Stationary solutions in the Rayleigh–Taylor instability for spatially periodic flow

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Stationary solutions in the Rayleigh–Taylor instability for spatially periodic flow with C_4 symmetry (square lattice of bubbles and jets, lattice period$=2\pi/k$, k is the wave number) are investigated analytically. A one-parameter family of solutions S is found. The solutions are shown to converge on S as a function of the order of the approximation, and it is proved that the family S exists and is unique. We parametrize the solutions with the radius of curvature R at the stagnation point on a bubble. The ascent velocity $v(R)$ of a bubble is found as a function of the parameter. Each point of S corresponds to an exact solution. The family has a termination point with $kR_{\rm cr}\approx 3.0$ and $v_{\rm cr}(R_{\rm cr}) \approx 0.99\sqrt{g/k}$, where g is the acceleration of gravity. A solitary jet is another terminal point of S and is localized in the limit $kR\rightarrow\infty$ with $v(R)\rightarrow 4\sqrt{g/k}/\sqrt{kR}$. A comparison is made to the case of a stationary two-dimensional flow. © *1995 American Institute of Physics.*

1. INTRODUCTION

The instability of the boundary between a heavy top and light bottom liquid is known as a Rayleigh–Taylor instability. This instability arises in a compressible medium when the pressure gradient is oriented opposite to the density gradient. In recent experiments[1] and in the earliest investigation[2] it was shown that when the density of the bottom liquid is zero and the top liquid is incompressible, the "linear" stage of the Rayleigh–Taylor instability with growth rate $\tau^{-1} = \sqrt{2\pi g/\lambda}$, where g is the acceleration of gravity and λ is the wavelength of the perturbation, lasts for only a short time and nonlinear effects come into play even with perturbation amplitudes (of the bubble) of the order of 0.3λ. A periodic system of "bubbles" and "jets" forms, the bubble velocity approaches a constant value asymptotically, and steady motion is established (Fig. 1).

Finding a stationary solution in the theory of Rayleigh–Taylor instability is a classic problem. It was initially assumed that this solution is unique, i.e., it corresponds to a point in the state space. In 1955 Layzer,[3] proceeding from this assumption, investigated the stationary solution for two-dimensional flow. But in 1957 Garabedian and later Birkhoff[4] hypothesized that this problem has a one-parameter family of solutions. However, the first quantitative results were obtained only very recently.[5] The approach developed in Ref. 5 made it possible to prove convincingly, on the basis of simple considerations about the spatial periodicity of a two-dimensional flow of bubbles and jets, the existence of a one-parameter family F in the solution of the problem of stationary flow in Rayleigh–Taylor instability.

2. STATIONARY SOLUTIONS. UNIQUENESS PROBLEM

We present some qualitative arguments for the existence of a one-parameter family of stationary solutions. Consider two-dimensional (x,z) undetached flow of an incompressible liquid over some profile.[6] The velocity of the liquid at infinity ($z\rightarrow+\infty$) is fixed and equals $-v^*$, and the width of the flow is λ. The flow profile describes both a bubble and a jet, and it is a smooth function satisfying the following conditions (Fig. 1). First, the flow has a stagnation point (top of the bubble) and, second, the width of the jet at infinite depth $z\rightarrow-\infty$ must approach zero. Consider motion in a gravitational field with a linear pressure distribution $p=-\rho g z$, where ρ is the density of the liquid. Since the velocity of the liquid (flow potential) and width λ of the flow are fixed in the limit $z\rightarrow+\infty$, the acceleration of gravity g, whose dimensions are $[(v^*)^2/\lambda]$, is a free parameter. Therefore, in accordance with the similarity laws, our flow is associated with a family of pressure distributions corresponding to different values of g, i.e., it is a one-parameter family. It is clear from physical considerations that this family must have two terminal points. The first ($g\rightarrow\infty$) corresponds to a solitary jet (the radius of curvature of the bubble is infinite), and the second g^* limits the values of g from below, distinguishing from the entire range of possible values of the parameter only those values which describe a smooth profile with a physical, nonzero radius of curvature of the bubble, i.e., $g\in[g^*,+\infty)$. Generally speaking, since our problem contains three independent dimensional parameters g, λ, and v^*, the "inverse" problem of flow around an object with g and λ fixed can be studied, and the velocity v^* (with dimensions $\sqrt{g\lambda}$) or Froude's number $v^*/\sqrt{g\lambda}$ can be varied.

In Ref. 5 it is shown, in an analysis of the two-dimensional stationary flow in very high orders of nonlinearity, that the dimension of the stationary solution is indeed $F=1D$ and not $F=0D$, i.e., in some space of states F is a curve (a function of one parameter) and not an isolated point.

All preceding investigations were limited, however, to analysis of only two-dimensional flow, since great difficulties arise even in numerical modeling of the three-dimensional problem.[7] We note that the existence and uniqueness theorem for the boundary-value problem for Laplace's equation holds for two- and three-dimensional cases, and the arguments presented above about the dimension of the stationary solution can also be extended to three-dimensional flow. Here, the symmetry of the flow itself is found to be important; it must ensure that a unique characteristic length scale is obtained in

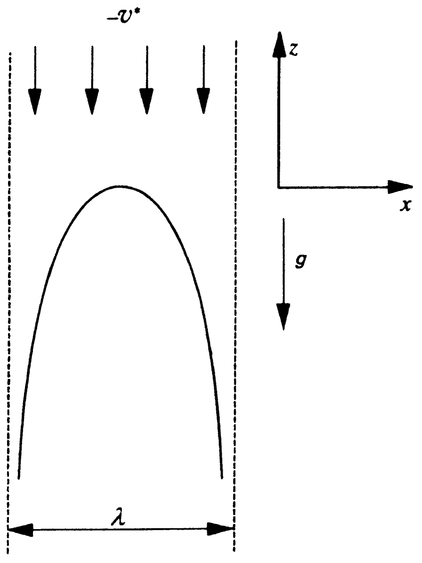

FIG. 1. Stationary flow—asymptotic stage of the development of the Rayleigh–Taylor instability.

the problem of flow around an object. It is easy to show that for this to occur, the order of the flow symmetry axis, directed along the z axis, must be greater than 3: $n=4.6$, i.e., the solutions of the problem of a stationary state in Rayleigh–Taylor instability in three-dimensional space must form a one-parameter family for the case in which the flow is invariant with respect to a possible symmetry axis oriented along the z axis: 4,6 (we consider only "simple" planar unit cells).

In the present paper we investigate analytically the stationary solution for a three-dimensional, spatially periodic flow. The objective of this investigation is to find a one-parameter family of solutions in the three-dimensional case (x,y,z) and to prove that this family is unique.

We consider stationary flow in a space with translational symmetry in the (x,y) plane $\{x\to x+\lambda m,\ y\to y+\lambda n\}$ and the 4 symmetry axis directed along the z axis $\{x\to -x,\ y\to -y,\ x\to y\}$. The square lattice in the (x,y) plane forms a spatially periodic flow of bubbles and jets (Fig. 2). We now transform to a coordinate system moving together with a bubble at constant velocity v. The potential $\Phi(x,y,z)$ of this flow is determined by Laplace's equation, periodic conditions, and the boundary condition at infinity,

$$\Delta\Phi=0,\quad \Phi(x+\lambda m,y+\lambda n,z)=\Phi(x,y,z),$$

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=+\infty}=-v,$$

and two conditions at the free boundary of the liquid $z=z^*(x,y)$:

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=z^*}-\left.\left(\frac{\partial z^*}{\partial x}\frac{\partial\Phi}{\partial x}+\frac{\partial z^*}{\partial y}\frac{\partial\Phi}{\partial y}\right)\right|_{z=z^*}=0,$$

$$\left.\frac{1}{2}(\nabla\Phi)^2+gz\right|_{z=z^*}=0.$$

According to the first (kinematic) condition, there is no fluid flow through the free boundary. The second (dynamic) boundary condition (Bernoulli's equation) expresses the fact that the pressure is constant on the free surface. In the dimensionless coordinates $\{kx\to x,\ ky\to y,\ kz\to z,\ \sqrt{kg}t\to t,\ v/\sqrt{g/k}\to v\}$, where $k=2\pi/\lambda$ is the wave vector, the Fourier expansion of the potential has the form

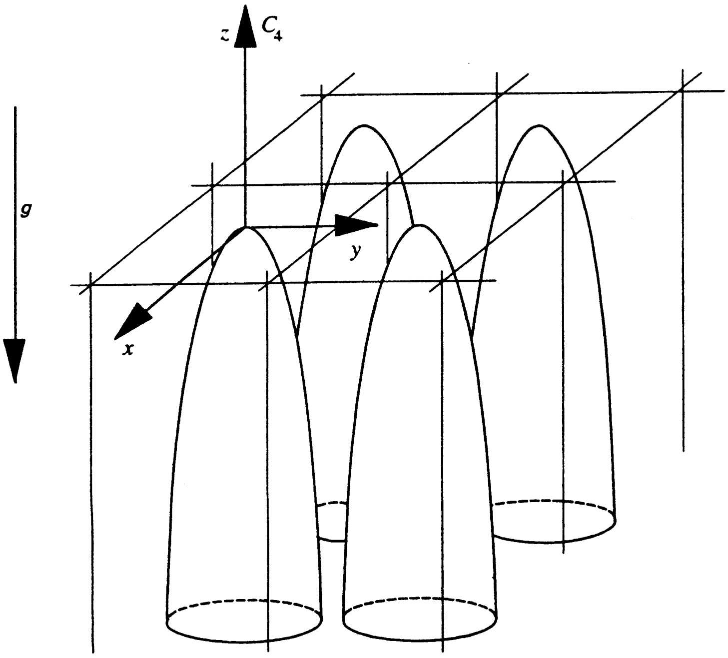

FIG. 2. Spatially periodic flow of bubbles and jets. The stationary flow is invariant with respect to translations in the xy plane and rotations relative to the C_4 axis, which is directed along the z axis.

$$\Phi=\sum_{mn}\Phi_{mn}\left(\cos mx\cos ny\,\frac{\exp(-\sqrt{m^2+n^2}z)}{\sqrt{m^2+n^2}}+z\right). \tag{2.1}$$

The matrix of unknown amplitudes $\{\hat{\Phi}\}$ is symmetric: $\Phi_{mn}=\Phi_{nm}$; without loss of generality, we set $\Phi_{00}=0$. In the laboratory coordinate system the velocity of the bubble is $v=-\Sigma_{m,n}\Phi_{mn}$. In a moving coordinate system, however, the top of the bubble (0,0,0) is a stagnation point (Fig. 2). Expanding the potential in powers of x, y, and z near the stagnation point, we substitute this potential into the boundary conditions and obtain in this manner a system of equations for the unknown Fourier amplitudes. We introduce together with the amplitudes the moments $\{M\}$ as linear functions of $\{\Phi_{mn}\}$:

$$M(\alpha,\beta,\gamma)=\sum_{m,n}\Phi_{mn}m^\alpha n^\beta(\sqrt{m^2+n^2})^\gamma, \tag{2.2}$$

then

$$M(0,0,0)=\sum_{m,n}\Phi_{mn}=-v. \tag{2.2a}$$

Not all moments determined in this manner are linearly independent. As a result of the symmetry,

$$M(\alpha,\beta,\gamma)=M(\beta,\alpha,\gamma),\quad M(\alpha+2,\beta,\gamma-1)+M(\alpha,\beta+2,\gamma-1)=M(\alpha,\beta,\gamma+1). \tag{2.3}$$

Let $z_d=\Sigma_{i,j}\gamma_{ij}x^{2i}y^{2j}$ be the free surface corresponding to the dynamic condition, and let $z_k=\Sigma_{i,j}\beta_{ij}x^{2i}y^{2j}$ correspond to the kinematic condition (on account of the symmetry, $\gamma_{ij}=\gamma_{ji}$, $\beta_{ij}=\beta_{ji}$). Then, the expansions of the boundary conditions near the stagnation point have the form

$$\sum_{N=1}^{\infty}\sum_{s+p+q=N}\overline{(spq)}x^{2s}y^{2p}z_d^q=0,$$

$$\sum_{n=1}^{\infty}\sum_{s+p+q=N}(spq)x^{2s}y^{2p}z_k^q=0. \tag{2.4}$$

The functions corresponding to the dynamic boundary condition $\overline{(spq)}$ are, generally speaking, quadratic in the moments of the functions. The relations for them have the form

$$\overline{(000)}=0,\quad \overline{(001)}=2,$$

$$\begin{aligned}\overline{(spq)}=&\frac{(-1)^{s+p+q}}{(2s)!(2p)!q!}\sum_{i=0}^{s}\sum_{j=0}^{p}\sum_{r=0}^{q}C_q^r(C_{2s}^{2i}C_{2p}^{2j}\\&\times M(2i,2j,r)M(2s-2i,2p-2j,q-r)\\&-C_{2s}^{2i+1}C_{2p}^{2j}\\&\times M(2i+2,2j,r-1)M(2s-2i,2p-2j,q-r\\&-1)C_{2s}^{2i}C_{2p}^{2j+1}M(2i,2j+2,r-1)M(2s-2i,2p\\&-2j,q-r-1)-2M(2s,2p,q)M(0,0,0),\end{aligned} \tag{2.5}$$

$\overline{(spq)}=\overline{(psq)}$ in view of the symmetry, and in addition here

$$C_s^i=\frac{s!}{i!(s-i)!},\quad i\leqslant s,\quad C_s^i=0,\quad i>s.$$

The (spq) determined from the expansion of the kinematic conditions are functions of β_{ij} and linear functions of the moments. Indeed, in order N of the expansion the kinematic condition has the form

$$\begin{aligned}&-\left(\sum_{s+p+q=N}(spq)_0x^{2s}y^{2p}z_k^q\right)+R_N(000)_1\\&+Q_N(000)_2\sum_{l=1}^{N-1}\left\{R_l\sum_{s+p+q=N-l}(spq)_1x^{2s}y^{2p}z_k^q\right.\\&\left.+Q_l\sum_{s+p+q=N-l}(spq)_2x^{2s}y^{2p}z_k^q\right\}\\&=\sum_{s+p+q=N}(spq)x^{2s}y^{2p}z_k^q,\end{aligned}$$

where

$$(000)_0=0,\quad (spq)_0=\frac{(-1)^{s+p+q}}{(2s)!(2p)!q!}M(2s,2p,q),$$

$$(spq)_1=\frac{(-1)^{s+p+q}}{(2s+1)!(2p)!q!}M(2s+2,2p,q-1),$$

$$(spq)_2=\frac{(-1)^{s+p+q}}{(2s)!(2p+1)!q!}M(2s,2p+2,q-1),$$

$$R_l=\sum_{i=0}^{l}2i\beta_{i,l-i}x^{2i}y^{2(l-i)},\quad Q_l=\sum_{i=0}^{l}2(l-i)\beta_{i,l-j}x^{2i}y^{2(l-i)},\quad R_l+Q_l=2l\sum_{i=0}^{l}\beta_{i,l-i}x^{2i}y^{2(l-i)}. \tag{2.6}$$

We underscore that $(spq)_1=(psq)_2$ and $(000)_1=(000)_2=-\frac{1}{2}(001)_0=\frac{1}{2}M(001)$, but $(spq)_1\neq(psq)_1$ and $(spq)_2\neq(psq)_2$ for all other s, p, and q.

It follows from the expansions of the dynamic and kinematic conditions that for $N\geqslant 2$

$$\overline{(001)}\gamma_{i+j=N}=F_N(M^2,\{\gamma_{ij}\}_{i+j=1,\dots,N-1}),$$

and

$$(001)_0\beta_{i+j=N}=\frac{1}{N+1}G_N(M,\{\beta_{ij}\}_{i+j=1,\dots,N-1}), \tag{2.7}$$

where F_N are quadratic and G_N are linear in the moments of the functions, and they are polynomials in γ_{ij} and β_{ij} with $1\leqslant i+j\leqslant N-1$, respectively.

For $N=1$,

$$\overline{(001)}\gamma_{10}=-\overline{(100)}=-\frac{1}{4}M^2(001),$$

$$(001)_0\beta_{10}=-\frac{(001)_0}{2}=-\frac{1}{8}M(002). \qquad (2.7a)$$

The solution of the system of algebraic equations

$$z_d=z_k=z^* \quad \text{or} \quad \gamma_{ij}=\beta_{ij} \quad \text{for} \quad \forall i,j,i+j=N =1,2,3,\ldots,\infty, \qquad (2.8)$$

determines the matrix of unknown amplitudes $\{\hat{\Phi}\}$ and the form of the free surface z^*. We seek a solution of this infinite-dimensional system and investigate its convergence by the method of successive approximations. In this fashion, we obtain the sequence of equations

$$\{\gamma_{ij}(\{M\})=\beta_{ij}(\{M\}), \quad i+j\leqslant N\}, \qquad (2.8a)$$

whose order increases linearly as a function of the order N of the approximations.

In the first approximation, $\gamma_{10}=\beta_{10}\Rightarrow M^3(001)=M(002)$.

The structure of the equations (2.7) is such that, as is easily shown, all β_{ij} are "dimensionless" as functions of the moments, i.e., $\beta_{i+j=N}\approx M^{K(N)}/M^{K(N)}$, and $K(N)$ is a positive integer that depends on the order N of the approximation. Moreover, it is easy to prove that since $\overline{(001)} = 2$, $(001)_0 = -M(001)$ and $\overline{(001)}/(001)_0 \approx M^{-1}$ and $\forall i,j,i+j\geqslant 2$, all equations $\gamma_{ij}=\beta_{ij}$ for $\forall i,j,i+j\geqslant 2$ will be homogeneous in the moments. Therefore, for arbitrary N the first equation $\gamma_{10}=\beta_{10}$ will be the only equation of the system (2.8a) that is inhomogeneous in the moments.

We underscore two obvious properties of the system (2.8) which are important for what follows. First, according to Eq. (2.7), the form of the equations of this system does not change as the order of the approximation increases. Second, the equations $\gamma_{ij}=\beta_{ij}$ depend only on the moments and not (by the definition of the moments (2.2), where the summation extends over all Φ_{mn}) on which amplitudes from the set $\{\Phi_{mn}\}$ are chosen to approximate the boundary conditions in order N.

The following recurrence relation simplifies the derivation of the equations in the approximation of order N. Consider the series $\Sigma_{s,p,q}\{spq\}x^{2s}y^{2p}z^q$ with arbitrary coefficients $\{spq\}$. If $z=\Sigma_j^\infty \pi_j=\Sigma_j^\infty\Sigma_{i=0}^j\varphi_{i,j-i}x^{2i}y^{2(j-i)}$, then in order k_0 in x and y this series has the form

$$\sum_{q=1}^{k_0}\{00q\}\left(\sum_{j=1}^{k_0}\pi_j\right)^q+\sum_{q=1}^{k_0}\{sk_0-s0\}x^{2s}y^{2(k_0-s)}$$
$$+\sum_{i=1}^{k_0-1}\sum_{q=1}^{k_0-i}\sum_{s=0}^{i}\{si-sq\}x^{2s}y^{2(i-s)}\left(\sum_{j=1}^{k_0-i}\pi_j\right)^q,$$

where only terms of order j_0 in x and y are chosen in $(\Sigma_{j=1}^{j_0}\pi_j)^q$.

Using this relation we easily obtain successive approximations of γ_{ij} and β_{ij} only as a functions of the moments $\{M\}$ from the expansion of the dynamic condition

$$\overline{(001)}\gamma_{10}=-\overline{(100)},$$
$$\overline{(001)}^3\gamma_{11}=-[2\overline{(002)}\,\overline{(100)}^2-2\overline{(001)}\,\overline{(100)}\,\overline{(101)}+\overline{(001)}^2\overline{(110)}],$$
$$\overline{(001)}^3\gamma_{20}=-[\overline{(002)}\,\overline{(100)}^2-\overline{(001)}\,\overline{(100)}\,\overline{(101)}+\overline{(001)}^2\overline{(110)}], \qquad (2.9)$$

and kinematic condition

$$(001)_0\beta_{10}=-\tfrac{1}{2}(100)_0(001)_0^3,$$
$$(001)_0^3\beta_{11}=-\tfrac{1}{6}((002)_0(100)_0^2-2(001)_0(100)_0(101)_0+2(001)_0^2(110)_0-2(100)_0^2(001)_1+4(001)_0(100)_0(010)_1),$$
$$(001)_0^3\beta_{20}=-\tfrac{1}{12}((002)_0(100)_0^2-2(001)_0(100)_0(101)_0+4(001)_0^2(200)_0-2(100)_0^2(001)_1+4(001)_0(100)_0(100)_1), \qquad (2.9a)$$

respectively. The expressions for γ_{ij} and β_{ij} with $i+j=3$ are quite complicated and will not be presented here.

We choose among the moments $\{M\}$ the linearly independent moments (2.3). In first order there are two linearly independent moments $M(001)\equiv M_1$ and $M(002)\equiv M_2$. But, on the basis of Eq. (2.3), one could chose instead the moments $M(20-1)=(1/2)M(001)$ or $M(200)=(1/2)M(002)$. In second order the moments $M(220)$, $M(400)$, $M(22-1)$, and $M(40-1)$ are added to the moments M_1 and M_2 (or, for example, the moment $M(003)=M(22-1)+M(40-1)$). In third order the moments $M(420)$, $M(600)$, $M(42-1)$, and $M(60-1)$ appear, and so on.

We now express the functions $\overline{(spq)}$, $(spq)_0$, and $(spq)_1$ [Eqs. (2.5) and (2.6)] that appear in (2.9) and (2.9a) in terms of the following linearly independent moments: M_1, M_2, $M(220)$, $M(400)$, $M(22-1)$, $M(40-1)$, $M(420)$, $M(600)$, $M(42-1)$, $M(60-1)$. Then, the equations (2.8) have the form

$$\gamma_{10}=\beta_{10}\Rightarrow M_1^3-M_2=0,$$
$$\gamma_{11}=\beta_{11}\Rightarrow 8M_1^2M(220)-4M_1M_2(15M(22-1)+M(40-1))+9M_2^3=0,$$
$$\gamma_{20}=\beta_{20}\Rightarrow 8M_1^2M(400)-4M_1M_2(3M(22-1)+17M(40-1))+27M_2^3=0,$$
$$\gamma_{21}=\beta_{21}\Rightarrow M_1^2M_2^2(1059M(220)+199M(400))-40M_1^2M_2(15M^2(22-1)+4M(22-1)M(40-1)-3M^2(40-1))+M_1M_2^3(543M(22-1)-227M(40-1))-12M_1^3M_2(M(60-1)+67M(42-1))-16M_1^3(M(22-1)(18M(220)$$

$$+5M(400))+M(40-1)(10M(220)+M(400)))+48M_1^4M(420)-135M_2^5=0,$$

$$\gamma_{30}=\beta_{30}\Rightarrow 15M_1^2M_2^2(9M(220)+181M(400))+40M_1^2M_2(9M^2(22-1)+36M(22-1)M(40-1)-13M^2(40-1))-15M_1M_2^3(147M(22-1)-17M(40-1))-36M_1^3M_2(23M(60-1)+5M(42-1))-80M_1M(400)(3M(22-1)+7M(40-1))+48M_1^4M(600)-675M_2^5=0. \quad (2.10)$$

The equations $\gamma_{ij}=\beta_{ij}$ obtained for $i+j=2,3$ in the first three orders of approximation are, as expected, homogeneous in the moments: third order for $i+j=2$ and fifth order for $i+j=3$. As follows from the derivation, in the higher orders the first two moments M_1 and M_2 (or, as follows from Eq. (2.2), the leading amplitudes Φ_{10}, Φ_{11}, and Φ_{20}), appear in them.

3. DIMENSION OF THE SET OF SOLUTIONS IN THE PROBLEM OF THE STATIONARY STATE

A solution of any system of equations exists, generally speaking, when the number of variables N_t is not less than the number of equations N_ε in this system, $N_t \geqslant N_\varepsilon$. The difference $\Delta N(t,\varepsilon)=N_t-N_\varepsilon$, however, determines the number of free parameters or the dimension of the solution of the system. If $\Delta N(t,\varepsilon)=0$, then the solution of the system of equations is a point in the space of the variables of the system, for $\Delta N(t,\varepsilon)=1$ the solution is a curve, and for $\Delta N(t,\varepsilon)=2$ the solution is a surface.

We seek a solution of the system (2.8) $\{\gamma_{ij}=\beta_{ij}, i+j=N=1,\ldots,\infty\}$. It is interesting that the equations of this system, which were derived in "dimensionless" variables, do not explicitly contain any additional parameters. The dimension of the solution of the system (2.8) is therefore not known in advance. Since we seek a solution of our infinite-dimensional system by the method of successive approximations (finite-dimensional approximation of the problem of determining a function in function space), the dimension $\Delta N(t,\varepsilon)$ of the solution must be such that convergence with increasing N occurs on the set of solutions of the system of equations (2.8a) $\{\gamma_{ij}=\beta_{ij}, i+j\leqslant N\}$.

As we have already mentioned above, for an approximation of order N, the equations (2.8) for all N depend only on the moments $\{M\}$ and not on the specific choice of variables—the unknown Fourier amplitudes Φ_{nm} from the set $\{\hat{\Phi}\}$—and thus on the number N_t of these variables. These equations therefore make it possible to find in order N a solution of any dimension.

Setting $N_t=N_\varepsilon$, we obtain in each order N of approximation of the solution a point (or points) in the space of the Fourier amplitudes Φ_{mn}. If the dimension of the solution of our system (2.8) $\Delta N(t,\varepsilon)=0$, then as N increases, these solutions of (2.8) (points) must converge to an isolated limit (or limits).

Setting $N_t=N_\varepsilon+1$, we find in each order N a solution as a curve—a function of one parameter in the space Φ_{mn}. If the dimension of the solution of the system (2.8) $\Delta N(t,\varepsilon)=1$, then the solutions (2.8a) must form a family of curves that converges with increasing N to a one-parameter function in the space Φ_{mn}.

If, however, the dimension of the solution of the system (2.8) $\Delta N(t,\varepsilon)=0$ and in solving Eq. (2.8a) we set $N_t=N_\varepsilon+1$, then as N increases, the curves obtained will converge only in a neighborhood of an isolated point (solution).

On the basis of qualitative considerations for a three-dimensional spatially periodic flow with C_4 symmetry, we seek a one-parameter family S of stationary solutions.

Let such a family of solutions S exist and be unique. If p is a parameter, then there must exist a functional limit with respect to this parameter of the Fourier amplitudes $\Phi_{mn}(p)$ as the order N of the approximation increases:

$$\{\hat{\Phi}(p)\}=\lim_{N\to\infty}\{\hat{\Phi}(p)\}_N. \quad (3.1)$$

The absolute value of the amplitudes $\Phi_{mn}(p)$ must decrease as the number $m+n$ of the amplitude increases:

$$\|\Phi_{mn}(p)|_{m+n=l+1}-\Phi_{mn}(p)|_{m+n=l}\| \ll \|\Phi_{mn}(p)|_{m+n=l}\| \quad (3.1a)$$

for all m, n, and l. Such behavior of $\Phi_{mn}(p)$ would indicate convergence and it would prove the existence of a unique one-parameter family S of solutions for a three-dimensional spatial flow of bubbles and jets. The family S itself would then be completely determined by the matrix $\{\hat{\Phi}(p)\}$. Each point of this one-parameter family S would correspond to an exact solution. We show below that the absolute value of the amplitudes $\Phi_{mn}(p)$ decays exponentially as $m+n$ increases.

Since in finding the one-parameter family of solutions S the number of equations N_ε in the system (2.8a) in each order N and the number of variables N_t (unknown Fourier amplitudes Φ_{mn}) must be related by $N_t=N_\varepsilon+1$, we obtain, for example, in the first approximation $N=1$ a single equation ($N_\varepsilon=1$) $\gamma_{10}=\beta_{10}$ and two variables $\{\Phi_{10},\Phi_{20}\}$ (or, equivalently, $\{\Phi_{10},\Phi_{11}\}$), i.e., $N_t=2$. One of these variables, Φ_{10}, is the first-order amplitude and the other (Φ_{11} or Φ_{20}) is a second-order amplitude. In the second approximation $N=2$, we obtain a system of three equations ($N_\varepsilon=3$): $\{\gamma_{10}=\beta_{10}, \gamma_{11}=\beta_{11}, \gamma_{20}=\beta_{20}\}$ and four variables $N_t=4$: $\{\Phi_{10},\Phi_{11},\Phi_{20},\Phi_{30}\}$ (or $\{\Phi_{10},\Phi_{11},\Phi_{20},\Phi_{21}\}$). In the third approximation $N=3$ we obtain a system of five equations ($N_\varepsilon=5$): $\{\gamma_{10}=\beta_{10}, \gamma_{11}=\beta_{11}, \gamma_{20}=\beta_{20}, \gamma_{21}=\beta_{21}, \gamma_{30}=\beta_{30}\}$, and six variables: $\{\Phi_{10},\Phi_{11},\Phi_{20},\Phi_{31},\Phi_{30},\Phi_{40}\}$ (either $\{\Phi_{10},\Phi_{11},\Phi_{20},\Phi_{31},\Phi_{30},\Phi_{31}\}$, or $\{\Phi_{10},\Phi_{11},\Phi_{20},\Phi_{31},\Phi_{30},\Phi_{22}\}$). This process can be continued. The choice of the additional amplitude is arbitrary. Generally speaking, in Nth order any $\{\hat{\Phi}\}$ and more than one amplitude of the next ($N+1$)st order can be chosen as the additional amplitude. But this, in accordance with Eqs. (3.1) and (3.1a), would reduce the domain of convergence.

In the Nth approximation the value of the parameter for which all additional amplitudes vanish corresponds to the solution of the system (2.8a) in the case $N_t = N_\varepsilon$ and $\Delta N(t,\varepsilon)=0$. This value of the parameter will obviously be the point of intersection of the curves in the N-th approximation which correspond to different choices of the additional amplitude. The case $\Delta N(t,\varepsilon)=0$ in each order N, in principle, therefore requires no further analysis, since it is automatically contained in the solution of the system (2.8a) in the case $N_t = N_\varepsilon + 1$.

We now consider in greater detail a method for solving the system (2.8a). In each order N the number of linearly independent moments $\{M(\alpha,\beta,\gamma)\}$ is greater than the number of linearly independent amplitudes $\{\Phi_{mn}\}$. Since

$$M_1 = \sum_{m,n} \Phi_{mn}\sqrt{m^2+n^2} = 2\Phi_{10} + \sqrt{2}\Phi_{11} + 4\Phi_{20} + \sum_{m+n>2} \Phi_{mn}\sqrt{m^2+n^2},$$

$$M_2 = \sum_{m,n} \Phi_{mn}(m^2+n^2) = 2\Phi_{10} + 2\Phi_{11} + 8\Phi_{20} + \sum_{m+n>2} \Phi_{mn}(m^2+n^2),$$

we easily obtain for each moment $M(\alpha,\beta,\gamma)$ in the case $\Phi_{20}\neq 0$

$$M(\alpha,\beta,\gamma) = M_1\left(a-\frac{b}{4}\right) + \frac{1}{2}M_2\left(-a+\frac{b}{2}\right) + \Phi_{11}\times\left(2^{\gamma/2} - a(\sqrt{2}-1) - \frac{b}{2}\frac{(\sqrt{2}-1)}{\sqrt{2}}\right) + \sum_{m+n>2}\Phi_{mn}\left(m^\alpha n^\beta(\sqrt{m^2+n^2})^\gamma - \left(a-\frac{b}{4}\right)(\sqrt{m^2+n^2}) - \frac{1}{2}\left(-a+\frac{b}{2}\right)\times(m^2+n^2)\right), \tag{3.2}$$

where $a = 0^\alpha + 0^\beta$ and $b = 2^\gamma(2^\alpha 0^\beta + 2^\beta 0^\alpha)$. The case $\Phi_{20}\equiv 0$ will in fact appear only in the first approximation $N=1$ if in solving the system (2.8a) the amplitude Φ_{11} is chosen as the additional amplitude. Then

$$M(\alpha,\beta,\gamma) = M_1\frac{(a-2^{\gamma/2})}{2-\sqrt{2}} + M_2\frac{(-\sqrt{2}a + 2^{\gamma/2+1})}{2(2-\sqrt{2})} + \sum_{m+n>2}\Phi_{mn}\left(m^\alpha n^\beta(\sqrt{m^2+n^2})^\gamma - \frac{(a-2^{\gamma/2})}{2-\sqrt{2}}(\sqrt{m^2+n^2}) - \frac{(-\sqrt{2}a+2^{\gamma/2+1})}{2(2-\sqrt{2})}(m^2+n^2)\right). \tag{3.2a}$$

Substituting the expression (3.2) into the system (2.8a), we obtain in order N a system of equations in the variables

$$\{U_{ij}(M_1,\Phi_{11},M_2,\{\Phi_{mn}\}_{m+n>2}) = 0, 1\leqslant i+j\leqslant N\}. \tag{3.3}$$

Since the transformation (3.2) is linear, all homogeneous equations of the system (2.8a) become homogeneous in the variables $\{M_1,\Phi_{11},M_2,\{\Phi_{mn}\}_{m+n>2}\}$ of the equations of the system (3.3), and the first equation $\gamma_{10}=\beta_{10}$ will remain unchanged. The degree of nonlinearity of the system (3.3) in the variables Φ_{mn} is comparatively low.

On the other hand, the radius R of curvature of the free surface of a bubble at the stagnation point and the velocity v of the bubble in the laboratory coordinate system are physical variables that describe our steady flow (the surface is described by two radii of curvature, but in this case, in view of the symmetry, they are identical and equal to R). The radius of curvature R is obviously related to the moments M_1 and M_2: $R = -1/2\beta_{10}(\Phi_{mn})$ or $R = 4M_1/M_2$.

We also call attention to the fact that all amplitudes and, according to Eq. (2.2), the moments $\{M\}$ have the dimensions v of the bubble velocity. We define

$$m(\alpha,\beta,\gamma) = \frac{M(\alpha,\beta,\gamma)}{v}, \quad m_1 = \frac{M_1}{v}, \quad m_2 = \frac{M_2}{v}, \quad \varphi_{mn} = \frac{\Phi_{mn}}{v}, \tag{3.4}$$

introducing the velocity v as a scale factor. This transformation changes the form of only the first inhomogeneous equation ($M_1^3 - M_2 = 0$) of the system (2.8a) or (3.3), which will contain, besides the variables m_1 and m_2, the velocity v. However, this approximation will not affect the form of the homogeneous equations of the system (2.8a) or (3.3) and therefore Eq. (3.3) will have the form

$$m_1^3 - v^2 m_2 = 0,$$
$$U_{ij}(m_1,\varphi_{11},m_2,\{\varphi_{mn}\}_{m+n>2}) = 0, 2\leqslant i+j\leqslant N. \tag{3.5}$$

The condition (2.2a) becomes

$$m(0,0,0) = \sum_{m+n>1}\varphi_{mn} = -1. \tag{3.5a}$$

Together with the expression for the radius of curvature of the surface $R = 4m_1/m_2$, the first of the equations (3.5) $v^2 = m_1^3/m_2$ defines the change of variables from the variable moments $\{m_1,m_2\}$ to the physical variables $\{v,R\}$, the velocity v playing the role of "length" and the radius of curvature R playing the role of "angle":

$$m_1 = -\frac{2}{v\sqrt{R}}, \quad m_2 = -\frac{8}{v(\sqrt{R})^3}. \tag{3.6}$$

Therefore, in order N, we finally obtain on the basis of Eqs. (3.2), (3.4) and (3.6) a system of equations in the variables $\{m_1,\varphi_{11},m_2,\{\varphi_{mn}\}_{m+n>2}\}$:

$$U_{10} = m(0,0,0) + 1 = \sum_{m+n\geqslant 1}\varphi_{mn} + 1 = 0,$$

$$U_{ij}(m_1,\varphi_{11},m_2,\{\varphi_{mn}\}_{m+n>2})=0, 2\le i+j\le N, \quad (3.7)$$

where the velocity v and the radius of curvature R in Nth order are related to the solution (3.7) by the transformation (3.6). The first equation in Eq. (3.7) is once again inhomogeneous, but now this equation is linear.

Different variables can be chosen as the parameter p of our family S of stationary solutions. For example, the parameter can be an amplitude or a moment. We choose as the parameter of the family S a physical variable—the radius of curvature—$p=R$.

We set the number of variables N_t in the system (3.7) to one greater than the number of equations N_ε, i.e., $N_t=N_\varepsilon+1$. Solving any $N_\varepsilon-1$ equations of the system (3.7), we obtain the amplitudes $\varphi_{mn}=\varphi_{mn}(m_1,m_2)$ as functions of the variables m_1 and m_2. Substituting these expressions into the remaining equation (3.7), we obtain a function $f(m_1,m_2)=0$, which determines a curve in the (m_1,m_2) plane and, after the transformation (3.6), the velocity $v=v(R)$ and then the amplitude $\varphi_{mn}=\varphi_{mn}(v(R),R)$ as functions of the parameter R in the Nth approximation. As N increases, convergence must occur to some "ideal" functions of the velocity and amplitudes corresponding to $N=\infty$.

We note that among the solutions of the system (2.8a) or (2.10), there are always "identity" solutions $\{M_1\equiv0, M_2\equiv0\}$ or $\{m_1\equiv0, m_2\equiv0\}$. For such solutions the Jacobian of the transition from variables $\{m_1,m_2\}$ to variables $\{v,R\}$ is zero; this means that branch points appear on the curve in the (m_1,m_2) or (v,R) plane. In what follows we shall not study such "identity" solutions.

4. ONE-PARAMETER FAMILY OF SOLUTIONS

Our investigation was limited to the first three orders of approximation with $N=i+j=1,2,3$, and with $N_\varepsilon=1,3,5$ and $N_t=2,4,6$, respectively, in view of the complexity of the system (2.8a), which for orders higher than third must be solved numerically.

1) In the first approximation ($N=1$), the system (2.8a) consists of one equation ($N_\varepsilon=1$)—the linear inhomogeneous equation (3.7).

We choose as the variables of the system (2.8a) the Fourier amplitudes $\{\Phi_{10},\Phi_{20}\}$ ($N_t=2$). Then, after the transformations (3.2), (3.4), and (3.6), we obtain

$$\varphi_{10}=(2m_1-m_2)/2=-2(R-2)/(vR^{3/2}),$$

$$\varphi_{20}=(-m_1+m_2)/4=(R-4)/(2vR^{3/2}),$$

whence we easily find as the solution of the linear equation of the system (3.7) the velocity and amplitudes as functions of the parameter R:

$$m(000)+1=2(\varphi_{10}+\varphi_{20})+1=(3m_1-m_2)/2+1=0:$$

$$v(R)=(3R-4)/R^{3/2}, \quad \varphi_{10}(R)=-2(R-2)/(3R-4),$$

$$\varphi_{20}(R)=(R-4)/2(3R-4),$$

and, correspondingly,

$$v(R)=(3R-4)/R^{3/2}, \quad \Phi_{10}(R)=-2(R-2)/R^{3/2},$$

$$\Phi_{20}(R)=(R-4)/2R^{3/2}. \quad (4.1a)$$

Choosing the Fourier amplitudes $\{\Phi_{10},\Phi_{11}\}$ as the variables (in this case $\Phi_{20}\equiv0$) results in [after the transformations (3.2a), (3.4), and (3.6)] a somewhat different dependence of the velocity and amplitudes on the radius of curvature:

$$v(R)=((2+\sqrt{2})R-4\sqrt{2})/R^{3/2},$$

$$\Phi_{10}(R)=-((2+\sqrt{2})R-4(1+\sqrt{2}))/R^{3/2},$$

$$\Phi_{11}(R)=(2+\sqrt{2})(R-4)/R^{3/2}. \quad (4.1b)$$

We now analyze the expressions (4.1a) and (4.1b). First, irrespective of the choice of the additional amplitude, the velocity and the amplitude in the first approximation $N=1$ have the form $v(R)$, $\Phi_{mn}(R)\approx W(R)/R^{1/2}$, where $W(R)$ is a rational function of R. In the limit $R\to\infty$ $v(R)\to3/\sqrt{R}$, $\varphi_{10}(R)\to-2/3$, $\varphi_{20}(R)\to1/6$ in Eq. (4.1a) and $v(R)\to(2+\sqrt{2})/\sqrt{R}\approx3.414/\sqrt{R}$, $\varphi_{10}(R)\to-1$, $\varphi_{11}(R)\to1$ in Eq. (4.1b).

Moreover, the "additional" amplitudes $\Phi_{20}(R)$ and $\Phi_{11}(R)$ in Eqs. (4.1a) and (4.1b) vanish (the velocity and the amplitudes from Eqs. (4.1a) and (4.1b) become equal) at the same value of the parameter $R_0=4$: $\Phi_{20}(4)=\Phi_{11}(4)=0$. This corresponds to a zero-parameter solution ($\Delta N(t,\varepsilon)=0$) of the system (2.8a) in the first approximation $N=1$:

$$v(4)=1, \quad \Phi_{10}(4)=-1/2, \quad \Phi_{20}(4)=\Phi_{11}(4)=0.$$

2) In the second-order approximation, the velocity and amplitudes can be determined as functions of the parameter by solving the system (2.8a), which for $N=2$ consists of three equations ($N_\varepsilon=3$) for four variables ($N_t=N_\varepsilon+1)=4$ ($\{\Phi_{10},\Phi_{11},\Phi_{20},\Phi_{30}\}$ or $\{\Phi_{10},\Phi_{11},\Phi_{20},\Phi_{21}\}$) or, taking into account the relations (3.2) and (3.4), from the system of equations

$$U_{10}=m(0,0,0)+1=0,$$

$$U_{20}(m_1,\varphi_{11},m_2,\{\varphi_{mn}\}_{m+n=3})=0,$$

$$U_{11}(m_1,\varphi_{11},m_2,\{\varphi_{mn}\}_{m+n=3})=0, \quad (4.2)$$

for the unknowns $\{m_1,\varphi_{11},m_2,\varphi_{30}\}$ (or $\{m_1,\varphi_{11},m_2,\varphi_{21}\}$). As follows from Eq. (2.10), the variables $\{\varphi_{mn}\}$ appear in the equations of this system linearly. We solve any two of the three equations (4.2) for $\{\varphi_{11},\varphi_{30}\}$ (or $\{\varphi_{11},\varphi_{21}\}$, respectively), and then, using Eq. (3.6), we transform from the variables $\{m_1,m_2\}$ to the variables $\{v,R\}$ in the remaining equation (4.2). Irrespective of the choice of the additional amplitude, the velocity as a function of the radius of curvature will be determined by the expression

$$v(R)p(R)\sqrt{R}+q(R)=0,$$

where $q(R)$ and $p(R)$ are polynomials in R, and both polynomials have the same degree. Hence, we easily find $v(R)=-q(R)/p(R)\sqrt{R}$.

In choosing Φ_{21} as the "additional" amplitude, the functions $q(R)$ and $p(R)$ have the form

$$q_{[21]}(R)=3(-100800+80640\sqrt{2}-45696\sqrt{5}$$
$$+34272\sqrt{10}+(47760-42840\sqrt{2}$$
$$+22176\sqrt{5}-19320\sqrt{10})R+(10360$$

TABLE I. Asymptotic behavior of the velocity ($v(R)$) and the amplitudes ($\Phi_{mn}(R)/v(R)$) as functions of the parameter R in the limit $R\to\infty$.

	Φ_{11}	Φ_{20}	Φ_{21}	Φ_{30}	Φ_{22}	Φ_{31}	Φ_{40}
$v(R)\sqrt{R}$	$2+\sqrt{2}$	3	3.80	4/3	4.06	3.97	7/2
$\Phi_{10}(R)/v(R)$	−1	−2/3	−7/6	−3/4	−4/3	−5/4	−4/5
$\Phi_{11}(R)/v(R)$	1	0	4/3	$3/5R$	16/9	3/2	$56.84/R$
$\Phi_{20}(R)/v(R)$	0	1/6	1/6	3/10	1/3	3/10	2/5
$\Phi_{21}(R)/v(R)$	0	0	−1/6	0	−4/9	−3/10	$-7.05/R$
$\Phi_{30}(R)/v(R)$	0	0	0	−1/20	$0.70/R$	−1/20	−4/35
$\Phi_{22}(R)/v(R)$	0	0	0	0	1/9	0	0
$\Phi_{31}(R)/v(R)$	0	0	0	0	0	1/20	0
$\Phi_{40}(R)/v(R)$	0	0	0	0	0	0	1/70

$$-3580\sqrt{2}+4912\sqrt{5}-1264\sqrt{10})R^2 + (-1430+420\sqrt{2}-664\sqrt{5}+180\sqrt{10})R^3 + (35+15\sqrt{5})R^4),$$

$$p_{[21]}(R)=(-30240+20160\sqrt{2}-14112\sqrt{5} +8568\sqrt{10})R^2+(3870-2370\sqrt{2} +1780\sqrt{5}-1032\sqrt{10})R^3+(-125+70\sqrt{2} -55\sqrt{5}+30\sqrt{10})R^4,$$

$$v(R)=-q_{[21]}(R)/p_{[21]}(R)\sqrt{R}.$$

If, however, Φ_{30} is chosen as the additional amplitude, then

$$q_{[30]}(R)=(-199584-108864\sqrt{2})+(313200 +224856\sqrt{2})R-(215076+156480\sqrt{2})R^2 +(52574+37573\sqrt{2})R^3-(4962 +3460\sqrt{2})R^4+(140+100\sqrt{2})R^5,$$

$$p_{[30]}(R)=(34776+25704\sqrt{2})R^2-(12588 +9075\sqrt{2})R^3+(1396+971\sqrt{2})R^4 -(42+30\sqrt{2})R^5),$$

$$v(R)=-q_{[30]}(R)/p_{[30]}(R)\sqrt{R}.$$

The expressions for the amplitudes $\Phi_{mn}(R)$ are quite complicated and will not be presented here. We merely note that just as in the first-order approximation, the ratio $\Phi_{mn}(R)/v(R)$ is a rational function irrespective of the choice of the additional amplitude. If Φ_{21} is chosen as the additional amplitude, then $\Phi_{mn}(R)/v(R)=P_7(R)/Q_7(R)$ for all $\Phi_{mn}(R)$. When Φ_{30} is chosen as the additional amplitude, $\Phi_{mn}(R)/v(R)=P_8(r)/Q_8(R)$ for all $\Phi_{mn}(R)$, except $\Phi_{11}(R)/v(R)=P_7(r)/Q_8(R)$ (here and below $P_i(R)$ and $Q_i(R)$ are polynomials of degree i). The asymptotic behavior of the ratio $\Phi_{mn}(R)/v(R)$ in the limit $R\to\infty$ is presented in Table I.

The zero-parameter solution of the system (2.8a) in the second-order approximation ($N=2$) corresponds to solutions of the system (4.2), in which the "additional" variables $\varphi_{21}(R)\equiv\varphi_{30}(R)\equiv 0$, and the number of unknowns $\{m_1,\varphi_{11},m_2\}$ equals the number of equations: $\Delta N(t,e)=0$. Two of the possible solutions are complex conjugates of one another; for their real parts

$$R_1=10.641,\quad v(R_1)=0.792,$$

$$[\Phi_{10}(R_1)/v(R_1)]=-0.574,$$

$$[\Phi_{11}(R_1)/v(R_1)]=-0.133,\quad [\Phi_{20}(R_1)/v(R_1)]=0.140,$$

$$R_2=20.158,\quad v(R_2)=0.746,$$

$$[\Phi_{10}(R_2)/v(R_2)]=-1.096,$$

$$[\Phi_{11}(R_2)/v(R_2)]=1.346,\quad [\Phi_{20}(R_2)/v(R_2)]=-0.078$$

only the solution with $R_1=10.641$ is physically meaningful.

3) In the third-order approximation the system (2.8a) consists of five equations ($N_\varepsilon=5$) and six unknowns, since $N_t=N_\varepsilon+1=6$. As the "additional" amplitude, we choose one fourth-order amplitude: Φ_{22}, Φ_{31}, or Φ_{40}. Taking into account the relations (3.2) and (3.4), the system (2.8a) in this case is equivalent to the system (3.7) with $N=3$:

$$U_{10}=m(0,0,0)+1=0,$$

$$U_{20}(m_1,\varphi_{11},m_2,\{\varphi_{mn}\}_{m+n\leqslant 4})=0,$$

$$U_{11}(m_1,\varphi_{11},m_2,\{\varphi_{mn}\}_{m+n\leqslant 4})=0,$$

$$U_{21}(m_1,\varphi_{11},m_2,\{\varphi_{mm}\}_{m+n\leqslant 4})=0,$$

$$U_{30}(m_1,\varphi_{11},m_2,\{\varphi_{mn}\}_{m+n\leqslant 4})=0 \tag{4.3}$$

with one of the following sets of variables: $\{m_1,\varphi_{11},m_2,\varphi_{21},\varphi_{30},\varphi_{22}\}$, $\{m_1,\varphi_{11},m_2,\varphi_{21},\varphi_{30},\varphi_{31}\}$, or $\{m_1,\varphi_{11},m_2,\varphi_{21},\varphi_{30},\varphi_{40}\}$. In accordance with Eqs. (2.8a), (2.10), and (3.2) the equations $U_{10}=0$, $U_{11}=0$, and $U_{20}=0$ of this system are linear and the equations $U_{21}=0$ and $U_{30}=0$ are quadratic in the variables $\{\varphi_{mn}\}$. We shall solve the linear subsystem $U_{10}=0$, $U_{11}=0$, and $U_{20}=0$ for any three of the variables $\{\varphi_{mn}\}$. As one can easily see from Eq. (2.10), the determinant of this subsystem is proportional to a function which is quadratic in $\{m_1,m_2\}$. In $\{v,R\}$, its zeros are therefore roots of a quadratic equation in R. It is of interest that for any choice of the variables $\{\varphi_{mn}\}$, the zeros of the determinant of the subsystem (4.3) correspond to real values of the parameter R^*. These values of the parameter, however, are not distinguished in any way from a physical standpoint, and a solution of the system (4.3) exists for $R=R^*$. Thus, solving the linear subsystem $U_{10}=0$, $U_{11}=0$, and $U_{20}=0$ for any three of the variables $\{\varphi_{mn}\}$, we substitute the expressions obtained into the equations $U_{21}=0$ and $U_{30}=0$, which now depend on φ_{m_1,n_2} and the remaining, fourth, variable φ_{mn} as follows:

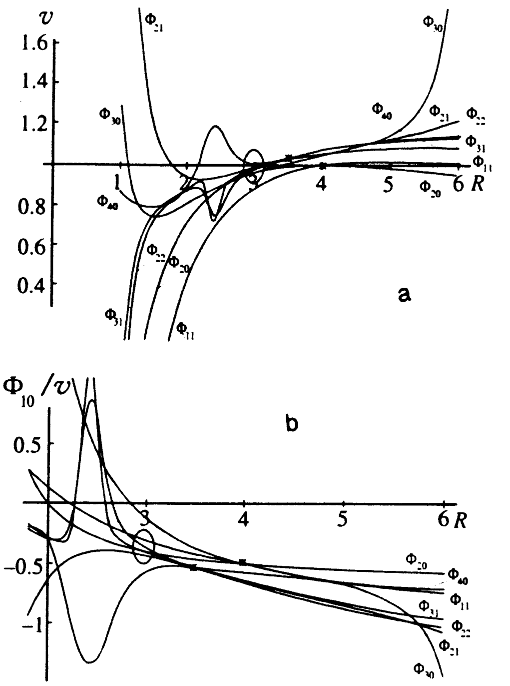

FIG. 3. One-parameter family of solutions. The velocity $v(R)$ (a) and the amplitude $\Phi_{10}((R)/v(R)$ (b) are shown as functions of one parameter—the radius of curvature R of a bubble at the stagnation point in the first three orders of approximation $N=1$, 2, and 3. The different parameter dependences of the velocity and the amplitudes correspond to the choice of the "additional amplitude" $\Phi_{mn}(R)$ in each order N of the approximation. The asterisks designate zero-parameter solutions in the first $R_0=4$ and third $R_0=3.486$ approximations. The circles mark the region of the critical value of the parameter $R_{cr}=3\pm0.3$. The velocity and the radius of curvature are expressed in dimensional coordinates $|R|=1/k, |v|=\sqrt{g/k}$.

$$A_1(m_1,m_2)\varphi_{mn}^2+C_1(m_1,m_2)\varphi_{mn}+D_1(m_1,m_2)=0,$$

$$A_2(m_1,m_2)\varphi_{mn}^2+C_2(m_1,m_2)\varphi_{mn}+D_2(m_1,m_2)=0, \tag{4.3a}$$

where $A_i(m_1,m_2), C_i(m_1,m_2)$, and $D_i(m_1,m_2)$ are polynomials in m_1 and m_2. We easily find from Eq. (4.3a)

$$\varphi_{mn}=-\frac{A_1(m_1,m_2)D_2(m_1,m_2)-A_2(m_1,m_2)D_1(m_1,m_2)}{A_1(m_1,m_2)C_2(m_1,m_2)-A_2(m_1,m_2)C_1(m_1,m_2)}.$$

The condition that the system (4.3a) be compatible,

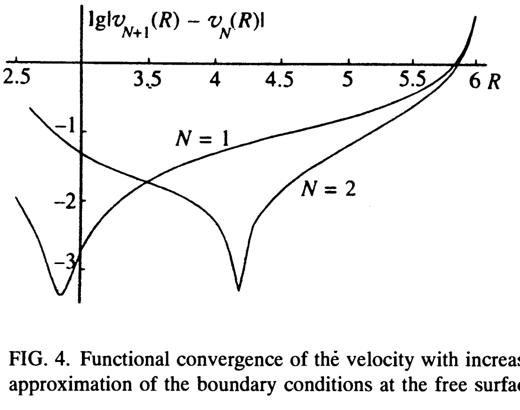

FIG. 4. Functional convergence of the velocity with increasing order of the approximation of the boundary conditions at the free surface of the liquid.

$$(A_1D_2-A_2D_1)^2+A_1D_2C_2^2+A_2D_1C_1^2-(A_1D_2+A_2D_1)C_1C_2=0$$

determines in the third-order approximation the solution (curve) of Eq. (4.3) in the $\{m_1,m_2\}$ or $\{v,R\}$ plane. Irrespective of the choice of the additional amplitude for $N=3$, $v(R)$ is the solution of the equation

$$v^4(R)R^{14}P_8(R)+v^3(R)R^{21/2}P_{12}(R)+v^2(R)R^7P_{16}(R)+v(R)R^{7/2}P_{20}(R)+P_{23}(R)=0, \tag{4.3b}$$

where the $P_i(R)$ are, once again, polynomials of degree i in R. Only the specific form of these polynomials depends on the choice of the additional amplitude [this is easily understood by analyzing Eq. (2.10)]. We note that the exact expressions for $P_i(R)$ in Eq. (4.3b) are extremely complicated. It is interesting, however, that over practically the entire range of R, to determine the solutions of Eq. (4.3) it is sufficient to expand first the expressions (2.2)–(2.10) and (3.2)–(3.7) with prescribed accuracy. This would both greatly simplify the expressions for the polynomials $P_i(R)$ and make it much easier to find the solution $v(R)$ in Eq. (4.3b). Extended (high!) precision is required here only near the values of the parameter R^* for which the determinant of the linear system (4.3) vanishes.

We now consider the solution of the system (4.3). It follows from Eq. (4.3b) that as $R\to\infty$, one solution $v(R)$ of this equation approaches zero, $v(R)\approx 1/\sqrt{R}$, while the three other solutions behave as $\sqrt{R}$. For small values of the parameter, however, irrespective of the choice of the additional amplitude, two of the four possible solutions of Eq. (4.3b) are always complex conjugates of one another, and physical values of the amplitudes of the real solutions $\Phi_{mn}(R)$ correspond to only one. For a specific choice of the "additional" amplitude in Eq. (2.8a), the functions $v(R)$ and $\Phi_{mn}(R)/v(R)$ for small values of the parameter R and the asymptotic behavior of these functions in the limit $R\to\infty$ are displayed in Fig. 3 and in Table I, respectively.

The zero-parameter solution of the system (2.8a) with $N=3$ corresponds to the solution of the system (4.3) with the variables $\{m_1,\varphi_{11},m_2,\varphi_{21},\varphi_{30}\}(\Delta N(t,\varepsilon)=0)$. In this case the system (4.3) has five real solutions: $R_1=1.995$, $R_2=3.486$, $R_3=20.067$, $R_4=32.762$, $R_5=51.715$. Of these solutions, the one with

$$R_2=3.486, \quad v(R_2)=1.029,$$

$$\Phi_{10}(R_2)/v(R_2)=-0.525,$$

$$\Phi_{11}(R_2)/v(R_2)=0.084, \quad \Phi_{20}(R_2)/v(R_2)=0.065,$$

$$\Phi_{21}(R_2)/v(R_2)=-0.082, \quad \Phi_{30}(R_2)/v(R_2)=0.001.$$

corresponds to physical values of the amplitudes. As is clearly seen from Fig. 3, this value of the parameter is the point of intersection of the curves $v(R)$ (and, naturally, $\Phi_{mn}(R)/v(R)$) which correspond to different choices of the additional amplitude for $N=3$.

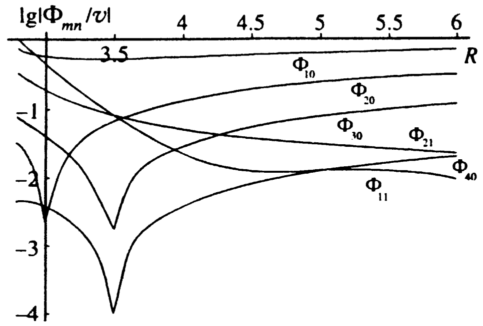

FIG. 5. Decrease of the absolute value of the amplitudes with increasing number of the harmonic. Third-order approximation of the boundary conditions. The "additional amplitude" is $\Phi_{40}(R)$.

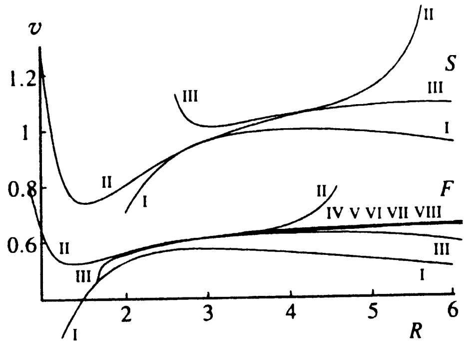

FIG. 6. One-parameter family of solutions for three-dimensional (*S*) and two-dimensional (*F*) flows. The roman numerals designate the order of the approximation of the boundary conditions.

5. DISCUSSION. QUESTIONS OF CONVERGENCE

The solution of the system (2.8) is exact and satisfies Laplace's equation, periodic boundary conditions, the boundary condition at infinity, and dynamic and kinematic conditions on the free surface of the liquid. We have derived analytically the velocity and the amplitudes as functions of a parameter—the radius of curvature of a bubble at the stagnation point—in the first three orders of approximation, i.e., we have found the solutions of the system (2.8a) with $N=1$, 2, and 3, respectively. The most important question is the functional convergence of these solutions as the order of the approximation increases, i.e., whether or not the solutions obtained satisfy (3.1) and (3.1a).

The condition (3.1) means that for curves of each amplitude and velocity, the curves must not differ by too much, and as the order of the approximation increases, they must approach the boundary conditions on the free surface of the liquid. We now discuss in greater detail the behavior of the velocity $v(R)$ as a function of N, noting immediately that any amplitude $\Phi_{mn}(R)$ behaves similarly. In each order of approximation the velocity $v(R)$ is represented by several curves, corresponding to a possible choice of the additional amplitude for the solution (2.8a). In the first approximation, there are two such curves (the additional second-order amplitudes are Φ_{11} and Φ_{20}); in second order there also two curves (Φ_{21} and Φ_{30}); and, in third order there are three curves (Φ_{22}, Φ_{31}, or Φ_{40}) (Fig. 3a). All curves lie close together. From their relative position it should be noted, first, that the curves are close to one another in the first approximation. Second, the curves corresponding to different orders of approximation can also be classified according to symmetry, i.e., according to the symmetry of the additional amplitude Φ_{mn}: $\{\Phi_{20}, \Phi_{30}$, and $\Phi_{40}\}$, $\{\Phi_{21}$ and $\Phi_{31}\}$, and finally $\{\Phi_{11}$ and $\Phi_{22}\}$ (Fig. 3a). Figure 4 demonstrates the convergence of the curves with increasing order N of the approximation of the boundary conditions on the free surface of the liquid. For simplicity here, we show the convergence of curves with the same symmetry as the order of the approximation increases; the additional amplitudes are Φ_{20}, Φ_{30}, and Φ_{40}, respectively. For the functions $\log|v_{N+1}(R)-v_N(R)|$ presented in Fig. 4, $v_N(R)$ converges exponentially as N increases; this indicates that the "ideal" solution ($N=\infty$) is a smooth function of the parameter. The amplitudes $\Phi_{mn}(R)$ behave similarly and they do not have any additional peculiarities.

As we have already mentioned above, it follows from physical considerations that the "ideal" ($N=\infty$) one-parameter family of solutions should have two singular points. One point corresponds to a solitary jet (if the parameter is the radius of curvature of the bubble at the stagnation point, which is $R=\infty$). The other one, $R=R_{\rm cr}$, is an upper bound on the Froude number of order unity. This point is probably the termination point of the "ideal" solution with $N=\infty$. For finite values of N, however, as $R_{\rm cr}$ is approached, the functional convergence of the velocity and amplitudes with increasing order of the approximation worsens markedly. Analyzing in our case the functions $v(R)$ and $\Phi_{mn}(R)$, we obtain in the third-order approximation the following crical value of the parameter:

$$R_{\rm cr}=3.0\pm0.3, \quad v_{\rm cr}=0.9887\pm0.0093. \tag{5.1}$$

We emphasize that this value of the velocity agrees satisfactorily with the experimental data obtained in Ref. 8 for the ascent velocity of a bubble for which the ratios μ of the densities of the heavy top and light bottom liquids are $\mu=0.2$ with $v=0.92$ and $\mu=0.1$ with $v=0.93$.

The amplitudes averaged over all third-order curves at the crical point $R_{\rm cr}=3.1$ are.

$$\Phi_{10\rm cr}=(-0.4452\pm0.0502)v_{\rm cr},$$

$$\Phi_{11\rm cr}=(-0.0461\pm0.1601)v_{\rm cr},$$

$$\Phi_{20\rm cr}=(-0.0060\pm0.0036)v_{\rm cr},$$

$$\Phi_{21\rm cr}=(-0.0747\pm0.0467)v_{\rm cr},$$

$$\Phi_{30\rm cr}=(0.0160\pm0.0110)v_{\rm cr},$$

$$\Phi_{22\rm cr}=(-0.0126\pm0.0103)v_{\rm cr},$$

$$\Phi_{31\rm cr}=(-0.0055\pm0.0045)v_{\rm cr},$$

$$\Phi_{40\rm cr}=(-0.0013\pm0.0011)v_{\rm cr}, \tag{5.1a}$$

and the free surface is given by

$$z^*(x,y)=(-0.1687\pm0.0018)(x^2+y^2)+(-0.0082$$

$$\pm 0.0009)(x^4+y^4)+(0.0063$$
$$\pm 0.0045)x^2y^2+(-0.0010\pm 0.0002)(x^6$$
$$+y^6)+(0.0020\pm 0.0007)(x^4y^2+x^2y^4). \quad (5.1b)$$

For comparison, if the amplitude Φ_{40} is taken as the "additional" amplitude, then for $R=3.0$ we have

$$v_{cr}=1.0111, \quad \Phi_{10cr}=-0.5661v_{cr},$$
$$\Phi_{11cr}=0.4317v_{cr}, \quad \Phi_{20cr}=-0.0023v_{cr},$$
$$\Phi_{21cr}=-0.1840v_{cr}, \quad \Phi_{30cr}=0.0404v_{cr},$$
$$\Phi_{40cr}=-0.0039v_{cr},$$

and the free surface is given by

$$z^*(x,y)=-0.1667(x^2+y^2)-0.0115(x^4+y^4)$$
$$+0.0209x^2y^2-0.0018(x^6+y^6)$$
$$+0.0046(x^4y^2+x^2y^4).$$

Therefore, the condition (3.1) holds for $R \geqslant R_{cr}$.

The decrease in the absolute value of the amplitudes $\Phi_{mn}(R)$ with increasing amplitude number $m+n$ [condition (3.1a)] is most easily demonstrated by constructing for $R \geqslant R_{cr}$ in the leading third-order approximation the function $\log|\Phi_{mn}(R)/v(R)|$. Irrespective of the choice of the "additional" amplitude, the first amplitude $\Phi_{10}(R)$ is always largest in absolute value, and the $|\Phi_{mn}(R)|$ decrease exponentially with increasing $m+n$ (Fig. 5). In addition, on account of the symmetry of the "additional" amplitude (which generally determines the form of the free surface) the higher-order harmonics of different symmetry decay. For example, choosing $\Phi_{40}(R)$ as the "additional" amplitude, the absolute value of the amplitudes $\Phi_{l0}(R)$ decreases exponentially with increasing l (Fig. 5), and the amplitudes $\Phi_{11}(R)$ and $\Phi_{21}(R)$, in addition, are small compared to the amplitudes $\Phi_{20}(R)$ and $\Phi_{30}(R)$, respectively (Fig. 5). Similarly, choosing $\Phi_{31}(R)$ as the "additional" amplitude, the amplitudes $\Phi_{20}(R)$ and $\Phi_{30}(R)$ will be small compared to amplitudes of the same order.

Thus, the condition (3.1a) also holds for $R \geqslant R_{cr}$.

The fact that the decay of the higher-order harmonics depends on the symmetry of the additional amplitude means that even for radii of curvature $R \gg R_{cr}$, right up to values $R \to \infty$ [asymptotic behavior of the amplitudes in the limit $R \to \infty$ (Table I)].

The point $R=\infty$ is another finite point of the one-parameter solution and corresponds to a solitary jet.

In other words, in low orders of approximation a solution exists only for $R_{cr} \leqslant R \leqslant R_{max}$ (for very large values of R, the jet is localized in a small region, and this requires that a large number of harmonics be introduced to obtain a correct description of the free surface). In particular, the second-order approximation ($N=2$) is bounded by the values (Fig. 3)

$$R_{max} \approx 6, \quad v_{max}=1.2. \quad (5.2)$$

The solution for the solitary jet therefore cannot be obtained in the first three orders of approximation. However, because the amplitudes decay exponentially with increasing order of the harmonic, the velocity of steady flow defined as $v(R)=-\Sigma_{m,n}\Phi_{mn}(R)$ depends least on the order of the approximations. In the limit $R \to \infty$, the asymptotic velocities (Table I) therefore form a series which converges as N increases:

$$v_N(R) \underset{N\to\infty}{\to} C_N/\sqrt{R}$$

so that

$$\lim_{N\to\infty} C_N=4.$$

In addition, it can be expected that in the limit $R \to \infty$ the velocity of a solitary jet will approach zero as follows:

$$v(R)=\lim_{N\to\infty} v_N(R)=4/\sqrt{R}.$$

Completing our discussion of convergence for the one-parameter family of solutions (2.8a), we note that the zero-parameteric solutions obtained for this system in the first three orders of approximation obviously do not converge to a limit (or limits) with increasing order of approximation (Fig. 3).

Finally, we shall compare the basic features of our one-parameter family S for a three-dimensional C_4 "lattice" of bubbles and jets and the family F for a periodic plane flow[5] (Fig. 6).[1)] In both cases functional convergence of the families with respect to a parameter and the critical singular points (finite point of the solution and a solitary jet) exist. For the same wavelength λ of the disturbance, the bubble ascent velocity as a function of the radius of curvature is higher for a three-dimensional flow. The radii of curvature at the critical point in the three- and two-dimensional cases differ by the factor $R_{cr,3D}/R_{cr,2D} \approx 1.5$ and the velocity ratio is $v_{cr,3D}/v_{cr,2D} \approx 1.7$ (Fig. 6). In the limit $R \to \infty$ (solitary jet) the bubble ascent velocity approaches zero as $\approx 1/\sqrt{R}$: $v_{3D} \to 4/\sqrt{R}$ in the three-dimensional case and $v_{2D} \to 2/\sqrt{R}$ in the two dimensional case[5] (see footnote also).

6. CONCLUSIONS

It follows from what has been said above that the dimension of the stationary solution in the Rayleigh–Taylor instability is completely determined by the number of independent physical dimensional parameters. In the cases presented, in view of the symmetry of the problems, these parameters are g, λ, and v^*; the number of parameters is the same; and, the dimension of the set of stationary solutions is $1D$ for both two- and three-dimensional periodic flows.

The question of which solution from the continuum of solutions will be observable remains open. Garabedian,[4] who was the first to hypothesize the one-parameter nature of the family of solutions in the Rayleigh–Taylor instability, suggested that only one solution will be realized: the solution that corresponds to the maximum Froude number of the flow; all other solutions are in reality unstable. In accordance with Ref. 5 and the results presented in the present paper, this conjecture means that the observable shape of the free surface and bubble ascent velocity will be completely determined by the critical value of the parameter $[R_{\rm cr},\ v_{\rm cr}(R_{\rm cr})]$. This is also supported by the results of a computational experiment.[5,8] However, it has never been shown that only the "end" solution is stable and all others are unstable, and the problem of the stability of a one-parameter family has not, generally speaking, been studied.

The investigation of the evolution of the continuum of solutions obtained, the criterion of the stability of the solutions, and the choice of a particular solution are largely questions for future investigations of the classical problem of the Rayleigh–Taylor instability and the problem of why and how water pours out of an overturned cup.

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[1)]Besides the results already published in Ref. 5, Fig. 6 also contains new data on the velocity of two-dimensional steady flow as a function of the radius of curvature in very high (VI, VII, and VIII) orders of approximation; these were obtained on the basis of the formalism developed in Ref. 5.

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