The "frozen-in" concept in a collisionless plasma and in turbulent processes in tokomaks

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An attempt is made to understand the principles of turbulent transport in tokomaks, starting with the general principles of nonlinear physics. The key point is the conservation of the magnetic field topology. The freezing-in of the generalized vorticity (a special case of which is the magnetic field) is derived from the Poincaré relative integral invariant. The inertia of the electrons, in contrast to their thermal motion, does not destroy the freezing-in constants, but only changes their form. Heat and particle transport are caused by a group of particles with nonintegrable trajectories; a group of electrons with integrable trajectories plays the role of preserver of the magnetic-field topology and interferes with the appearance of turbulent resistance. It is shown that, although the basis for Onsager symmetry in turbulent processes is insufficient, simplified models can possess symmetry. But in the general case fluxes appear even in the absence of gradients of the thermodynamic quantities, which agrees with experimentally observed pinching of particles and heat. The asymmetric loss of ions results from the toroidal rotation of the plasma, which affects the pinching of particles and the bootstrap current when it interacts with the rippled magnetic field of the coils. The existence of a regime with electrostatic confinement of the ions is predicted.

1. INTRODUCTION

Following the experimental success of tokomaks at the end of the 1960s (Ref. 1) their study became a major thrust in plasma physics, giving rise to thousands of theoretical papers, which were often very detailed. Unfortunately, the physics of tokomaks has been broken up into a multitude of special cases, strung out one after the other. Since the system is highly nonlinear, there are no rules for "assembling" special solutions into a general system. It is therefore natural to attempt to understand the principles of the operation of tokomaks by starting, if possible, from the general principles of nonlinear physics. The presents effort belongs to this class.

Even the first work of Tamm and Sakharov[2–4] gave a very realistic estimate of the possibilities of tokomaks, aside from turbulent transport of heat, particles, and angular momentum. Turbulent transport remains the principal problem of tokomak physics and magnetic confinement generally,[5] and this is what we will concentrate on. We will disregard neoclassical effects, since the experimental values of the heat and particle transport in large tokomaks are far in excess of the neoclassical transport due to Coulomb collisions.[6,7]

In the opinion of the author, one of the most important questions is the apparent contradiction between the Spitzer Coulomb conductivity exhibited in experiments and the turbulent diffusion of the plasma, which is two orders of magnitude greater than the Coulomb value. The point is that in the simplest one-fluid hydrodynamics[8] magnetic field diffusion and plasma diffusion are determined by a single process, friction between electrons and ions, and this is a single coefficient; to use the Coulomb value in one equation but to replace it with an anomalous coefficient in the other, without explanation, is bad, even if this agrees with experiments. The correct resolution of the paradox by means of the introduction of an anomalous transverse conductivity while the normal conductivity is retained in the longitudinal direction is well known; the reason for keeping the normal longitudinal conductivity is explained in Sec. 3 in terms of the concept of a set of nonresonant particles, a "preserver" of magnetic field topology, while the resonant particles supply the transverse diffusion of the plasma.

This is preceded by Sec. 2, in which the reason for the existence of the concept of a frozen-in magnetic field is clarified. It is well known that energy conservation can be proved and the causes explained by using the translational invariants of the Hamiltonian in time, whereas it can also be proved "experimentally" by differentiating the energy with respect to time. This applies to many other invariants as well. The freezing-in of the magnetic field is derived "experimentally" in most textbooks (see, e.g., Ref. 8). As will be shown, the very important concept of freezing-in arises from the Poincaré relative integral invariant. The resulting generalizations are also useful for understanding the operation of tokomaks.

If we include the off-diagonal fluxes, such as, e.g., the auxiliary toroidal current proportional to the radial gradient of the electron temperature (the bootstrap current[6,7]) or the radial particle flux, which is proportional to the radial gradient of the electron temperature or to the toroidal electric field (pinching), then dozens of transport coefficients arise. The desire to reduce the number of independent off-diagonal coefficients by a factor of two by attributing Onsager symmetry to them is natural. The main limitations result from the incompressibility of the phase flux, which follows from the Hamiltonian property of the system. As will be shown in Sec. 4, symmetry can arise in the simplest situations, but in the general case it is lacking, and furthermore fluxes can exist even in the absence of gradients of the thermodynamic quantities. Turbulence is active and can create thermal fluxes the way a home refrigerator creates them. Turbulent equilibrium distributions differ from uniform Coulomb distributions.

While there are dozens of fluxes, inevitably several of these are the most important, and they are the only ones that need to be investigated. Section 5 is devoted to transport by vortices and their relationship with rotation and the bootstrap current. It contains no significantly new ideas and is supplied in order to complete the picture. The complexity of the interaction between the vortices and particles means that it is impossible to calculate either the shape of the vortices or the transport coefficients. Next a conjecture is described regarding a mechanism for the appearance of off-diagonal terms in a turbulent plasma. Asymmetric loss of ions gives rise to toroidal rotation of the plasma. The nonconservation of the toroidal momentum due to rippling of the coils interacting with the rotation leads to pinching of the particles and a new mechanism for the bootstrap current.

Section 6 is devoted to confinement regimes. The existence of a regime with electrostatic confinement of the ions is predicted; in it the ion temperature is nonzero at the plasma boundary. The equilibrium distribution of the ions suppresses some of the instabilities and thus improves confinement. The transition to this regime has a threshold, since it requires traversal of a Coulomb barrier: the cooling of ions by electrons at the boundary.

2. ORIGIN OF THE CONCEPT OF FREEZING-IN FROM THE POINCARÉ RELATIVE INTEGRAL INVARIANT

Let us elucidate the fundamental physical reasons for the frozen-in law.

In the differential formulation freezing-in means that the evolution of the magnetic field $\mathbf{B}$ is described by the equation

$$\mathbf{B}_t=[\nabla[\mathbf{vB}]], \tag{1}$$

where $\mathbf{v}$ is the plasma velocity. In the equivalent integral formulation the flux of the magnetic field $\mathbf{B}$ through any closed contour transported with the plasma velocity $\mathbf{v}$ is conserved.

Equation (1) can be derived by taking the curl of the hydrodynamic equation for the electron motion and setting the electron mass equal to zero. In this connection people frequently say that including finite mass destroys freezing-in and causes plasma to leak out of the confinement system or leads to reconnection. Furthermore, when finite mass is taken into account the equation of motion of each species of the plasma can be represented in the form of the conservation of a generalized vorticity Ω:

$$\Omega_t=[\nabla[\mathbf{v}\Omega]], \tag{2}$$

where $\Omega=[\nabla\mathbf{p}],\mathbf{p}=m\mathbf{v}+e\mathbf{A}/c$ is the generalized particle momentum. When the mechanical part dominates in the generalized momentum, we obtain the theorem of Kelvin on the circulation in an ideal fluid, which was well known a century ago; when the electromagnetic part dominates we obtain the freezing-in of the magnetic field. Conservation of generalized vorticity was also pointed out in the well-known review of Braginskii,[10] and in the western literature by Lynden-Bell;[11] it was found independently by Imshennik, and then rediscovered in connection with electron hydrodynamics.[12] In the West the work of Sudan[13] is best known.

However, the conservation of the generalized vorticity remains little known, which is responsible for many misunderstandings. For example, although the inclusion of a small electron inertia implies that the magnetic field deviates slightly from the frozen-in condition, these changes cannot accumulate even over a long period of time because of the conservation of generalized vorticity. When their inertia is taken into account the electrons are "glued" not to magnetic field lines but to lines of constant Ω. The constant of the motion has not disappeared, its definition has changed. Consequently, the numerous attempts to explain enhanced electron thermal conductivity or reconnection of magnetic field lines in terms of the electron inertia, without treating thermal drifts, are dubious. Conservation of the generalized vorticity is responsible for the potential behavior of the high-frequency pressure, also known as the ponderomotive force.

The new notation also helps to clarify the fundamental reason for the origin of the frozen-in concept. The conservation of generalized vorticity is not accidental; it arises from the canonical form of the Hamiltonian equation for a fluid particle, which was pointed out in Refs. 14 and 15. In fact, the hydrodynamic equations of motion when the pressure depends only on the density can be derived from the Hamiltonian

$$HP(\mathbf{q})+e\phi(\mathbf{q})+\frac{(\mathbf{p}-e\mathbf{A}/c)^2}{2m}, \tag{3}$$

where $P(\mathbf{q})$ is the normalized pressure and $\phi(\mathbf{q})$ and $\mathbf{A}(\mathbf{q})$ are the electrostatic and vector potentials. The equations have the canonical form

$$\dot{\mathbf{p}}(\mathbf{q})=-\delta H/\delta\mathbf{q};\quad \dot{\mathbf{q}}=\delta H/\delta\mathbf{p}. \tag{4}$$

Then from (4) the conservation of the Poincaré relative integral invariant[16] follows:

$$I=\oint \mathbf{p}d\mathbf{q}, \tag{5}$$

where the integration contour is transported by the phase flux. (This exact invariant should not be confused with the approximate adiabatic invariant which stems from the Poincaré invariant and has exactly the same form. The difference lies in the integration contour: in the case of the adiabatic invariant the integral is taken over a periodic particle trajectory. The periodic trajectory is only approximately the same as the contour transported by the phase flux.) Since in hydrodynamics the generalized momentum is a function of space and time, it is possible to carry out a projection from the six-dimensional phase space into ordinary three-dimensional configuration space. After that the contour integral can be transformed into the flux $[\nabla\mathbf{p}]$ through the surface spread across the contour, and this is the integral formulation for the freezing-in of the variable:

$$\Omega=[\nabla\mathbf{p}].$$

These descriptions require no calculations, but experience shows that they are difficult to understand, due to the transition from phase flux and ordinary differential equations to hydrodynamics and partial differential equations.

Note that this integral of the generalized vorticity is a consequence of the form of the Poisson brackets (4), and does not depend on the form of the Hamiltonian; these are called Casimir integrals. In particular, this means that the frozen-in condition is also conserved in the curved space near a black hole, when the magnetic field acquires mass. It is also possible to introduce freezing-in into a quark-gluon plasma describes by the Yang–Mills equations.[17]

This constant cannot be destroyed by any additions to the Hamiltonian; in our case, that of a collisionless plasma, the only chance is the treatment of particle thermal motion, i.e., kinetics. This analysis is presented in the next section. A brief version was sketched in Ref. 18.

We note also the possibility of using freezing-in directly in kinetics, without hydrodynamic averaging. The content of a conservation law is determined by its ability to restrict motion. The Poincaré invariant in 6+1-dimensional space is very arbitrary and imposes serious restrictions only together with additional assumptions, such as the transition to hydrodynamics. A different possibility results from drift kinetics. If the Larmor radius is small comparison with the characteristic length scale of the fields and if the frequencies are low, then the transverse adiabatic invariant is conserved:

$$\mu=\frac{v_\perp^2}{B}. \tag{6}$$

This immediately reduces the number of dimensions by two. If we now consider quasiperiodic motion of a particle between two magnetic mirrors and introduce conservation of the longitudinal adiabatic invariant

$$J=\oint v_\parallel dl, \tag{7}$$

then the centers of the banana orbits lie on hypersurfaces of dimension 2+1, which introduces important limitations. This is because a line divides two-dimensional space, so that the number of particles inside a contour transported by the particles is conserved. Thus, we have now gone over to the drift approximation; in the Poincaré invariant it is only necessary to retain the magnetic part of the momentum, and it assumes the extremely simple form of a Lagrangian invariant. The quantity

$$L=\frac{B}{n_{\mu,J}} \tag{8}$$

is conserved along trajectories.

This Lagrangian invariant is very convenient for analyzing the structure of the equations of turbulent transport (see Sec. 4). We should keep in mind that this invariant and both of the adiabatic invariants are derived from the Poincaré invariant.

3. THE FREEZING-IN OF THE MAGNETIC FIELD IN THE KINETIC DESCRIPTION OF COLLISIONLESS PLASMA TURBULENCE IN A TOKOMAK

In this section we consider the problem of the preservation and destruction of the freezing-in of a magnetic field in collisionless tokomak plasmas. The key concept of a "preserver" for magnetic field topology is introduced. In a tokomak the role of the preserver can be performed by untrapped electrons, those which are not in resonance with the turbulence and which therefore perform integrable motion. The longitudinal conductivity then diverges, while the transverse conductivity is finite. As a result the poloidal magnetic field is frozen-in to the toroidal field, while the particle and heat diffusion are determined by the turbulence, in agreement with experiment.

The concept of the freezing-in of the magnetic field is central to the idea of magnetic plasma confinement. The magnetic field itself plays the role of a skeleton to which the tokomak plasma is attached; the poloidal magnetic field stores a significant part of the energy, so it is natural to begin the study of the turbulent transport with its effect on the conductivity. An interesting property is the absence of turbulent resistivity in the experiments, although the plasma diffusion is turbulent. The solution is to use two conductivities instead of one: the classical longitudinal conductivity and an anomalous transverse conductivity.[9,19,21] A further complication (and one of the main subjects of this paper) is the customary use of the hydrodynamic concept of freezing-in far beyond its limits of applicability in ideal hydrodynamics, under conditions such that the Vlasov equation is appropriate.

In the differential formulation freezing-in means that the evolution of the magnetic field $\mathbf{B}$ is described by the equation

$$\mathbf{B}_t=[\nabla[\mathbf{vB}]], \tag{9}$$

where $\mathbf{v}$ is the plasma velocity. In the equivalent integral formulation the flux of the magnetic field $\mathbf{B}$ through an arbitrary closed contour transported with the plasma velocity $\mathbf{v}$ is conserved. Unfortunately, under actual conditions it is not clear how we should interpret the "plasma velocity $\mathbf{v}$," since the particles move with the individual velocity of the thermal drifts. Generally speaking, these are not small in comparison with $\mathbf{v}$, not to mention the large thermal velocity of the particles parallel to the magnetic field. If by $\mathbf{v}$ we mean the average velocity of particles of a single species, then terms appear in the pressure tensor which violate the freezing-in condition (9), while in the experiments the plasma behaves as if this does not occur. This question is analyzed in the present section. It can also be posed by asking, what effects destroy the freezing-in (9) in a collisionless plasma?

As was pointed out in the previous section, a Casimir integral cannot be destroyed by any terms added to the Hamiltonian; in our case the only possibility is to take into account the thermal motion of the particles, i.e., kinetics. Before treating kinetics realistically we introduce the important concept of a preserver for the freezing-in in the simplest example.[20] Assume that, e.g., the plasma electrons consist of cold ($T=0$) and hot components. Then it is easy to see that the generalized vorticity of the cold component is conserved, independent of the behavior of the hot particles. Thus, the cold component is a preserver for the topology of the generalized vorticity, and if we neglect the inertia of the cold electrons, a preserver for the topology of the magnetic field. This remains true, however, only until multistreaming first occurs, which in a cold flow occurs very readily. In a collisionless plasma the circulation about a contour is conserved even after multistreaming starts, but its sign changes because the contour gets turned inside-out. The general idea for the destruction of the freezing-in is as follows. In the absence of multistreaming the magnetic part of the vorticity changes into mechanical vorticity reversibly. But multistreaming introduces irreversibility and causes the freezing-in to be destroyed if there is no group of particles acting as a preserver within which multistreaming cannot occur. Cold storage of the freezing-in operates very briefly, and in actual fact is of little use. Above we have discussed multistreaming in hydrodynamics, but similar irreversibility also arises in kinetic multistreaming.

Let us try to apply the concept of a preserver to the tokomak plasma. Its property is the distribution of magnetic field lines in a system of nested toroidal surfaces which undergo three-dimensional perturbations owing to the turbulence. It is natural to expect that the turbulence does not give rise to mixing throughout the entire phase space, so that there exists a group of particles (most likely the fast untrapped electrons) which perform integrable motion and do not multistream; these are not in resonance with the perturbations, so this set acts as a preserver. Consequently, the magnetic field diffusion is much slower than the particle and heat diffusion, i.e.,

$$\mathbf{B}_t=[\nabla[\mathbf{uB}]], \tag{10}$$

where $\mathbf{u}$ is no longer the plasma velocity. We can say that the poloidal and toroidal magnetic field are frozen to one another, but not to the plasma. This explains the important experimental fact that the plasma conductivity in tokomaks is classical, whereas the other processes are anomalous. This can be given a transparent explanation: in the plasma the conductivities of different groups of particles add like resistances in parallel. If a particular group does not scatter off the turbulence, it is the one that makes the principal contribution to the conductivity. In the other transport coefficients the main contribution comes from just those particles which scatter most effectively; these coefficients are determined by the turbulence. We can replace the intuitive claims with another, closely related to them, but one which can be proved and is quite well known (see, e.g., Refs. 9, 19, and 21).

Perfect conductivity parallel to the magnetic field suffices for conservation of the magnetic field topology; the transverse conductivity can be arbitrary. There may be no direct connection between the transverse current and the transverse momentum flux (in a tokomak this is just what occurs).

In order to prove this, let us consider the equations of motion for electrons, neglecting their mass in the presence of a force $\mathbf{F}_\perp$ perpendicular to the magnetic field and otherwise arbitrary, plus an arbitrary potential force $\nabla\phi$:

$$\mathbf{E}+[\mathbf{vB}]=\mathbf{F}_\perp/e+\nabla\phi/e.$$

Perfect conductivity parallel to $\mathbf{B}$ implies that there is no rotational component parallel to $\mathbf{B}$. We introduce

$$\mathbf{u}=\mathbf{v}+[\mathbf{BF}_\perp]/ecB^2.$$

Then the equation of motion assumes the form

$$\mathbf{E}+[\mathbf{uB}]=\nabla\phi,$$

and after taking the curl of this expression we find the condition for conservation of topology:

$$\mathbf{B}_t=[\nabla[\mathbf{uB}]].$$

In those regions where $\mathbf{B}$ vanishes all components of the conductivity should become infinite.

A similar assertion for the plasma in toroidal systems is contained in Refs. 9, 19, and 21, and a description for the case of magnetic field generation by pressure and density gradients in Ref. 20.

As regards preservers, this has the following implications. It can be shown that the presence of fast untrapped electrons which are not resonant with the turbulence leads in an obvious way to an infinite conductivity and to conservation of the magnetic field topology. In other words, nonresonance and infinite conductivity are completely equivalent. In fact, thermal pressure forces act on the electrons, but they are either perpendicular to the magnetic field (like centrifugal forces) or derivable from a potential (like repulsion parallel to the field due to a change in the field strength). Consequently, the argument given above is necessary.

The following consideration is also useful. The Spitzer conductivities parallel and transverse to the field are quantities of the same order, since the Coulomb field of an ion is isotropic. Drift perturbations which give rise to turbulent transport are highly elongated parallel to the magnetic field, so that the transverse resistivity naturally dominates the longitudinal. A similar idea appears in Ref. 22.

Finally, the drift perturbations have a tendency to be localized in the transverse direction into vortices, also called islands. These contain all the transverse harmonics, so it is more difficult to ensure the absence of resonances in the transverse direction than in the longitudinal.

Some time ago it was noted[23] that many experimental facts can be explained by taking the electron–electron collision frequency to be anomalously large. In terms of results this is close to the picture presented above.

The reasoning about the existence of a preserver appears to be almost universal for tokomaks, so it is worthwhile to mention an example in which it fails. Sawtooth oscillations are related to the development of local current-layer singularities.[5,24,25] Large gradients can cause all the electrons passing through the region of the current layer to undergo nonadiabatic perturbations, so that the magnetic field ceases to be frozen-in in this region. Large-scale hydrodynamic instabilities, which make use of the magnetic field energy,[26,27] naturally give rise to current layers and destroy the frozen-in property.

Magnetic reconnection is a typical process in toroidal pinches with magnetic field reversal, so there may be no preserver in pinches. In stellarators the situation is similar to tokomaks.

In regions where there are no magnetic surfaces the longitudinal particle motion can have a continuous frequency spectrum even when there are no perturbations, so there are no nonresonant particles and there is no reason to expect a preserver to exist.

The meaning of a topology preserver is that the turbulent variations $\delta\mathbf{B}$ of the magnetic field all have the same frozen-in form:

$$\delta\mathbf{B}=[\nabla[\delta\mathbf{r}\mathbf{B}]].$$

The term "iso-frozen-in" is used in analogy with isohelicity, introduced by Arnol'd[16] for the Euler equation of an ideal fluid. This should not be confused with the well-known result[28] in which the destruction of magnetic surfaces for arbitrary (not uniformly frozen-in) small $\delta\mathbf{B}$ was considered.

It is essential to treat the fact that perturbations are frozen-in to an equal degree in models of electron thermal transport along a braided magnetic field, reviewed by, e.g., Isichenko.[29] The complete agreement with experiment regarding conservation of magnetic field topology implies that runaway electrons are confined better than the rest.

It can also be shown that Eq. (10) has little content because of the uncertainty in the velocity $\mathbf{u}$. This is untrue in the general case, but as regards tokomaks with their strong longitudinal field it simply means that $\mathbf{u}=0$ holds over times long compared to drift times but less than the classical skin time.

In this section we have shown that in order to preserve the magnetic field topology it is enough to have an infinite longitudinal conductivity (this result is not new), and we have presented arguments as to why a preserver for the frozen-in condition in a tokomak brings about this condition. We have disregarded the electron inertia and deviations of the electrons from resonance, and we have assumed that there are magnetic surfaces but no current layers.

4. THE SYMMETRY OF THE TURBULENT TRANSPORT COEFFICIENTS

In this section we will show that turbulence not only destroys Onsager symmetry but gives rise to fluxes in the absence of gradients in the thermodynamic quantities.

In the absence of turbulent transport, when the main contribution to transport comes from Coulomb collisions, the neoclassical coefficients[6,7] have Onsager symmetry.[10] That is, the fluxes of particle number, heat, charge, toroidal angular momentum, etc., are proportional to gradients of the thermodynamic variables:

$$q_i=a_{ik}\nabla\varphi_k, \tag{11}$$

and the coefficients a_{ik} are symmetric or antisymmetry.[30] The treatment of turbulence usually reduces to an additive correction to the transport matrix:

$$q_i=(a_{ik}+T_{ik})\nabla\varphi_k. \tag{12}$$

There exist dozens of papers such as Refs. 31–33 in which turbulent transport in tokomaks possesses symmetry, an dozens of papers in which it is shown that there is no symmetry, e.g., Refs. 34 and 35 (see also the discussion in Ref. 36). We will show that there may be symmetry for simplified turbulence models, but in the general case not only is symmetry destroyed, the matrix equation (12) itself is incorrect: turbulence gives rise to fluxes in the absence of gradients in the thermodynamic variables. This assertion is the main result of the present section.

The presence or absence of symmetry is not specific to turbulence in tokomaks. It is a general problem of turbulence, so we will begin our treatment by using a minimum of restrictions. The main restriction is the incompressibility of the phase flux which follows from the Hamiltonian property of the system (Liouville's theorem).We will first see how the one-dimensional transport equation originates. Suppose that on a two-dimensional plane incompressible interchanges take place, i.e.,

$$\frac{df(x,y)}{dt}=0,$$

and the mean distribution function f_0 depends only on x. Then for a flux q we have

$$q_x=\langle\delta f\delta x\rangle=\partial f_0/\partial x\langle\delta x^2\rangle,$$

whence

$$\frac{\partial f_0}{\partial t}=\frac{\partial q}{\partial x}=\frac{\partial}{\partial x}D_{xx}\frac{\partial f_0}{\partial x}.$$

This happens, e.g., with quasilinear diffusion due to waves.[8] It is useful to note that the quasilinear diffusion coefficient is uniquely determined from the energy conservation condition,

$$\gamma W=D_{xx}\frac{mv^2}{2}\frac{\partial f}{\partial v}.$$

Here γ is the Landau damping rate and W is the wave energy density.

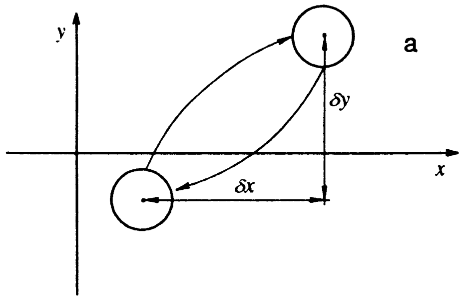
If the diffusion is substantially two-dimensional and the turbulence is anisotropic but not gyrotropic, there exist fundamental interchanges of the form shown in Fig. 1a,from which it is possible to construct a general incompressible interchange. For this we have

$$q_x=\langle\delta f\delta x\rangle=\frac{\partial f_0}{\partial x}\langle\delta x^2\rangle+\frac{\partial f_0}{\partial y}\langle\delta x\delta y\rangle,$$

$$q_y=\langle\delta f\delta y\rangle=\frac{\partial f_0}{\partial x}\langle\delta x\delta y\rangle+\frac{\partial f_0}{\partial y}\langle\delta y^2\rangle.$$

We see that in this special case the transport matrix is symmetric and can be diagonalized by a rotation.

For the special case of two-dimensional gyrotropic but isotropic turbulence the fundamental interchange looks like a local rotation through a small angle. This transformation is shown in Fig. 1b. Estimates similar to those above show that the transport matrix is antisymmetric in this case. In the more general case the sum of a symmetric and antisymmetric matrix has no symmetry. The real situation, however, can be even worse. The reason is that in the multidimensional case



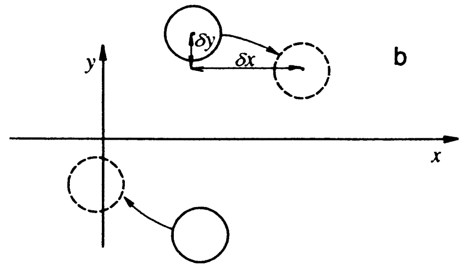


FIG. 1. a—Incompressible interchange typical of anisotropic but not gyrotropic turbulence. b—Incompressible interchange typical of isotropic and gyrotropic turbulence.

(and the phase space of the Vlasov equation is six-dimensional) two-dimensional transformations can be compressible due to balancing of the compressions in different directions. Consequently, there can be fluxes even when the thermodynamic variables have zero gradients. For real tokomak conditions this is hard to show, so we consider four successively more complicated examples.

The simplest example is a home refrigerator, which produces a heat flux in the absence of an initial temperature gradient. A refrigerator is active, as is turbulence, so such fluxes are not forbidden.

The example of a refrigerator may seem artificial, so we consider adiabatic mixing in an initially isothermal atmosphere in thermodynamic equilibrium. Upwelling volumes of air expand and hence cool, while descending volumes contract and heat up, so that a heat flux develops which is directed downward. Of course, an isothermal atmosphere is conductively stable and there has to be an external source of turbulence, e.g., a convective wind. The equation for the heat flux can easily be derived from the condition that is vanish in an isentropic atmosphere $s=\text{const}$:

$$q=D\,\frac{\partial s}{\partial z}.$$

The coefficient D is determined by the turbulence. Of course, we make no pretense that this process is important in the atmosphere, but it is completely transparent and physical. In tokomaks the analogous process would be called a thermal pinch.

A similar example is easily constructed for the interchange instability in a Z pinch.[37] We restrict ourselves to the simplest case, in which the interchanges take place in the corona of the pinch, where the temperature, current, and the electric field vanish, the density is constant, and the magnetic field strength falls off inversely with the radius. Assuming that the conductivity is ideal and using the frozen-in condition for axisymmetric interchanges, we see that a particle flux toward the axis (pinch) develops, where the flux has the form

$$q_n=D\,\frac{\partial(nr^2)}{\partial r}.$$

Here the factor r^2 appears because the magnetic field is proportional to the radius and the length of a field line is also.

Finally, the interchange instability can occur in a tokomak near rational surfaces, e.g., in the region where the stability safety factor is $q=1$. The specific volume per unit magnetic flux in a tokomak is a weaker function of the minor radius than in pinches, so the pinching that develops because of this mechanism is of less interest.

These four examples have been given in order to illustrate the fundamental assertion that fluxes are possible in the absence of gradients of the thermodynamic variables, not to mention the destruction of the Onsager symmetry. A treatment has been carried out in as much detail as could be desired of the transition layer of a magnetoelectrostatic confinement system maintained at marginal stability.[38] In that work what may have been the first nonuniform turbulent equilibrium distribution ever in plasma physics was constructed.

Thus, in the general case turbulent processes in tokomaks for a half-dozen quantities (the density, electron and ion heat flux, toroidal rotation velocity, poloidal magnetic field, radial electric field, and so on) are determined by the full matrix plus terms in the absence of gradients, i.e., several dozen independent coefficients, which cannot realistically be calculated. Fortunately, in this situation there is no need to calculate all the coefficients. It suffices to include only the dominant ones. The following sections are devoted to an attempt to distinguish the dominant turbulent fluxes.

5. PINCHING AND THE BOOTSTRAP CURRENT

Following the clarification in Sec. 3 of the reasons for the absence of a turbulent electrical resistivity, we now discuss turbulent diffusion of electrons and ions. Note that heat and particle diffusion occur essentially in all turbulence models, and so are not very useful in choosing the correct theory.

Less trivial are the experimentally observed absence of turbulent resistivity and particle pinching, and also the possible effect of turbulence on the bootstrap current. Pinching and the bootstrap current are related to the balance of toroidal momentum, so it is natural to consider them together. Pinching and the bootstrap current are the two main off-diagonal processes in a tokomak.

Thus far we have avoided selecting a specific turbulence model. When nonlinearity is taken into account, drift instabilities give rise to highly elongated (parallel to the magnetic field) quasi-two-dimensional vortices, which can be regarded as the structural elements of the turbulence. The majority of theoretical treatments focus on these vortices (see the reviews in Refs. 39 and 40), so we give a short summary of the results.

1. In the hydrodynamic description the family of stable two-dimensional vortices has an infinite number of parameters, since the number of frozen-in constants of motion is infinite.[40]

2. A large number of one- and two-parameter exact analytical solutions have been found whose three-dimensional stability has not been determined, and for which even the two-dimensional stability is controversial.[41]

3. Vortex theory is based on a hydrodynamic understanding of the freezing-in phenomenon. The transition to kinetics causes instability of previously stable vortices even in the presence of a topology preserver. Furthermore, the concept of a preserver introduced in the present work is inapplicable to vortices, since in the derivation we have neglected particle inertia.

4. The principles of nonlinear physics require that only those vortices which are attractors be chosen and included in the theory. For a tokomak this treatment should be based on the Vlasov equation. No such attempt has even been made, so there is no point in talking about serious (nonmodel) calculation of turbulent processes in vortices. Moreover, to the best of our knowledge no attractor in an unstable medium described by Hamilton's equations has even been analyzed.

In an unstable medium stable localized solutions cannot exist, but the region where entropy is produced may be localized. An example is wind waves on water. The instability grows at a rate proportional to a small parameter, the ratio of the air density to that of the water, but the waves grow to an amplitude of order unity since the damping is turned on only when combers break as a result of steepening above the Stokes limit. Combers are completely local; they occur precisely where potential motion is transformed into vortical and dissipation occurs. In a plasma the situation may be similar, except for one complicating factor: the breaking takes place in the kinetics. The small nonuniformity of the plasma implies that the fraction of phase space in which electrons and ions undergo nonintegrable motion is small. We have already introduced a group of particles with integrable motion in discussing the concept of a topology preserver.

It is well known experimentally that the scale of transverse turbulent drift pulsations amounts to several ion Larmor radii. This scale appears natural, since it allows the freezing-in of the gneralized ion momentum to be conserved. This also permits the resonance of the longitudinal ion motion with the transverse drift motion. Starting from the ideas presented above, we will assume that the pulsation amplitude is of order unity, i.e., the transverse displacement of the particles (and the magnetic field lines) is on the order of the transverse wavelength, and the perturbations are on the verge of changing of waves into vortices.[40] The passage through this boundary gives rise to damping and leads to nonlinear competition among the perturbations. In one-fluid hydrodynamics the vortices travel in both directions along the magnetic field with the same velocity. When particle kinetics is taken into account the two directions are no longer equivalent, and the result of the nonlinear competition may be that vortices survive in only one direction. This has important macroscopic consequences. For example, a resonant group of ions will be asymmetric with respect to the direction of the magnetic field, and as it propagates across magnetic surfaces it will transport longitudinal momentum and produce strong untwisting of the plasma. The efficiency of this thermal engine contains no apparent small parameters and can be large. It is natural to expect that it is just this rotation which is responsible for off-diagonal processes.

For this reason it is very important to determine experimentally whether there is asymmetry in the direction of vortex propagation. Unfortunately, the probes are oriented only in the transverse direction in most experiments.

The simplest consequence of the toroidal rotation—the interaction between the particles and the rippled field of the coils—changes the toroidal momentum of the particles. The change in the magnetic part of the momentum is familiar as pinching; the change in the mechanical part of the momentum gives rise to a new mechanism, the bootstrap current. A virtue of this mechanism is the possibility of regulating it by means of the corrugation of the field. Since the turbulence asymmetrically deforms the electron distribution function also, the classical bootstrap current induced by Coulomb collisions can also change.[6,7]

Thus, we do not attempt to calculate the turbulent transport coefficients, but point out their interconnectedness, which may be used in extracting the coefficients from the experimental data.[42] Note that the possibility of a bootstrap current itself is due to the absence of turbulent resistivity in the presence of the preserver. For this region it may turn out to be neoclassical,[6,7] but it should be considered even with the possible effective turbulence.

6. CONFINEMENT REGIMES

If we assume that the ion diffusion is much larger than the electron diffusion, which agrees in part with experiment and with theoretical considerations, and also neglect the electron–ion Coulomb heat exchange, then the possibility of purely electrostatic ion confinement arises naturally. This is the Boltzmann ion regime. The electrons are confined by the magnetic field. A characteristic feature of this regime is that the ion temperature is almost constant all the way to the boundary, and instabilities caused by the deviation of the ion distribution function from equilibrium are suppressed. In the coordinate system comoving with the ions the electric field is determined by the density gradient:

$$n \propto \exp(-e\phi/T_i).$$

There is no obstacle to having a high ion temperature all the way to the magnetic field separatrix and even beyond it; a high electron temperature beyond the separatrix is impossible because of the large thermal conductivity along the magnetic field. In order to reach the regime in which ions are electrostatically confined we must have an electron temperature which is still quite high so as to avoid Coulomb heat exchange between the ions and the electrons. A rough estimate yields

$$n\tau_i < T_e^{3/2} 10^{12}.$$

Here the electron temperature is in keV and the density is given in particle number per cubic centimeter. This condi-

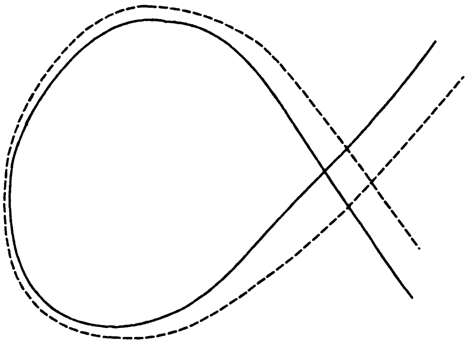


FIG. 2. Schematic representation of the splitting of the separatrices for ion and electron generalized vorticity.

tion is hard to satisfy in the separatrix region. We recall that the typical size of the drift turbulence pumped by the ions is several ion Larmor radii, so in a region of smaller extent failure of the ion Boltzmann regime to apply is no obstacle to suppressing the turbulence.

As regards the high ion temperature all the way to the boundary and the suppression of noise, this regime is similar to the well-known H-mode.[43] And if it is not the same, then—even better—it is possible to improve on it. It differs from the H-mode in not having any obvious connection to plasma rotation. The connection with rotation was studied in Ref. 44. Of course, we can assume that in experiment a strong electric field is required for rotation, produced by the density gradient and completely balanced by ion pressure since it is easier to measure the field than the ion velocity. We discard this possibility and assume that there is rotation and it affects confinement. The simplest description is derived from ideal two-fluid hydrodynamics. We assume that the generalized vorticity

$$\Omega=[\nabla\mathbf{p}],$$

is frozen-in to both the electrons and the ions, where $\mathbf{p}=m\mathbf{v}+e\mathbf{A}/c$ is the generalized particle momentum. Consider steady plasma rotation. Then the lines of the generalized vorticity lie on surfaces that are close to magnetic surfaces. Of course, the electron inertia can be ignored. The ion inertia also has only a small effect on the vector field Ω. This change, however, cannot be ignored near the separatrix. In the model particles can blow out only along the separatrices, and the rotation splits the electron and ion separatrices. But quasineutrality keeps the electrons and ions from moving along different trajectories, even close ones. This is shown schematically in Fig. 2.

Thus, plasma rotation suppresses the fluxes, since at the x point of the separatrix the particles are compelled to move across the surface of generalized vorticity, which is completely analogous to movement across magnetic surfaces. Unfortunately, it is difficult to calculate the diffusive resistivity of the x point as a function of the rotational velocity; in fact, a kinetic description of the particles is required.

In principle it is also possible to have a regime in which the plasma is pulled away from the separatrix. In this regime the electron temperature can also be high on the plasma surface. This is encountered in Z pinches.[45] This regime is far removed from tokomak experiments, but perhaps the plasma connections can be blown away by the transverse magnetic surfaces of the current from auxiliary electrodes, and this will create something like a vacuum gap which will promote the transition to the H-mode.[43]

7. CONCLUSION

The main results of this work stem from dividing all particles into two groups, those with integrable and those with nonintegrable behavior, rather than the usual division into trapped and untrapped.

Neglecting the nonintegrable particles and using ideal hydrodynamics, we have derived the freezing-in of the generalized vorticity from the Poincaré relative integral invariant.

Including both the integral and nonintegral particles enables us to introduce an important concept: a preserver for the magnetic field topology. It is important to emphasize that the usual relationship between particle diffusion and magnetic flux diffusion is disrupted. The particles acquire the ability to move across magnetic surfaces, despite the ideal conductivity. The Coulomb conductivity of the plasma is found to be combined with turbulent diffusion of particles, which is in complete accord with experiment.

We have shows that the turbulent transport does not simply fail to have Onsager symmetry (in special cases the symmetry can occur). More important is the appearance of fluxes in the absence of gradients of the thermodynamic variables; this is the second important result. The simple example of particle pinching is presented for interchange turbulence in the atmosphere and Z pinches. A mechanism closer to experiments was discovered in Ref. 46. We have described a mechanism by which the plasma rotation produced by turbulence affects pinching and the bootstrap current.

We have described a regime with electrostatic confinement of the ions, which shares certain features with the H-modes. Note that the plasma rotation causes the separatrices of the electron and ion generalized vorticity to be distinct, which in turn causes the diffusive resistance near the x point of a separatrix to increase.

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