Logarithmic radiative corrections to the dipole matrix elements in the hydrogen atom

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Corrections of order $\alpha(\alpha Z)^2$ are considered in the logarithmic approximation, the respective corrections to the energy levels in the Stark effect and to the radiative level widths are obtained, and the problems of the phases of the dipole matrix elements and of the relation between the spherical and parabolic eigenfunctions in hydrogen are examined.

1. INTRODUCTION

The accuracy with which one must calculate the dipole matrix elements in hydrogenlike systems is determined by the progress in experimental techniques in two areas: in measurements related to the radiative level widths (lifetimes of the levels), and in studies of the behavior of energy levels in moderate electric fields.

Precision spectroscopy of the hydrogen atom and other hydrogenlike systems is being intensively developed for metrological purposes. Recently Koze *et al.*[1] carried out a precision experiment to determine the fine-structure constant via the Stark effect for the levels of the hydrogen atom with principal quantum number $n \approx 40$ (see also Ref. 2). There are plans to measure the Stark splitting of levels with an accuracy of about 10^{-7} in moderate electric fields, where the linear Stark effect is dominant and the Stark shifts are proportional to the respective dipole matrix elements.

A more common avenue of research deals with precision measurements of the Rydberg constant. Recent experiments[3–6] measured two-photon transitions in hydrogen to levels with $n=1,2$. Data processing requires knowing with a high degree of accuracy the Lamb splitting of the hydrogen levels with $n=2$. Here the most exact value of this quantity was obtained by measuring the ratio of the width of the $2p_{1/2}$ level to the splitting of levels with $n=2$ (see Ref. 7), which naturally requires knowing with a high degree of accuracy the lifetime of the $2p_{1/2}$ level, whose width in the first approximation is determined by the dipole formula. The most exact calculation of the Lamb shift of the hydrogen-atom ground state can be done[8–10] using the experimental value for the splitting at $n=2$. We note also that the lifetime of the $2p_{1/2}$ level in the helium ion has been measured with a high degree of accuracy.[11]

This paper is devoted to calculating the radiative corrections of relative order $\alpha(Z\alpha)^2 \log(Z\alpha)$ to dipole matrix elements in hydrogenlike ions. Numerical results are given for levels with $n=2$. Section 2 discusses the general expression for the radiative corrections to the dipole matrix element in the logarithmic approximation [in $\log(Z\alpha)$]. The lifetimes in hydrogenlike systems and the energies of Stark levels are examined as possible applications. For example, Sec. 3 is devoted to finding the radiative corrections to dipole partial widths and in particular, to the width of the $2p_{1/2}$ level in the hydrogen atom and the helium ion. Section 4 is devoted entirely to the Stark effect in moderate fields. Special attention is paid to the relation between the base wave functions of the hydrogen atoms in spherical and parabolic coordinates.

2. DIPOLE MATRIX ELEMENTS

Here we consider the radiative corrections to the dipole matrix element. To avoid cumbersome expressions we seek the corrections to the quantity

$$\mathbf{d}_{1n} = \langle 1, l=0 | e\mathbf{r} | n, l=1 \rangle;$$

below we will see that the transition to arbitrary s-states presents no difficulties. First we note that in the logarithmic approximation in the Fried–Yennie gauge,[12,13] the radiative corrections to an electron line have the softest low-energy asymptotics,[13–15] and $\log(Z\alpha)$ is contained only in the correction to the wave function of the s-state:

$$\delta\psi_{1s}(\mathbf{r}) = \sum_{q\neq 1} \psi_{qs}(\mathbf{r}) \frac{\langle qs | \Sigma_C^{(1)}(E_{1s}) | 1s \rangle}{E_{1s} - E_{qs}}, \tag{1}$$

where $\Sigma_C^{(1)}$ is the single-loop operator of the self-energy of an electron in the Coulomb field of the nucleus, summation is over all states of the discrete and continuous spectrum, and the corrections to the wave functions of a level with $l \neq 0$ and to the dipole operator do not incorporate logarithmic contributions.

The matrix element in Eq. (1) can easily be found1) (see, e.g., Ref. 16):

$$\langle qs | \Sigma_C^{(1)} | 1s \rangle = \frac{\psi_{qs}(0)}{\psi_{1s}(0)} \frac{4}{3\pi} \alpha(Z\alpha)^4 m \log \frac{1}{(Z\alpha)^2}, \tag{2}$$

where we have explicitly allowed for the fact that only momenta of the atomic order are of interest and, hence, in the case of a continuous spectrum the characteristic wave numbers k are of order $\gamma = Z\alpha m$. The wave functions of the s-states of the discrete and continuous spectrum are defined in such a way that their values at zero, $\psi_{qs}(0)$, are real:

$$\left(\frac{\psi_{ns}(0)}{\psi_{1s}(0)}\right)^2 = \frac{1}{n^3} \tag{3}$$

and

$$\left(\frac{\psi_{ks}(0)}{\psi_{1s}(0)}\right)^2 = \frac{2\pi}{\gamma} \frac{k/\gamma}{1-\exp(-2\pi\gamma/k)}. \tag{4}$$

It is convenient to introduce the normalized ratio $\mathscr{D}$ of dipole matrix elements by the following relation:

$$\mathbf{d}_{1n}\mathscr{D}_{qn}=\mathbf{d}_{qn}\frac{\psi_{1s}(0)}{\psi_{qs}(0)}\,. \tag{5}$$

Below we will see that, considered a vector, the correction to the dipole matrix element is directed along the initial matrix element. Hence in what follows we write the relative factor as $\delta\mathbf{d}_{1n}/\mathbf{d}_{1n}$. Note that the arbitrariness in selecting the phase of the wave function in the form of an additional factor depending on the orbital quantum number l and the azimuthal quantum number m (see the definitions in Refs. 17–19 and 20, 21) has no effect on the magnitude of $\mathscr{D}_{qn}$, while the reality of the various expressions for the discrete spectrum is ensured by the fact that the phases of the wave functions are independent of the principal quantum number.

Because the normalization of (5) is relative we can use the reduced matrix elements of Refs. 20 and 21 for the discrete spectrum without any additional factors:

$$\begin{aligned}\langle n',l=0\|er\|n,l=1\rangle &= i\,\frac{4(-1)^{n'-1}n^2(n')^4\sqrt{(n^2-1)n}}{(n^2-n'^2)^2}\\ &\times\left(\frac{n-n'}{n+n'}\right)^{n+n'-2}\left(\frac{\psi_{n's}(0)}{\psi_{1s}(0)}\right)\\ &\times\left\{F\left(2-n,1-n',2,\ -\frac{4nn'}{(n-n')^2}\right)\right.\\ &\left.-\left(\frac{n-n'}{n+n'}\right)^2F\left(-n,1-n',2,-\frac{4nn'}{(n-n')^2}\right)\right\}\end{aligned} \tag{6}$$

for $n\neq n'$.

This differs somewhat from the expression in Ref. 21 but is convenient for building an analytic continuation and thus finding the result of s-states of the continuous spectrum. Note, for one thing, that the hypergeometric functions in Eq. (6) can be reduced to polynomials for both real and arbitrary complex values of n'.

For equal values of the principal quantum number we have

$$\langle n,l=0\|er\|n,l=1\rangle=-\,i\cdot\frac{3}{2}n^2\sqrt{(n^2-1)n}\left(\frac{\psi_{ns}(0)}{\psi_{1s}(0)}\right). \tag{7}$$

Note that the result for $n=n'$ is given in Refs. 17–19 and 21 with an incorrect sign. The error in sign is due to the calculation of the integral of the radial wave functions,

$$\mathscr{R}^{n',l}_{n,l-1}=\int_0^\infty dr\,r^3R_{n,l-1}(r)R_{n'l}(r), \tag{8}$$

to which the authors of Refs. 17–19 and 21 assign, in atomic units, a value of $(3/2)n\sqrt{n^2-l^2}$, while actually this integral is negative. For instance, for the particular case of $n=2$, which we will need in further numerical calculations, we can easily see that

$$\mathscr{R}^{2,1}_{2,0}=\int_0^\infty dr\,r^3\left[\frac{1}{2\sqrt{6}}e^{-r/2}r\right]\left[\frac{1}{\sqrt{2}}e^{-r/2}\left(1-\frac{r}{2}\right)\right]=-3\sqrt{3}.$$

The method by which the integral in (8) can be calculated for $n=n'$ in the general case is described in detail in Ref. 21.

The origin of this misprint is easily traced. The integral in (8) for $n\neq n'$ was found by Gordon.[22] Bethe[17] added to Gordon's results an expression for the case where $n=n'$ [Eq. (41.5)] with a misprint and with a reference to Sec. 34 devoted to the Stark effect, in which the result for the integral under discussion is given with the correct sign. Later (see Ref. 18) the reference to the section devoted to the Stark effect was dropped, while the expression with the wrong sign and the reference to Ref. 17 or 18 were reproduced in Refs. 19 and 21. In Ref. 21 there is also a misprint in the expression for the reduced matrix elements [Eq. (52.3)], where $(-1)^{n'-1}$ is printed instead of $(-1)^{n'-l}$ (cf. Refs. 17–19 and 22). The sign of the matrix element is rarely needed and the above-noted misprints have no effect on the final result.

The normalization factor in (5) is

$$\langle 1,l=0\|e\mathbf{r}\|n,l=1\rangle=i\cdot 16n\left(\frac{n}{n^2-1}\right)^{5/2}\left(\frac{n-1}{n+1}\right)^n. \tag{9}$$

For discrete states the ratio defined by (5) has the form

$$\begin{aligned}\mathscr{D}_{n'n}&=\frac{(-1)^{n+1}}{4}\frac{(n^2-1)^3}{n^2}\left(\frac{n+1}{n-1}\right)^n\frac{n'^4}{(n^2-n'^2)^2}\\ &\times\left(\frac{n'-n}{n'+n}\right)^{n+n'-2}\left\{F\left(2-n,\,1-n',\,2,\,-\frac{4nn'}{(n-n')^2}\right)\right.\\ &\left.-\left(\frac{n-n'}{n+n'}\right)^2F\left(-n,\,1-n',\,2,\,-\frac{4nn'}{(n-n')^2}\right)\right\}\end{aligned} \tag{10}$$

if $n\neq n'$ and

$$\mathscr{D}_{nn}=-\frac{3}{32}\frac{(n^2-1)^3}{n}\left(\frac{n+1}{n-1}\right)^n \tag{11}$$

if $n=n'$, and the correction to the dipole matrix element is

$$\begin{aligned}\frac{\delta\mathbf{d}_{1n}}{\mathbf{d}_{1n}}&=\frac{4}{3\pi}\alpha(Z\alpha)^4\log\frac{1}{(Z\alpha)^2}\\ &\times\sum_{q\neq 1}\left(\frac{\psi_{qs}(0)}{\psi_{1s}(0)}\right)^2\frac{m}{E_{1s}-E_{qs}}\mathscr{D}_{qn}\,.\end{aligned} \tag{12}$$

Note that the continuous-spectrum wave functions are well known (see, e.g., Ref. 20), and the dipole matrix elements $\mathscr{D}_{qn}$ for the continuous spectrum are obtained by analytic continuation of (10).

However, a simpler way to find the analytic continuation is to use the dimensionless variable $t=\gamma/k$ instead of the wave number k. In terms of this variable Eq. (12) assumes the form

$$\frac{\delta \mathbf{d}_{1n}}{\mathbf{d}_{1n}}=-\frac{8\alpha}{3\pi}(Z\alpha)^2\log\frac{1}{(Z\alpha)^2}$$

$$\times\left[\sum_{n'\neq 1}\frac{1}{n}\frac{1}{n'^2-1}\mathscr{D}_{n'n}\right.$$

$$\left.+\int_0^\infty\frac{dt}{t(t^2+1)}\frac{1}{1-e^{-2\pi t}}\mathscr{D}_{tn}\right], \tag{13}$$

where

$$\mathscr{D}_{tn}=\frac{(-1)^{n+1}}{4}\frac{(n^2-1)^3}{n}\left(\frac{n+1}{n-1}\right)^n\frac{t^4}{(n^2+t^2)^2}$$

$$\times\exp\left\{2[i(n-2)-t]\tan^{-1}\left(\frac{n}{t}\right)\right\}$$

$$\times\left\{F\left(2-n,1+it,2,\frac{4int}{(n+it)^2}\right)-\left(\frac{n+it}{n-it}\right)^2\right.$$

$$\left.\times F\left(-n,1+it,2,\frac{4int}{(n+it)^2}\right)\right\} \tag{14}$$

is the analytic continuation of (5) in t (compare this with the photoionization amplitudes in Refs. 17,18, and 21 and with the explicit expressions in Ref. 22):

$$n'\rightarrow -it. \tag{15}$$

The expression for the correction to an arbitrary dipole matrix element is

$$\frac{\delta \mathbf{d}_{n'n}}{\mathbf{d}_{n'n}}=-\frac{8\alpha}{3\pi}(Z\alpha)^2\log\frac{1}{(Z\alpha)^2}$$

$$\times\left[\sum_{n''\neq n'}\frac{1}{n''^3(1/n'^2-1/n''^2)}\frac{\mathscr{D}_{n''n}}{\mathscr{D}_{n'n}}\right.$$

$$\left.+\int_0^\infty\frac{dt}{t^3(1/n'^2+1/t^2)(1-e^{-2\pi t})}\frac{\mathscr{D}_{tn}}{\mathscr{D}_{n'n}}\right]. \tag{16}$$

To conclude our discussion of dipole matrix elements, here is a simple sum rule that enables tracing the relative phases of the matrix elements:

$$\sum\psi_{qs}(0)\langle qs|\mathbf{d}|np\rangle=0, \tag{17}$$

which becomes obvious if one replaces the value of the wave function at zero with the quantity $\delta(\mathbf{r})|qs\rangle$ and multiplies from the right by an arbitrary wave function $\langle\phi|$. Clearly, summation over all states leads to a zero matrix element $\langle\phi|\delta(\mathbf{r})\mathbf{d}|np\rangle$. In terms of normalized ratios of dipole matrix elements (5), the sum rule (17) assumes the form

$$\sum\psi_{qs}^2(0)\mathscr{D}_{qn}=0. \tag{18}$$

3. CORRECTIONS TO RADIATIVE LEVEL WIDTHS

As noted in the Introduction, there are two types of problems that require knowing the dipole matrix elements with a high degree of accuracy. Matrix elements that are diagonal in the principal quantum number arise in Stark-effect calculations, while off-diagonal matrix elements appear in problems with emission (decay of bound states) or absorption of a photon.

To avoid cumbersome equations, in this section we study radiative corrections to the lifetime and width of the $2p_{1/2}$ level in a hydrogenlike system. This problem is important for hydrogen (for which the ratio of the level width to Lamb splitting at $n=2$ has been measured[7]) and helium (for which the level lifetime has been measured[11]). Generalization to the partial widths for the decay of other levels presents no difficulties. Note that logarithmic radiative corrections are related to the energy and wave function of s-states, and in the dipole approximation only the transitions between s and p levels contain contributions of relative order $\alpha(Z\alpha)^2\log(Z\alpha)$.

3.1. Lifetime of the $2p_{1/2}$ level in the hydrogen atom

In the first approximation, the lifetime of the excited level in the hydrogen atom is determined by a formula for dipole transitions, and the width of the $2p_{1/2}$ level is

$$\Gamma_0=\frac{4\omega^3}{3}|\mathbf{d}_{12}|^2, \tag{19}$$

or

$$\Gamma_0^{\mathrm{H}}=\frac{2^{10}\pi}{3^8}\alpha^3\mathrm{Ry}\frac{m_R}{m}, \tag{20}$$

where ω is the frequency, $\mathbf{d}_{12}$ is the dipole matrix element of the $2p_{1/2}\rightarrow 1s_{1/2}$ transition, Ry is the Rydberg constant of the hydrogen atom with an infinitely heavy nucleus, m_R is the reduced mass, m is the electron mass, and we have explicitly allowed for the fact that Z for hydrogen is equal to unity. The dependence of the dipole width on the nuclear mass and charge is examined in Sec. 3.2. Below we discuss the relativistic and radiative corrections.

The leading relativistic corrections of order $(Z\alpha)^2$ were found in Ref. 7. Later, in Ref. 23, it was pointed out that the radiative corrections to this decay of order $\alpha(Z\alpha)^2$ are important for analyzing the results of the experiment reported in Ref. 7. Clearly, the corrections to (19) can be produced by the corrections to the transition energy (frequency) and the corrections to the dipole matrix elements and by allowing for other decay modes. In Ref. 23 the exact transition frequencies were taken into account, and the width of the $2p_{1/2}$ level was found to be

$$\Gamma^{\mathrm{H}}=\Gamma_0^{\mathrm{H}}\left\{1+(Z\alpha)^2\left[\log\frac{9}{8}-\frac{32\alpha}{3\pi}\left(\log\frac{1}{(Z\alpha)^2}\right.\right.\right.$$

$$\left.\left.\left.-\log k_0(1,0)+\frac{1}{8}\log k_0(2,1)+\frac{1}{64}+\frac{19}{30}\right)\right]\right\}, \tag{21}$$

where we have used the standard notation $\log k_0(n,l)$ for the Bethe logarithms.

The relativistic corrections were also found by Drake[11] in a discussion of the lifetime of the $2p_{1/2}$ level in the helium ion, but the coefficient of $(Z\alpha)^2$ proved to be twice as large. In view of the discrepancy between the results of Refs. 7 and

23 and those of Ref. 11, we did an independent calculation, and the result coincided with that of Refs. 7 and 23.[2)]

We calculate the radiative corrections with logarithmic accuracy. Clearly, in the Fried–Yennie gauge the corrections to (19) related to other decay modes are nonlogarithmic. When the width of the $2s_{1/2}$ level is taken into account, a nonlogarithmic contribution of order $\alpha(Z\alpha)^2$ also appears.

The contribution to the dipole transition matrix element (12) considered in the previous section has the form

$$\delta\Gamma_{\text{dip}}=\Gamma_0\cdot 2\ \text{Re}\left(\frac{\delta\mathbf{d}_{12}}{\mathbf{d}_{12}}\right), \tag{22}$$

and in the case of the $2p$-level the sign of the real part can be dropped, i.e., the corresponding quantity proves to be real.

Direct numerical integration leads to the following result:

$$\delta\Gamma^{\text{H}}_{\text{dip}}=-\Gamma^{\text{H}}_0\cdot\frac{16}{3\pi}\alpha(Z\alpha)^2\log\frac{1}{(Z\alpha)^2}S, \tag{23}$$

where the contribution is separated into three terms:

$$S=(-1.8984...)+(0.2412...)+(0.1488), \tag{24}$$

with the first related to the level $n=2$, the second to the sum with $n>2$, and the third to the continuous spectrum.

The energy denominator can be written as

$$\frac{1}{E_{1s}-E_{qs}}=\frac{1}{E_{1s}}+\left(\frac{1}{E_{1s}-E_{qs}}-\frac{1}{E_{1s}}\right), \tag{25}$$

and the total sum splits into two parts, where the first can be calculated directly using the sum rule (18),

$$S^{(1)}=-1, \tag{26}$$

and the second, after numerical integration, proves to be equal to

$$S^{(2)}=(-0.474\,61...)+(0.021\,09...)+(-0.054\,91...), \tag{27}$$

with the separate terms interpreted in the same way as in Eq. (24). All three contributions in (27) are smaller than in Eq. (24), which means that separating the contribution with a simpler energy denominator has made it possible to sum the significant part of the excited-state contributions and, hence, that the correction (26), obtained via the sum rule (17) using the explicit expressions, provides a good preliminary estimate and indicates the scale and sign of the contribution.

The overall result for the correction associated with the dipole transition matrix element is

$$S=-1.5084..., \tag{28}$$

or

$$\delta\Gamma^{\text{H}}_{\text{dip}}=\Gamma^{\text{H}}_0\cdot\frac{16}{3\pi}\alpha(Z\alpha)^2\log\frac{1}{(Z\alpha)^2}(1.5084...). \tag{29}$$

The final expression for the decay width of the $2p_{1/2}$ level in the hydrogen atom that allows for the results of Refs. 7 and 23 [Eq. (21)] and the correction (29) found in the present work has the form

$$\Gamma^{\text{H}}=\Gamma^{\text{H}}_0\left\{1+(Z\alpha)^2\left[\log\frac{9}{8}-\frac{16}{3\pi}\alpha\log\frac{1}{(Z\alpha)^2}(0.491\,58...)\right]\right\}, \tag{30}$$

which corresponds to a lifetime of

$$\tau^{\text{H}}_{2p_{1/2}}=1.596\,1887(15)\cdot 10^{-9}\ \text{s}. \tag{31}$$

The logarithmic part of the decay-width radiative corrections of order $\alpha(Z\alpha)^2\log(Z\alpha)$ leads to a correction to the lifetime of order $-5.1\cdot 10^{-15}$ s, with the result that the constant contribution to the shift is estimated at $1.5\cdot 10^{-15}$ s.

Allowing for the experimental value[7] of the ratio of the width of the $2p_{1/2}$ level to the Lamb splitting of the $2s_{1/2}$ and $2p_{1/2}$ levels, we obtain the following result for the Lamb splitting of the levels in the hydrogen atom:

$$L^{\text{H}}(2s_{1/2}--2p_{1/2})=1057.8576(21)\ \text{MHz},$$

which agrees with the results of other precision measurements (Refs. 24–26) and with theoretical calculations (Refs. 16 and 27–29)[3)], when the root-mean-square charge radius is taken from Ref. 31.

3.2. Lifetime of the $2p_{1/2}$ level in the helium ion

The expression for the radiative width of $2p_{1/2}$ level in an ion with $Z\neq 1$ incorporates additional corrections to the recoil:[32]

$$\Gamma^{(Z)}_0=Z^4\Gamma_0\left[1+(Z-1)\frac{m_R}{M}\right]^2, \tag{32}$$

where M is the mass of the nucleus.

The physical meaning of the correction becomes obvious if we write the factors that allow for recoil in (20) and (32) in the form

$$\left(\frac{m_R}{m}\right)^3\left[1+2Z\frac{m}{M}+Z^2\left(\frac{m}{M}\right)^2\right] \tag{33}$$

and compare this with the coefficient of the Bethe logarithm $\log k_0(nl)$ in the expression for the Lamb shift.[33,34] Indeed, we can find the probability of photon emission without taking into account relativistic effects (i.e., specifying the leading order in the parameter $Z\alpha$) but allowing for nuclear emission (the imaginary part of the single-loop form-factor diagram for the nucleus) and the interference of the photon emissions by the nucleus and an electron (the imaginary part of the diagram representing the corrections for recoil). The power to which the reduced mass should be raised can easily be reconstructed from dimensionality considerations via the dipole formula (19) if we employ the fact that the correct nonrelativistic approximation for the wave function of an electron in an atom is the product of the free electron spinor and the Schrödinger wave function with the reduced mass, with the result that the dipole matrix element in (19) emerging in the Coulomb gauge because of the convolution of the large spinor components with the small ones is proportional to a reciprocal power of the electron mass rather than of the reduced mass.

Knowing the imaginary part, which is proportional to the decay probabilities, we can now reconstruct the real part. Here the Bethe logarithm emerges as an averaged logarithm of the difference in binding energies in units of $m(Z\alpha)^2$ with a weight function determined by the transition probability. Comparing the factor (33) with the corresponding quantity for the Lamb shift in Refs. 33 and 34, we immediately see that the coefficient of the Bethe logarithm is indeed proportional to (33). Note that the coefficient of the parametrically larger logarithm $\log(Z\alpha)$ in the Lamb shift does not coincide with (33) since actually there are two types of such logarithms. The first, the energy type, is closely related to the Bethe logarithm and leads to a coefficient with the factor (33); but, in addition to this, in calculating the interference contributions (corrections to recoil proportional to Z; see Ref. 33) there appears a logarithm that emerges in integration over momenta from the reciprocal Bohr radius to the electron mass.

The dependence on the nuclear charge and mass can also be reconstructed from the symmetry that exists between the light and heavy particles when one calculates the dipole matrix element, without calculating diagrams. Indeed, the two-particle nonrelativistic wave function is the product of the free spinors of the electron and the nucleus and the Schrödinger wave function with the reduced mass. The dipole matrix element in the Coulomb gauge in (19), which emerges only if we allow for small components, is proportional to the factor

$$\left(-\frac{e}{m}-\frac{Ze}{M}\right), \tag{34}$$

where the first minus is related to the negative electron charge, and the second to the fact that in the center-of-mass system the particles have opposite momenta. By squaring the factor (34) and allowing for the proportionality of the transition frequency in (20) to the reduced mass we arrive at (33). This reasoning is based on the fact that the nuclear spin is $\frac{1}{2}$. It is clear, however, that if we ignore hyperfine splitting, the corrections in (34) inversely proportional to the nuclear mass cannot depend on the magnitude of the spin.[4)]

Now let us return to our discussion of the corrections to the dipole formula. If we allow for the logarithmic correction (30) found above, the expression for the decay width assumes the form

$$\Gamma^{(Z)}_{2p_{1/2}}=\Gamma^{(Z)}_{0}\left[1+(Z\alpha)^2\left(\log\frac{9}{8}-\frac{16}{3\pi}\alpha\right.\right.$$
$$\left.\left.\times\log\frac{1}{(Z\alpha)^2}(0.49158...)\right)\right], \tag{35}$$

which corresponds to the following value for lifetime of the $2p_{1/2}$ level in the helium ion:

$$\tau^{\mathrm{He}^+}_{2p_{1/2}}=0.996\,927(4)\times 10^{-10}\ \mathrm{s}. \tag{36}$$

The slight change in the lifetime in comparison with the result $\tau=0.996\,891\times 10^{-10}$ s listed in Ref. 11 is also related to the elimination of the misprint in the coefficient of $(Z\alpha)^2$, whose correct value was found in Refs. 7 and 23 and was discussed in the previous section (see also the Appendix). The radiative correction (30), equal to

$$\delta\tau^{\mathrm{He}^+}_{2p_{1/2}}=0.000\,011(4)\times 10^{-10}\ \mathrm{s}$$

is still beyond the accuracy of the experiment described in Ref. 11:

$$\tau^{\mathrm{He}^+}_{2p_{1/2}}=0.99717(75)\times 10^{-10}\ \mathrm{s}.$$

4. CORRECTIONS TO THE ENERGY LEVELS OF THE HYDROGEN ATOM IN AN ELECTRIC FIELD

The precision experiments described in Refs. 1 and 2 studied the Stark effect in a weak electric field for the highly excited levels of the nonrelativistic hydrogen atom. The corrections connected with the level shift (the Lamb shift) have been thoroughly studied (see, e.g., Ref. 18). Here we are interested in the case of an electric field that is weak in the sense of a Schrödinger atom but strong in comparison to the fine structure and the Lamb shift; below we call such fields moderate. When the field strength becomes high compared to the Lamb shift, the corrections become quadratic in the small parameter. However, there are corrections to the dipole matrix elements linear in α. These corrections are always small, whereas the level shifts in fairly low fields constitutes the primary effect. The smallness of the corrections made them unimportant prior to the suggestions put forward in Refs. 1 and 2.

In moderate fields, where the linear Stark effect dominates and is much stronger than the fine-structure splitting, all corrections are small, i.e., corrections that allow for the variations in the wave functions and level shifts due to the fine structure,[17,35–37] the relativistic effects of the Dirac equation,[38] the ordinary transverse photon (hyperfine structure),[39] the recoil effects, and QED effects. Obviously, the corrections to the matrix elements of the Hamiltonian can be considered separately, and the level shifts (see, e.g., Ref. 18), the levels widths,[40] and the corrections to the dipole matrix elements correspond to different small parameters[5)] and can also be examined separately.

The matrix elements of the Hamiltonian in the case of a field directed parallel to the z axis are proportional to the z-component of the corresponding matrix element of the dipole moment. Since the correction is a scalar, its relative magnitude is as usual determined by Eqs. (12) and (16).

To allow for the logarithmic radiative corrections we need only examine the corrections to the Schrödinger matrix elements. As noted earlier, such contributions emerge when one allows for the radiative effects in the wave function of the s-states (Eq. (1)). Corrections of order $\alpha(Z\alpha)^2\log Z\alpha$ exist for spherical wave functions in the matrix elements between the s- and p-states. When we go over to the parabolic wave functions, which diagonalize the Hamiltonian of the interaction with a moderate electric field for states with a fixed n, it is sufficient to find the diagonal matrix elements of the perturbation, equal to

$$\delta d_z(n_1)=\langle n_1,n_2|n,l=0\rangle\delta d_{znn}\langle n,l=1|n_1,n_2\rangle$$
$$+\langle n_1,n_2|n,l=1\rangle\delta d_{znn}^{*}\langle n,l=0|n_1,n_2\rangle, \qquad (37)$$

where n_1 and n_2 are the parabolic quantum numbers,

$$n_1+n_2+|m|+1=n,$$

and in (37) we have explicitly allowed for the fact that the azimuthal quantum number m is equal to zero and discarded it in our notation.

The matrix elements corresponding to the transformation from spherical wave function to parabolic have been discussed in Refs. 36 and 41 (see also Refs. 20 and 42). An important problem in finding these elements is that of the phases of the wave functions. Below we first examine the problem of radiative corrections in the particular case $n=2$, where it is possible to avoid the question of the explicit choice of phases, and then return to the case of arbitrary values of n and find the corrections to the dipole matrix element (37) in the general case.

4.1. The Stark effect for levels with $n=2$

In this section we discuss the corrections for the case with $n=2$, in which the dipole matrix element between the s- and p-states proves to be the only matrix element that is nonzero in a Schrödinger atom. According to Eq. (16), the correction to this matrix element is

$$\frac{\delta d_{z22}}{d_{z22}}=-\frac{8\alpha}{3\pi}(Z\alpha)^2\log\frac{1}{(Z\alpha)^2}$$
$$\times\left[\sum_{n''\neq 2}\frac{1}{n''^3}\frac{1}{1/4-1/n''^2}\frac{\mathscr{D}_{n''2}}{\mathscr{D}_{22}}\right.$$
$$\left.+\int_0^\infty\frac{dt}{t(t^2/4+1)}\frac{1}{1-e^{-2\pi t}}\frac{\mathscr{D}_{t2}}{\mathscr{D}_{22}}\right], \qquad (38)$$

where the expressions for the ratios $\mathscr{D}_{qn}$ defined in (5) are specified in Eqs. (10) and (14).

Note that the correction (38) can be represented as a common factor in (37). The remaining part is simply the dipole matrix element in the parabolic wave functions, to which the Stark level shift is proportional to within a real factor. As a result the expression for the linear Stark effect is multiplied by the factor

$$1+\frac{\delta d_{z22}}{d_{z22}} \qquad (39)$$

or, numerically,

$$1+\frac{\alpha(Z\alpha)^2}{\pi}\log\frac{1}{(Z\alpha)^2}(0.1111...). \qquad (40)$$

This dramatic difference between the numerical coefficients (29) and (40) is of a general nature and can be understood by employing the standard sum rules for oscillator strengths. Usually the dipole matrix element diagonal in the principal quantum number n is much larger than the off-diagonal.[17,18] In calculating the level width the correction to an off-diagonal matrix element (in the case of (29), between the states $1s$ and $2p$) incorporates the coefficient diagonal in n (see Eq. (16)), with the result that numerically the coefficient proves to be large. In the Stark effect the opposite situation emerges: the correction to the diagonal matrix element incorporates only off-diagonal terms, as a result of which numerically the coefficient is small.

TABLE I. Relative phases in the definitions of the wave functions of the hydrogen atom.

Reference	$R_{nl}(r)$	$Y_{lm}(\vartheta,\varphi)$	$\phi_{n_1n_2m}(\xi,\eta,\varphi)$
Bethe [17]	1	$(-1)^m$	1
Bethe and Salpeter [18]	1	$(-1)^m$	1
Condon and Shortley [19]	1	1	1
Landau and Lifshitz [20]	1	i^l	1
Berestetskiĭ *et al.* [21]	1	i^l	1
Rojansky [36]	-1	$(-1)^{(m+\|m\|)/2}$	1
Biedenharn and Louck [42]	$(-1)^{n-l-1}$	1	$(-1)^{n_1+(m+\|m\|)/2}$

[a] *Note:* $R_{nl}(r)$ is the radial part of spherical functions, and $\phi_{n_1n_2m}(\xi,\eta,\varphi)$ are parabolic functions.

4.2. Phases of wave functions and the relation between the spherical and parabolic wave functions in the hydrogen atom

The matrix elements of the transformation from the spherical wave functions $\psi_{nlm}(r,\vartheta,\varphi)$ to the parabolic wave functions $\phi_{n_1n_2m}(\xi,\eta,\varphi)$ have been examined by Park[41] (see also the discussion of this problem in Ref. 20) from the viewpoint of the theory of angular momentum and the symmetry of the hydrogen atom. Both sources assert that the expansion coefficients in

$$\phi_{n_1n_2m}(\xi,\eta,\varphi)=\Sigma\langle nlm|n_1,n_2,m\rangle\psi_{nlm}(r,\vartheta,\varphi) \qquad (41)$$

coincide with the Clebsch–Gordan coefficients

$$\left(\frac{n-1}{2},\frac{n-1}{2},l,m\middle|\frac{n-1}{2},\frac{n-1}{2},\frac{m+n_1-n_2}{2},\frac{m-n_1+n_2}{2}\right). \qquad (42)$$

Here the problem of the phases of the wave functions becomes important, and, in particular, the problem of determining the spherical harmonics. With the usual definitions[19,20,43] these coefficients are chosen real. It is clear, however, that with an imaginary factor in $Y_{10}(\vartheta,\phi)$ adopted in Ref. 20 and real radial functions, the matrix element of the dipole operator between the s- and p-states is purely imaginary, too. But in parabolic coordinates all the matrix elements are diagonal and, obviously, real. Thus, in the definitions of Refs. 20 and 21 the matrix of the transformation between these two representations cannot be real, and this property is independent of the choice of phases for the parabolic wave functions and is related to the presence of the well-known factor i^l in the definition of the harmonics.

In the original paper of Park[41] all the wave functions and the Clebsch–Gordan coefficients are defined in accordance to Ref. 19, in which both the spherical harmonics $Y_{lm}^{CS}(\vartheta,\varphi)$ and the parabolic functions are products of real quantities and the standard factor $\exp\{im\varphi\}$. This property makes the definitions in Ref. 19 more convenient for the problem in question, and the definitions of the phases of the basic functions given in the papers and books cited below are listed in Table I.[6)]

Returning to Eq. (41), we easily see that in the definitions adopted in Ref. 19 and used in Ref. 41 the expression given by (41) cannot be correct. The point is that the spherical harmonics

$$Y^{CS}_{lm}(\vartheta,\varphi)=(-1)^{(m+|m|)/2}\sqrt{\frac{2l+1}{2}\frac{(l-|m|)!}{(l+|m|)!}}$$
$$\times P^{|m|}_{l}(\vartheta)\frac{e^{im\varphi}}{\sqrt{2\pi}} \tag{43}$$

have the property

$$(Y^{CS}_{lm}(\vartheta,\varphi))^{*}=(-1)^{m}Y^{CS}_{l,-m}(\vartheta,\varphi), \tag{44}$$

which has no analog for parabolic functions, while the Clebsch–Gordan coefficients do not change when the sign of m changes:

$$\left(\frac{n-1}{2},\frac{n-1}{2},l,m\middle|\frac{n-1}{2},\frac{n-1}{2},\frac{m+n_1-n_2}{2},\frac{m-n_1+n_2}{2}\right)$$
$$=\left(\frac{n-1}{2},=\frac{n-1}{2},l,-m\middle|\frac{n-1}{2},\right.$$
$$\left.\times\frac{n-1}{2},\frac{-m+n_1-n_2}{2},\frac{-m-n_1+n_2}{2}\right). \tag{45}$$

This can easily be verified by noting that changing the sign of m corresponds to changing the sign of all projections and interchanging the projections on the right–hand side of the notation for the coefficient or, in terms of $3j$–symbols, amounts to changing the sign of the projections and interchanging two neighboring columns.

The problem of the choice of wavefunction phases that yields expansion (41) will be considered shortly. Now we turn to explicit expressions.

The transition coefficients for states with arbitrary quantum numbers have been obtained explicitly by Rojansky,[36] who[7)] defined functions of radial and parabolic variables in the usual manner but instead of standard spherical harmonics used zonal, sectorial, and tesseral harmonics, i.e., functions in which the imaginary exponential function exp($im\varphi$) is replaced by cos$m\varphi$ and sin$m\varphi$. For the standard parabolic functions the exponential function can easily be constructed from sines and cosines, while the spherical functions are not so easily constructed, because one must ensure that the property (44) is satisfied, which leads to the phase factor $(-1)^{(m-|m|)/2}$. Another part of the factor emerges when we pass from associated Legendre polynomials to the spherical harmonics (43). In Table I the relative phase of the radial part of the wave function is given in accordance with the series expansion of spherical wave functions that leads to Rojansky's result in Ref. 36.

Clearly, the expression obtained in Ref. 36 through explicit calculations for the coefficients that link spherical and parabolic functions,

$$\langle nlm|n_1,n_2,m\rangle=l!\,\sqrt{2l+1}\,\sqrt{\frac{(n-l-1)!}{(n+l)!}}\,\sqrt{n_1!n_2!}\sum\frac{(-1)^{z+1}\sqrt{(|m|+n_1)!(|m|+n_2)!(l-|m|)!(l+|m|)!(n+l)!}}{(|m|+z)!\,(n_1-z)!\,z!\,(l-z)!(|m|+n_2-l+z)!\,(l-|m|-z)!},$$

where the sum is over all values of the index z for which the terms have meaning, coincides to within a phase factor with the known Racah representation[20,43,47] for the Clebsch–Gordan coefficients:

$$\langle nlm|n_1,n_2,m\rangle=(-1)^{n-l-n_1}\left(\frac{n-1}{2},\frac{n-1}{2},l,m\middle|\frac{n-1}{2},\frac{n-1}{2},\frac{m+n_1-n_2}{2},\frac{m-n_1+n_2}{2}\right).$$

Comparing the definitions of Ref. 36 with those of Ref. 19, we arrive at the following expression in terms of the definitions of Ref. 19 (see also the Appendix):

$$\langle nlm|n_1,n_2,m\rangle=(-1)^{n-l-n_1-1+(m+|m|)/2}\left(\frac{n-1}{2},\frac{n-1}{2},l,m\middle|\frac{n-1}{2},\frac{n-1}{2},\frac{m+n_1-n_2}{2},\frac{m-n_1+n_2}{2}\right). \tag{46}$$

Another approach to determining the coefficients is based on the symmetry of the hydrogen atom and the conservation of the Runge–Lenz–Pauli vector. The fact of symmetry was first used in the quantum case to find the energy of bound states,[48] and later in discussions of various problems. Biedenharn and Brussaard[49] used the components of angular momentum and the Runge–Lenz–Pauli vector to first build operators that change the quantum numbers l and m by unity for a fixed value of n. Then, using these operators, they constructed the spherical wave functions. The phases, which emerge in this approach in a natural way, differ from the standard phases of Ref. 19. The operator changing the parabolic quantum number n_1 by unity was built by Biedenharn and Louck.[42] This made it possible to carry out the program of consistently building the spherical and parabolic bases of eigenfunctions of the bound states of the hydrogen atom in such a way that the expansion (41) is valid if the coefficients are equal to the Clebsch–Gordan coefficients (42), as suggested by Park.[41]

The function with the greatest value of the azimuthal quantum number m for a given n was chosen. This state has the quantum numbers $l=m=n-1$ in the spherical base. On the other hand, for this value of m the Schrödinger hydrogen atom can have only one state, with the result that the respective wave function of the parabolic base has quantum numbers $n_1=n_2=0$ and $m=n-1$. If to this function we apply the operators that change the quantum numbers n_1, l, and m by unity, we can build consistent spherical and parabolic bases, and the matrix of transformation between the two will consist of the Clebsch–Gordan coefficients (42). The phases of the phase function that emerge in this approach differ from the standard phases and are listed in Table I. Note that the operator that changes the values of the orbital quantum number l incorporates the Runge–Lenz–Pauli vector and does not appear in the standard theory of angular momentum, which allows for the possibility of an arbitrary phase dependent on l. Comparing the definitions, we can also obtain the transformation coefficients linking the wave functions in the definitions of Ref. 19. The results coincide with (46).

Table I makes it possible to easily find the coefficients in the notation of Refs. 17 and 18,

$$\langle nlm|n_1,n_2,m\rangle=(-1)^{n-l-n_1-1}\left(\frac{n-1}{2},\frac{n-1}{2},l,m\right|\left.\frac{n-1}{2},\frac{n-1}{2},\frac{m+n_1-n_2}{2},\frac{m-n_1+n_2}{2}\right), \tag{47}$$

and of Refs. 20 and 21,

$$\langle n_1,n_2,m|nlm\rangle=(-i)^l(-1)^{n-l-n_1-1+(m+|m|)/2}\times\left(\frac{n-1}{2},\frac{n-1}{2},l,m\right|\left.\frac{n-1}{2},\frac{n-1}{2},\frac{m+n_1-n_2}{2},\frac{m-n_1+n_2}{2}\right). \tag{48}$$

The coefficients corresponding to the definitions of Refs. 17–19 satisfy the condition

$$\langle nlm|n_1,n_2,m\rangle=\langle nlm|n_1,n_2,m\rangle,$$

while for the wave functions of Refs. 20 and 21 we have

$$\langle n_1,n_2,m|nlm\rangle=(-1)^l\langle nlm|n_1,n_2,m\rangle.$$

The last relationship makes it possible to find the matrix elements of the inverse transformation from parabolic functions to spherical.

4.3. Radiative corrections to the Stark effect for levels with arbitrary values of *n*

The general expression for the correction in the case of an arbitrary value of n is given by Eq. (37). It includes the correction to the dipole matrix element, whose relative value (16) is independent of the definitions of spherical harmonics, while the normalization factor corresponding to the dipole matrix element d_{znn} can be found in Refs. 17–19 and 21 according to the choice of the phases in the harmonics. Note that one must correct the misprint in sign (see above). The coefficients of the transformation from the spherical functions to the parabolic have been defined in Eqs. (46)–(48). For the particular case of $l=0,1$ we give the values of the Clebsch–Gordan coefficients required by these equations:

$$\left(\frac{n-1}{2},\frac{n-1}{2},0,0\right|\left.\frac{n-1}{2},\frac{n-1}{2},\frac{n_1-n_2}{2},-\frac{n_1-n_2}{2}\right)=(-1)^{n_2}\frac{1}{\sqrt{n}},$$

and

$$\left(\frac{n-1}{2},\frac{n-1}{2},1,0\right|\left.\frac{n-1}{2},\frac{n-1}{2},\frac{n_1-n_2}{2},-\frac{n_1-n_2}{2}\right)=(-1)^{n_2}\frac{\sqrt{3}\,(n_1-n_2)}{\sqrt{n(n^2-1)}}.$$

These expressions make it possible to find the radiative correction to the diagonal matrix element in the parabolic-function base (or, what is the same, to the energy of levels in a moderate field) for any choice of phases in Refs. 17–21. With the definitions of Ref. 19 the matrix elements of the transformation between spherical and parabolic functions have the form

$$\begin{aligned}\langle n00|n_1,n_2,0\rangle&=\frac{1}{\sqrt{n}},\\ \langle n10|n_1,n_2,0\rangle&=-\frac{\sqrt{3}(n_1-n_2)}{\sqrt{n(n^2-1)}}.\end{aligned} \tag{49}$$

5. CONCLUSION

We have considered the radiative corrections of relative order $\alpha(Z\alpha)^2$ in the logarithmic approximation to the dipole matrix elements, which arise in the problems of emission and the Stark effect in the hydrogen atom. The exact expression for the radiative corrections generally depends on the details of the problem. The point is that the dipole matrix element usually appears as the low-energy limit of various quantities. For instance, in the Coulomb gauge the amplitude of emission of a photon with a polarization vector ε_j (cf. the Appendix) has the form

$$e\varepsilon\langle f|\gamma_i|i\rangle,$$

where the bra-vector corresponds to Dirac conjugation, and in the nonrelativistic limit becomes

$$e\varepsilon_j\omega_{if}\langle f|d_j|i\rangle.$$

In the Stark effect the energy of levels in a field F directed along the z axis is determined in the rediagonalization of the Hamiltonian $H^{lj}_{l'j'}(n,m)$ as

$$H^{lj}_{l'j'}(n,m)=eF\langle nljm|\gamma_0 z|nl'j'm\rangle.$$

Below is the full expression for the radiative corrections of order $\alpha(Z\alpha)^2$ to the matrix element, which occurs in emission problems:

$$\delta(\omega_{if}d_j(fi))=e\langle\delta^f|\gamma_j|i\rangle+e\langle f|\gamma_j|\delta^i\rangle$$
$$+e\langle f|\Lambda_j^C(E_f,E_i)|i\rangle,$$

where $\Lambda_j^C(E_f,E_i)$ is the single-loop vertex operator for emission of a real photon in the Coulomb field, and the corrections to the wave functions have the form

$$\langle\delta^{nljm}|=\sum_{q\neq nljm}\frac{\langle nljm|\Sigma_C^{(1)}(E_{nlj})|q\rangle}{E_{nlj}-E_q}\langle q|$$
$$+\frac{1}{2}\langle nljm|\frac{\partial\Sigma_C^{(1)}(E)}{\partial E}\bigg|_{E=E_{nlj}}|nljm\rangle\langle nljm|.$$

The term with the derivative appears in perturbation theory[50] if the perturbation is energy-dependent. The correction to the dipole matrix element emerging in the Stark effect has a similar form, with vertex contribution differing both in the Lorentz index and in the kinematic conditions $(E_f=E_i)$.

The calculations with logarithmic accuracy were based on the delta-like nature (in the Fried–Yennie gauge) of the coefficient of the logarithms and hence on the absence of logarithmic contributions with an energy derivative with a single-loop vertex and of corrections to the wave function of states with $l\neq 1$. The results can be generalized in an obvious manner to any perturbing potential that can be considered delta-like. For instance, by using this approach one can easily allow for single-loop vacuum polarization, the distribution of charge in a nucleus, etc. When the results are applied to Rydberg states (i.e., states with large values of the principal quantum number), in addition to explicit expressions for dipole transitions [Eq. (6)] the quasiclassical formulas of Ref. 51 may prove useful.

Among the possible applications considered above we have discussed the partial level widths and the Stark effect. The general expressions found in Sec. 2 also allow finding the line strengths in the presence of an electric field as well as in its absence.

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APPENDIX: THE RELATIONS BETWEEN SPHERICAL AND PARABOLIC FUNCTIONS

We follow Rojansky's paper[36] and use atomic units and the definitions of Ref. 19 or, what is the same, Ref. 20 with the factor i^l in the definition of spherical harmonics. Here we consider only positive values of m; the result for negative m can be obtained from the final formulas. The explicit representation of the spherical and parabolic wave functions of bound states in the hydrogen atom in the form of polynomials of the variables

$$\rho=\frac{r}{n},\quad \beta=\frac{1+\cos\vartheta}{2}$$

yields the following result:[8]

$$\psi_{nlm}=\frac{e^{-\rho}e^{im\varphi}}{\sqrt{\pi}\,n^2}\sin^m\vartheta\,(-1)^{m+1}\sqrt{(2l+1)\,\frac{(l-m)!}{(l+m)!}\,\frac{(n-l-1)!}{(n+l)!}}$$
$$\times\sum_{k_1,k_2}\frac{(-1)^{l+k_1-k_2+1}2^{l-m+k_1}(n+l)!\,(l+k_2)!}{(n-l-1-k_1)!\,(2l+1+k_1)!\,k_1!\,(l-k_2)!\,k_2!\,(k_2-m)!}\rho^{l+k_1}\beta^{k_2-m}, \tag{A1}$$

$$\phi_{nn_1m}=\frac{e^{-\rho}e^{im\varphi}}{\sqrt{\pi}\,n^2}\sin^m\vartheta\sqrt{\frac{n_1!\,n_2!}{(n_1+m)!\,(n_2+m)!}}$$
$$\times\sum_{p_1,p_2,p_3}\frac{(-1)^{p_1+p_2-p_3}2^{p_1+p_2}(n_1+m)!\,(n_2+m)!}{(n_1-p_1)!\,(m+p_1)!\,(n_2-p_2)!\,(m+p_2)!\,p_1!\,p_3!(p_2-p_3)!}\rho^{m+p_1+p_2}\beta^{p_1+p_3}, \tag{A2}$$

where the sums are over all values of the indices for which the terms have meaning. The double sum in Eq. (A1) has only one term of the form $(\rho\beta)^k\beta^{-m}$ at $k=l$ or, in terms of (A1), $k_1=0$ and $k_2=l$. Thus, the coefficient for the transformation (41) can be found by comparing the expansions (A1) and (A2) and taking into account the unique nature of the term $(\rho\beta)^l\beta^{-m}$ in the expansion (A1). As a result we have (46). This expression reproduces the original one in Ref. 36

(if i is substituted for p and p for n_1) but differs in the sign factor $(-1)^{m+1}$, which appears because of the different definitions of the wave functions in Ref. 19.

The result for negative values of m can be obtained by complex conjugation of the obtained expression9) with allowance for (44) and (45).

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1) Here the relativistic system of units, $\hbar = c = 1$, is employed.

2) The coefficient of log(9/8) also agrees with the calculation of the matrix element $\langle 1s_{1/2}|\gamma_i|2p_{1/2}\rangle$ in Ref. 11 in discussion of the experimental method. The author is grateful to G. W. F. Drake for a discussion of this problem.

3) According to a recent report,[30] the results of Ref. 28 are partially incorrect. The possible change in the entire theoretical expression will not exceed the error associated with the contributions yet to be calculated.

4) The author is grateful to K. Pachucki for a discussion of this problem.

5) In the case of the Lamb shift the expansion parameter is the ratio of this shift to the Stark splitting, but in moderate fields the first-order correction to the splitting of levels in the linear Stark effect is zero.

6) We discuss the notation adopted in Refs. 36 and 42 later when we discuss the results in the paper and the book.

7) The wave functions used by Rojansky[36] are defined according to Refs. 44–46, so that the generalized Laguerre polynomials [44,45] coincide with those adopted in Refs. 17–20, while the associated Legendre polynomials are introduced only for positive upper indices [44,46].

8) Note that in Ref. 36 expansions of spherical wave functions contain triple sums.

9) Note that in defining the parabolic functions in Ref. 19 the quantity $|m|$ enters all the expressions except the exponential function $e^{im\varphi}$.

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