda(y)=\zeta_1\lambda(y)-\epsilon$$

describes the dependence $\lambda(L)$ of the spectral data at the exit from the sample, $y=L$, on the initial values, which are described in turn by the solutions of the Whitham equations. This dependence for a fixed variable x is shown in Fig. 2. Depending on the signs of the constants qualitatively different lasing regimes are realized. The curves in Fig. 2 show that as the periodic field injected into the nonlinear medium evolves an abrupt change in the relation between the field parameters can develop, as a result of which the shape of the envelope of the generated field can change. Thus, if the roots $\lambda_{2,3}(0)$ are located close to a rapid rise in the curve *1* then the quasisoliton regime of the initial wave (23) can change into a quasiharmonic regime on leaving the sample. For the solution (24) the opposite transition can occur: a packet of quasiharmonic pulses can be transformed into a train of solitons.

Since nonlinear multiwave processes are usually studied in multipass systems it is of interest to investigate the effect of the feedback caused by the cavity. The standard design of a ring cavity is shown in the insert of Fig. 3. The transmission coefficient of the upper mirrors is T_r and that of the lower ones is unity. It is assumed that we can neglect the term with ζ_2 in the original system (1). The boundary conditions imposed by the cavity relate the values $U_I=R_+$ of the field at the entry to the cavity to the values U_T at the exit and take the form (see, e.g., Ref. 8)

$$R_+(0,x)=\sqrt{T_r}U_I+(1-T_r)\exp(i\delta_r)R_+(L,x+X_r),$$
$$U_T=\sqrt{T_r}R_+(L,x), \tag{37}$$

where δ_r and X_r are the cavity mismatch and the time for a field pulse to traverse the cavity. Here x and y play the role of the temporal and spatial variables respectively. Equations (8), (30), and (37) cannot be solved simultaneously in closed form. We will assume that the slow variation of the parameters λ with respect to y is completely determined by the dependence (8). In the approximation in which the field variables change slowly over a single pass through the cavity we find

$$\sqrt{T_r}U_I=[1-(1-T_r)\exp(i\Phi)-2X_r\partial_y]U_T,$$

$$\Phi=\phi_r+2\zeta_1\int_0^L|R_+(x,y)|^2dy. \quad (38)$$

In deriving (38) we have used the approximations $R_{+,n}(L,x)-R_{+,n}(0,x)\approx L\partial_yR_{+,n}(y=L,x)$, $R_{+,n}(L,x)-R_{+,n-1}(L,x)\approx X_r\partial_nR_{+,n}(y=L,x)$, where the subscript n refers to the nth pass through the cavity and ϕ_r is the cavity mismatch. Here $U_{I,T}$ are the field injected into the cavity and the transmitted field respectively. We further assume that a whole number of solitons falls within the length of the nonlinear medium. To find the conditions on the wave parameters we reduce (38) to a relation for the integrals of motion and average over a wave period. In the present work we restrict ourselves to the quasisoliton regime. We take $\lambda_1=-\lambda_4=ih$, $\lambda_2=-\lambda_3=id$, $\mathrm{Im}\,h=\mathrm{Im}\,d=0$, $h>d$. For the solution $\mu=i\chi$, $h>d\geq\chi>0$, $\epsilon=-1$, $h\to d$ we find

$$\sqrt{T_r}h_I=[(h-WL(2\zeta_1h-1)/h)^2+h^2(1-T_r)^2-2h(1-T_r)(h-WL(2\zeta_1h-1)/h)\cos\Phi]^{1/2}h_T,$$

$$\Phi=\phi_r+2\zeta_1L(1+h^2). \quad (39)$$

An analogous expression holds for $\mu=i\chi$, $h>\chi\geq d>0$, $\epsilon=1$, $d\to0$. In the derivation of (39) it was assumed that the speed of propagation of the pulses in the nonlinear medium is much less than the velocity outside it.

A plot of the multistable dependence of the transmitted wave parameter as a function of the wave parameter injected into the cavity is shown in Fig. 4. The branches having negative slopes correspond to unstable states.

The behavior of the periodic solution described above in a multistable system can be shown to persist in a more general solution of the model. Note that for the solution (24) there is another set of interchanges of regimes in a multistable medium. Thus, for similar values $\lambda_{2,3}$ a situation can arise in which switching occurs from the regime of quasiharmonic oscillations to the quasisoliton regime of periodic wave propagation.

Equation (39) also describes a new effect, multistable dependence of the parameters of a periodic wave and, in particular, of the soliton amplitude on the gain coefficient $G=WL$ (Fig. 3). It is well known that in ion lasers the quantity W is determined by the current and can vary over a wide range.[4] This allows effective control of the pulse-generation process in a bistable system. For $\epsilon=-1$ the lower branch goes to infinity. This gives rise to the regime of "hard" excitation.

When W increases, at some point the spectral parameter undergoes bifurcation. Assume that initially this parameter corresponds to a harmonic wave. Using the formalism of the monodromy matrix we can show that this bifurcation is responsible for a transition from a harmonic wave to a nonlinear wave described by an elliptic function of the first kind. This approach is of interest for analytical investigation of lasing when perturbation theory is inapplicable, e.g., processes by which N-phase nonlinear waves transform into $(N+1)$-phase waves.

6. DISCUSSION OF THE RESULTS OBTAINED

In the present work we have found for the first time quasiperiodic solutions of the model in question for $W=0$. A quasiperiodic wave arises naturally in many nonlinear optical multipass systems if the relaxation times are sufficiently long. Certain instabilities that occur in lasers evolve into quasiperiodic waves.[10] As shown in the present work, the dependence of the solution parameters on the amplification length and the nonlinear frequency modulation can lead to new effects, in particular soliton multistability.

Use of the inverse scattering method to study evolution models enables us to obtain new qualitative and quantitative information about the evolution of the fields. At the same time, its applicability is related to a number of restrictions on the physical parameters of the medium. In the present model a periodic solution is found for systems without pumping, $W=0$. At this stage we have been unable to generalize the periodic form of the ISM to the case $W\neq0$. However, in the soliton case the ISM can be generalized without difficulty to the nonzero-pump case by using the formalism of Ref. 7. Then, as in the present case, the spectral data (i.e., the soliton dimensions and velocity) in a ring cavity develop multistable dependence on the initial data. To describe this effect precisely we can use the solution in the "soliton limit" constructed above, including the dependence on the parameters and position. The conditions under which multistability of the parameters λ_k is observed numerically take the form

$$LW|\zeta_1|^2\leq5, \quad \lambda_k|\zeta_1|\leq20. \quad (40)$$

Several different kinds of multistability arise, depending on the sign of ϵ, W, and ζ_1. For model 5 the magnitude of ζ_1 is small in comparison with the other constants and critical field powers must be achieved in order to observe multistability of the field amplitude. Thus, for a CO_2 laser the power should be of order 2–4 TW/cm^2 (cf. Ref. 27). For the two-photon interaction $|\zeta_1|\approx1$ holds, and to observe amplitude multistability in cesium vapor under the conditions of Ref. 21 the fields should have a power of order 1–5 MW/cm^2. For model 1 the quantity ζ_1 depends on the carrier frequencies of the fields and can be chosen without difficulty to be much greater than unity, $|\zeta_1|\approx10$–100. This corresponds to reducing the critical values of the field strength by a factor of 10^2–10^4. The first of conditions (40) for small values of ζ_1 can be satisfied in multipass systems.

Consider the solution (23) with the condition $C=-\sqrt{P_0}$, which implies that in the absence of a field $R_+=0$ the system is in the ground state, i.e., the population satisfies $F_3=-1$. In this case the maximum of the field intensity is $\max|R_+|^2=4[\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}-\lambda_4^2]$. By choosing a sufficiently small value λ_4 we obtain small subthreshold values of the field strength. At the same time λ_1 must be large and satisfy (40). The multivalued behavior of λ_1 corresponds to multistable behavior of the characteristics of a periodic wave such

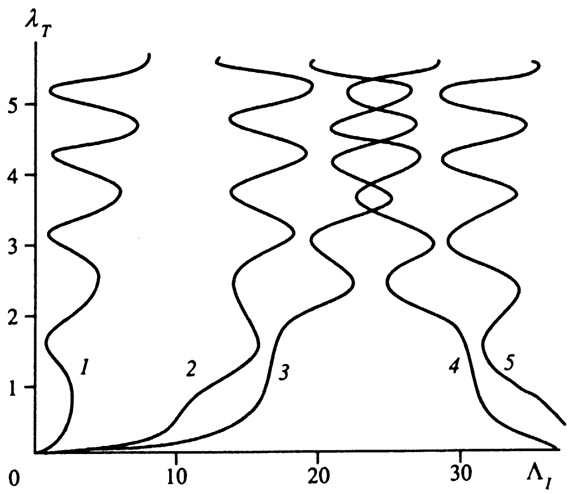

FIG. 4. Dependence of $\lambda_L=\lambda(L)$ on $\Lambda_I = \sqrt{T_r}\lambda(0)$, $T_r=0.1$, $\zeta_1=1$ in the ring cavity (see insert of Fig. 3): *1)* $g=WL=0$, *2)* $\epsilon=-1$, $g=10$; *3)* $\epsilon=-1$, $g=-10$; *4)* $\epsilon=1$, $g=10$; *5)* $\epsilon=1$, $g=-10$.

as the depth of modulation and the period of the nonlinear oscillations, as well as the nonlinear phase of the field.

We present an explicit special case of the periodic solution (24) of Eqs. (1), determined by "supercritical" values λ_k, $k=1-4$ but corresponding to a small power of the field. Assume $\gamma_1=\lambda_1=-\lambda_4$, $\gamma_2=\lambda_2=-\lambda_3$, $\gamma_1>\gamma_2\neq\chi_3>0$, $\nu_3=i\chi_3$, Im $\chi_3=0$. Using the formulas given above we find

$$\chi_3=\gamma_2 \ \mathrm{sn}(\gamma_1\theta,k_3), \quad k_3=\gamma_2/\gamma_1,$$

$$R_+=[\mathrm{dn}(\gamma_1\theta,k_3)-k_3 \ \mathrm{cn}(\gamma_1\theta,k_3)]\exp\{2i\zeta_1(8C-1+k_3^2)/(8\theta)+[E(\mathrm{am}\ \gamma_1\theta,k_3)-k_3\mathrm{sn}(\gamma_1\theta,k_3)/(4\gamma_1)]\}.$$

Here $\mathrm{dn}(\gamma_1\theta,k_3)$, $\mathrm{cn}(\gamma_1\theta,k_3)$, $\mathrm{sn}(\gamma_1\theta,k_3)$ are Jacobi elliptic functions and $E(\mathrm{am}\gamma_1\theta,k_3)$ is an elliptic integral of the second kind.[31] The soliton limit $k_3\rightarrow 1$ corresponds to vanishingly small amplitudes of the field, while the quantities $\gamma_{1,2}$ can be large enough to satisfy conditions (40) for soliton multistability. Thus, to obtain an optical trigger when periodic waves are used it is not necessary to have large values of the power, which actually shorten the lifetime for the optical device and require the application of complicated technology.

The imaginary part of the spectrum corresponds to soliton or periodic solutions, while a continuous real spectrum is associated with processes such as superradiance or Raman scattering.[7] In the case of Raman scattering the switching of the propagation regime of the Stokes field under pump-depletion conditions may be the reason for the appearance of soliton-like pulses of the Stokes field.

Numerical solution of the Whitham equations (30) for the case in which all the parameters λ_k as functions of y are determined by Eq. (8) shows that for $\epsilon\zeta_2<0$, $W>0$ there is an attractor $\lambda_k\rightarrow 1/\zeta_2$. The solution approaches an algebraic soliton. Assume that $W>0$ holds and $\lambda_k(y=0)$ are real. It can be shown that after a finite "time" y the spectral parameters acquire an imaginary correction. This process may be associated with the production of a soliton from the continuous spectrum.

The conditions for the applicability of the ISM require taking into account "friction" or "pumping" for $\epsilon=\pm 1$ respectively with specified coefficients in the last of Eqs. (1). For an ion laser, e.g., the quantity W is given by the magnitude of the current and can be varied over a broad range.[4] This allows us to "adjust" the parameters of the system to "precise" values. The size of the losses or of the pump can be adjusted to the precise variation of the cavity properties. In addition, preliminary numerical calculations reveal that the typical multistable dynamics should persist even when the variables undergo substantial (of order unity) deviations from their exact values. For the Raman-scattering or two-photon interaction models including nonzero pumping $W\neq 0$ may correspond to two situations. First, models of a two-photon laser with pumping of a two-level dipole-forbidden transition with nondepletion of the pump (or of one of the fields); second, models of the two-photon interaction with a transition having a constant difference between the populations and a constant external pumping of the intensity of one of the fields. The latter situation can be realized in a multipass system.

We will not dwell on the physical applications of the other models described in Sec. 2 with $W\neq 0$, since this can be done in analogy with those presented above.

The above results correspond to the quasisteady case. Here the quasistationary behavior of the periodic solution is to be understood as an adiabatically slow variation of the basic properties of the wave in the process of evolution under the influence of the soliton-soliton interaction and the perturbations. Often the main contribution to the perturbations comes from relaxation and diffraction effects. The influence of diffraction may be reduced by the choice of the geometry of the nonlinear media. In order to estimate the contribution made by relaxation to the medium we use the conditions of the experiment performed by Duncan *et al.*[11] to observe Raman scattering. In this work a cell of length 1 m containing hydrogen at room temperature and a pressure of 2–100 atm was used as the nonlinear medium. The laser beam used as a pump had a diameter of 60 cm and an energy of 10–20 mJ. The wavelengths of the fields were 532 nm and 683 nm. The transverse relaxation time was T_2=2.3 ns, i.e., much greater than the length of the injected pulses (20–40 ps). Under these conditions the slow deformation of the characteristics of the periodic wave due to the finite relaxation time may not result in loss of its quasistationary behavior in the adiabatic approximation. However, in view of the fact that this question has not been studied in detail here, we will assume that the condition for quasisteady behavior is $T_0\ll T_2$, where T_0 is the wave period. Such a relation between the characteristic time of the field and the relaxation time may holds even for a two-level solid-state laser with flashlamp pumping of the upper level.[4]

In addition to relaxation losses a ring cavity exhibits losses at the mirrors. These losses may be rendered negligible for all the fields taking part in the interaction by choosing the reflection coefficient of the mirrors close to unity,

$R=0.900-0.997$. The period T_0 undergoes an increase due to interaction between the solitons in the curse of the evolution. Let us estimate the cavity parameters for which we can observe multistability of a periodic wave in the quasisteady regime. Using Eqs. (34) and (36) we find the condition that the change in the shape of the wave over the time during which the multistable behavior develops be small:

$$L_0<(1-R)L \ln (\varrho^{-1}|_{y=L}),$$

where L_0 and L are the distance between peaks of the solitons and the length of the nonlinear medium respectively. This last condition may holds under the experimental conditions of Ref. 11. Note that the formulation of the problem on observing the evolution of a packet of solitons differs from that employed in Ref. 11. To investigate the soliton regime we must inject pulses of the pump and Stokes fields which are similar in energy.[20] Models of four-wave mixing use the induced Kerr nonlinearity resulting from two-photon interaction of the fields with the two-level transition.[18,21] Hence the estimates given above can also be used for models of the four-wave interaction. The combination of the soliton-soliton interaction and the relaxation can cause time-independent structures to develop.

This kind of stabilization is described in work on the propagation of individual (nonsoliton) pulses in a ring cavity with a saturated absorber,[15] a fiber lightguide with a particular type of nonlinearity,[16] and soliton multistability in a ring cavity.[17]

The single-phase solution constructed here can be used (as an exact solution) to study some systems of equations which are similar to (1) but are not integrable. These may include the Maxwell–Bloch equations treated in Ref. 36 which showed that a frequency shift proportional to the difference in level populations results in considerable modification of the soliton shape. This shift, which is due to the dipole-dipole interaction in a dense gas is proportional by virtue of (17) to the field strength for the single-phase self-similar solution. The single-phase solution of (1) with $\zeta_2\neq 0$, $\zeta_1=W=0$, $\epsilon=1$ is the solution of the model treated in Ref. 36, for which the ISM methodology is inappropriate.

The theory of integrable evolution models together with numerical calculation reveals that the asymptotic behavior of the waves is often determined by the dynamics of soliton or periodic waves. Hence spatially coherent structure can exist even for chaotic temporal behavior.[37] Multistable behavior of a periodic wave can bring about switching of the spatial structures, in some situations changing the regimes to chaotic. These processes can be studied using the technique developed above. When we take into account $\zeta_2\neq 0$ in the evolution of the field in a ring cavity the spectral parameters become dependent on x or on the number of traversals of the cavity (in the approximation used here). It can be shown that as the field evolves a transition from a one-band solution to a two-band solution, etc., occurs.

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