Nonlinear Faraday effect for ultrashort pulses

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A theoretical study is made of Faraday rotation of the polarization plane of optical solitons in tenuous media. A "derivative nonlinear Schrödinger equation" (DNSE) is obtained for the electric field intensity of a pulse in the low-frequency nonresonance limit. It is shown that the direction of rotation of the polarization plane of a DNSE soliton is opposite that of a linear plane wave. With the help of the proposed method of analytic continuation of the dispersion parameters to the complex plane we obtain a general rule which allows one to determine the angle of rotation of the polarization plane per unit length for arbitrary relations between the frequency and duration of the optical soliton—from envelope pulses to video pulses—without solving the master system of nonlinear equations. An analysis is given of the Macaluso–Corbino soliton effect, which is expressed by nontrivial dependences of the polarization plane rotation angle of the pulse on its frequency and inverse length, when the magnitudes of the latter quantities lie in the vicinity of the resonance frequencies of the medium. © *1995 American Institute of Physics.*

1. INTRODUCTION

It is well known that as a electromagnetic plane wave propagates in a medium in the direction of an external magnetic field $\mathbf{B}_0$ it undergoes rotation of its polarization plane (the Faraday effect). In recent years more and more attention has been given to the nonlinear Faraday effect in high-intensity light fields, which depends strongly on the intensity of the latter.[1–4] The nonlinear Faraday effect has been studied both in solids and in gases. It has been investigated mainly in setups using continuous laser illumination. The use of pulsed radiation can manifest the characteristic features of the pulsed nonlinear Faraday effect, which are not revealed by continuous radiation.

The creation of coherent pulses of ever shorter duration is one indicator of progress in laser physics. At the present time, under experimental conditions it has been possible to generate femtosecond[5] and picosecond[6] pulses with duration of the order of one period of the electromagnetic oscillations (video pulses). Following these experiments, several theoretical papers have appeared, dedicated to the interaction of such pulses with matter.[7–11] The most recent papers have examined soliton regimes of propagation of video pulses, analyzed their amplification and self-compression in nonequilibrium media, and studied the parametric generation of higher harmonics of the initial signals. The concept of an envelope for video pulses loses meaning since in the theoretical treatment of their interaction with matter in the wave equations and constitutive relations the approximation of slowly varying amplitudes and phases is inapplicable.[12] In such cases use has been made of the approximation of low density of the optically active atoms[13] and also very short or very long pulses (in comparison with the atomic periods).[8,9] The use of the latter approximations allows one to significantly simplify the theoretical investigation of the nonlinear problem.

The present paper is dedicated to a theoretical study of the pulsed Faraday effect for various splittings of the atomic lines in external magnetic fields. The cases of the normal and anomalous Zeeman effect, and also the Paschen–Back effect, are considered from a common standpoint. The most detailed analysis is given to the Faraday effect in pulses containing a small number of oscillations. In Sec. 2 the master system of Maxwell equations and constitutive relations is written down for the probability amplitudes of occupation of the Zeeman sublevels in the dipole approximation. Here account is taken of the optical transitions which form the main spectral series. In Sec. 3 on the basis of the low-frequency nonresonance approximation for the complex electric field intensity we obtain the "derivative nonlinear Schrödinger equation," which belongs to the class of equations that can be integrated by the inverse scattering transform method. On the basis of its single-soliton solution we carry out a detailed analysis of the pulsed Faraday effect in the low-frequency region. Section 4 is dedicated to a comparison of the pulsed Faraday effect and its linear analog. There an analysis is also given of the modulational instability of a plane wave in a nonlinear medium, which leads to the formation of solitons. Section 5 is concerned with methodology. The case of ultrashort pulses, investigated there, provides an additional argument for the method of analytic continuation of the dispersion parameters into the complex plane, which is proposed in Sec. 6. The latter method allows one to formulate a combination rule for finding the rotation angle per unit length for arbitrary frequencies and durations of an optical pulse. Section 7 presents an analysis of the Macaluso–Corbino effect, which has to do with the dependence of the Faraday rotation angle on frequency and pulse duration. The final section summarizes results and presents some generalizing conclusions.

2. THE MASTER EQUATIONS

Let us examine the Faraday effect for ultrashort pulses in gaseous media, not resorting to the slowly varying amplitude

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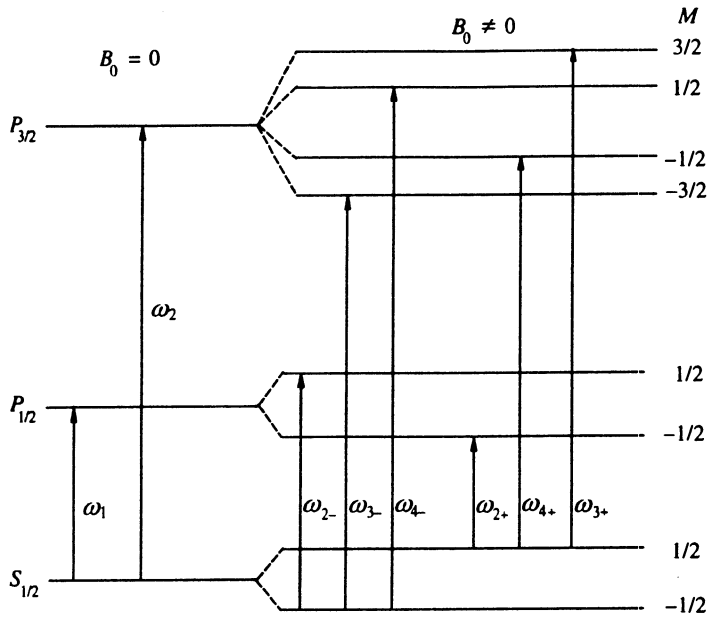

FIG. 1. Optical σ-transitions in the presence of anomalous Zeeman splitting for the main spectral series of the sodium atom.

and phase approximation. As the gaseous media, we will consider rarefied vapors of sodium and mercury. We will make use of the energy-level diagrams of the main spectral series of the sodium atom (the so-called D-lines with wavelengths 5890 and 5896 Å, Ref. 12) under conditions of anomalous Zeeman splitting (AZS) (Fig. 1) and the Paschen–Back effect (PBE) (Fig. 2). For completeness, we will also consider normal Zeeman splitting (NZS) (Fig. 3). The triplet splitting of the familiar optical line 2537 Å of mercury vapor[14] can serve as an example of the latter.

A study of the pulsed Faraday effect in gaseous media differs significantly from such a study in solid media for two main reasons. First, in gases the optical spectra of the atoms differ practically not at all from the spectra of the isolated atoms. Second, inhomogeneous broadening in gases is significantly less than in solids. In gases the lines of the Zeeman splittings are usually easy to distinguish. In solids, on the contrary, the inhomogeneous broadening of spectral lines significantly exceeds the distance between the Zeeman sublevels.[14] Usually, inhomogeneous broadening in gases is about 10% of the distance on the frequency scale between the nearest Zeeman components. If the frequency offset of the optical signal from the resonance line exceeds the magnitude of the inhomogeneous broadening, then the latter can be neglected in the zeroth approximation, which is what we will do. In this regard, cases in which inhomogeneous broadening has no qualitative effect on the dynamics of the pulses, whether in gases or in solids, will be noted below.

In the one-dimensional propagation of an electromagnetic wave along the $\mathbf{B}_0$ direction (along the z axis), only the σ transitions ($M_j - M_k = \pm 1$, where M_j is the projection of the total angular momentum of the atom in the jth state onto the z axis) contribute to the dynamics of the electric dipole interaction.[15] Thus, in the case of anomalous Zeeman splitting and the Paschen–Back effect, the Hilbert space of the atomic states, which is mapped into itself by electric dipole transitions, divides into two independent (invariant) subspaces. Within the first take place transitions from the lower Zeeman sublevel of the s state, and within the second take place transitions from the upper sublevel of the same state

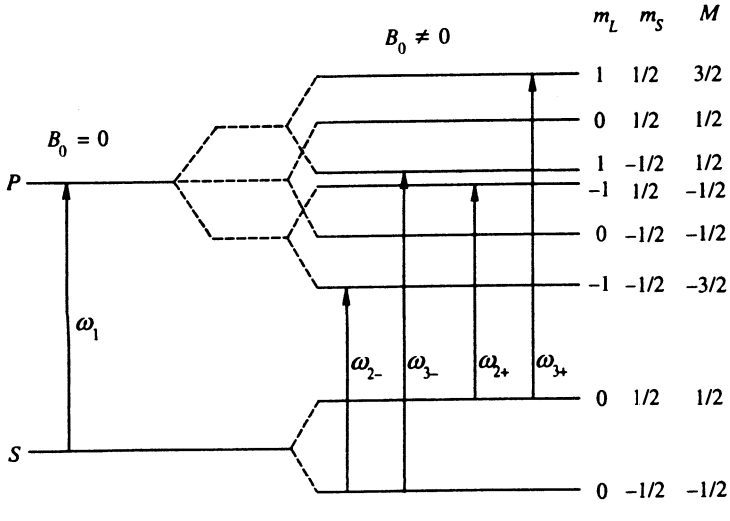

FIG. 2. Optical σ-transitions in the presence of the Paschen-Back effect for the sodium atom; $m_{S(L)}$ is the projection of the spin (orbital) angular momentum on the z axis: $\omega_{2-} = \omega_{2+}$, $\omega_{3-} = \omega_{3+}$.

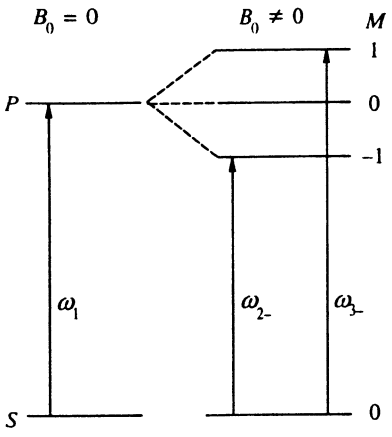

FIG. 3. Optical σ-transitions in the presence of normal Zeeman splitting for the mercury atom.

(Figs. 1 and 2). In the case of normal Zeeman splitting we have one invariant subspace, which coincides with the initial Hilbert space.

The Schrödinger equation for the probability amplitudes $a_j^{\pm}$ (the minus sign refers to the first subspace, and the plus sign refers to the second) is written in the form of a system of equations:

$$\frac{\partial b_1^{\pm}}{\partial t} = \frac{i}{\hbar} \sum_{j=2}^{N} (\mathbf{d}_{j\pm}\mathbf{E}) b_j^{\pm}, \tag{1}$$

$$\frac{\partial b_j^{\pm}}{\partial t} = -i\omega_{j\pm} b_j^{\pm} + \frac{i}{\hbar} (d_{j\pm}E) b_1^{\pm}, \quad j=2,\ldots,N. \tag{2}$$

Here $b_j^{\pm} = a_j^{\pm} \exp(-i\epsilon_1^{\pm} t/\hbar)$, $\epsilon_1^{\pm}$ is the energy of the Zeeman sublevel of the S-state, $\omega_{j\pm}$ and $d_{j\pm}$ are, respectively, the frequency and matrix element of the dipole moment of the quantum transition from the sublevel of the S-state to the jth state, $\mathbf{E}$ is the electric field vector of the pulse, and N is the number of levels in the invariant subspace (for AZS, PBE, and NZS, $N=4$, 3, 3, respectively). Below we allow for the fact that the σ transitions, which form symmetric Zeeman spectral lines relative to the unperturbed transitions, possess mutually complex-conjugate dipole moments with real and imaginary parts which are perpendicular and equal in magnitude.[15] Consequently, $|\mathbf{d}_{j+}| = |\mathbf{d}_{j-}| = d_{j1}$ (see Figs. 1, 2, 3). The expression for the dipole moment $\mathbf{d}_{jk}$ of the quantum transition between the jth and kth levels can be written in the form

$$\mathbf{d}_{jk} = \frac{1}{\sqrt{2}} d_{jk}[\mathbf{e}_x + i(M_j - M_k)\mathbf{e}_y], \tag{3}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit basis vectors of the corresponding axes of the Cartesian coordinate system. Here the kth level is assumed to be the ground level. Applying Eq. (3) to the quantum transition schemes depicted in Figs. 1, 2, and 3, we find for anomalous Zeeman splitting

$$\mathbf{d}_{2-} = d_{21}\mathbf{l}, \quad \mathbf{d}_{3-} = d_{31}\mathbf{l}^*, \quad \mathbf{d}_{4-} = d_{41}\mathbf{l},$$

$$\mathbf{d}_{2+} = d_{21}\mathbf{l}^*, \quad \mathbf{d}_{3+} = d_{31}\mathbf{l}, \quad \mathbf{d}_{4+} = d_{41}\mathbf{l}^*;$$

for the Paschen–Back effect

$$\mathbf{d}_{2-} = \mathbf{d}_{2+} = d_{21}\mathbf{l}^*, \quad \mathbf{d}_{3-} = \mathbf{d}_{3+} = d_{21}\mathbf{l};$$

and for normal Zeeman splitting

$$\mathbf{d}_{2-} = d_{21}\mathbf{l}^*, \quad \mathbf{d}_{3-} = d_{21}\mathbf{l}.$$

Here we have introduced the unit complex vector $\mathbf{l} = (\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}$.

We augment system (1), (2) by the Maxwell equation

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{4\pi n}{c^2}\frac{\partial^2}{\partial t^2} \sum_{k=+,-} \sum_{j=2}^{N} \mathbf{d}_{jk} b_j^{k*} b_1^k + \text{c.c.}, \tag{4}$$

where c is the speed of light and n is the concentration of the interesting atoms.

Below we assume that prior to the action of the pulse upon the medium, only the sublevels of the S-state that differ one from the other by the Larmor precession frequency $\omega_B = eB_0/mc$ have nonzero occupation probability. Here e and m are the charge and mass of the electron, respectively. If we assume that the medium is initially in thermodynamic equilibrium, then the given assumption is valid for temperatures $T \ll \hbar\omega_{1,2}/k_B$, where k_B is the Boltzmann constant (see Figs. 1–3). Setting $\omega_{1,2} \sim 10^{15}$ s^{-1}, we obtain $T \ll 10^4$ K. The latter condition is usually easily satisfied for gaseous media.

3. LOW-FREQUENCY NONRESONANCE PULSES

Following Refs. 8 and 9, we will examine system (1)–(4) first in the approximation

$$\mu^{-1} \equiv \omega_{1,2}\tau_{\min} \gg 1, \tag{5}$$

where $\tau_{\min}$ is the characteristic minimum time scale of the pulse. In this case, the solution of system (2) can be sought by the method of successive approximations in the small parameter μ, which is proportional to the derivative on the left-hand side of Eq. (2). In the zeroth approximation we can neglect the left-hand side of Eq. (2). We can then take it into account in the second approximation, etc. As a result, we obtain the expansion

$$b_j^{\pm} = \frac{1}{\hbar\omega_{j\pm}} (\mathbf{d}_{j\pm}\mathbf{E}) b_1^{\pm} + \frac{i}{\hbar\omega_{j\pm}^2}\frac{\partial}{\partial t}[(\mathbf{d}_{j\pm}\mathbf{E}) b_1^{\pm}] + \ldots \tag{6}$$

If we restrict ourselves to the first term (6), we will have a local temporal coupling between the dynamical parameters of the pulse and the medium. This locality corresponds to the nondispersive approximation. We set $b_1^{\pm} = b_{10}^{\pm}$ in the second term of the right-hand side of Eq. (6), where $b_{10}^{\pm}$ is the value of $b_1^{\pm}$ before the laser pulse enters the medium. The meaning of approximation (5) is that variations of the pulse parameters should be slow in comparison with the characteristic frequencies of the atomic processes. The pulse interacts weakly with the medium, exciting it only insignificantly. As a result, the amplitudes $b_1^{\pm}$ are "deformed" only weakly as a result of the action of the pulse. Thus, under condition (5) the interaction of the laser signal with the medium can be considered weakly nonlinear. Since the second term of expansion (6) is of a higher order of smallness than the first, the above-mentioned substitution satisfies condition (5).

We will give further attention to the integrals of the motion following from Eqs. (1) and (2) (the conservation laws of total probability in the invariant subspaces of the atomic states):

$$\sum_{j=1}^{N} |b_j^{\pm}|^2 = W_1^{\pm}, \quad W_1^{+} + W_1^{-} = 1, \tag{7}$$

where W_1^{+} and W_1^{-} are the probabilities of initial occupation of the upper and lower Zeeman sublevels of the electronic S-state, respectively. In normal Zeeman splitting we have only one level corresponding to the s-state. Invoking Eqs. (6) and (7), we find

$$|b_1^{\pm}|^2 = |b_{10}^{\pm}|^2 - \sum_{j=2}^{N} |b_j^{\pm}|^2 \approx W_1^{\pm}\left[1 - \frac{1}{2}\sum_{j=2}^{N}\left(\frac{d_{j1}}{\hbar\omega_{j\pm}}\right)^2 |\psi|^2\right], \tag{8}$$

where $\psi = E_x + iE_y$. Substituting Eqs. (6) and (8) on the right-hand side of Eq. (4), after multiplying by $\mathbf{l}$ we have

$$\frac{\partial^2\psi}{\partial z^2} - \frac{N_0^2}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{2N_0}{c}\frac{\partial^2}{\partial t^2}\left[i\alpha(0,0)\frac{\partial\psi}{\partial t} - \beta|\psi|^2\psi\right]. \tag{9}$$

Here we have introduced the static index of refraction

$$N_0 = \left(1 + \frac{4\pi n}{\hbar}\sum_{k=+,-}\sum_{j=2}^{N}\frac{d_{j1}^2}{\omega_{jk}}W_j^k\right)^{1/2}, \tag{10}$$

the low-frequency dispersion coefficient

$$\alpha(0,0) = \frac{2\pi n}{\hbar c N_0}\sum_{k=+,-}\sum_{j=2}^{N}(M_1^k - M_j^k)\left(\frac{d_{j1}}{\omega_{jk}}\right)^2 W_1^k \tag{11}$$

and the nonlinearity

$$\beta = \frac{\pi n}{\hbar^3 c N_0}\sum_{k=+,-}\sum_{j=2}^{N}\frac{d_{j1}^2}{\omega_{jk}}\sum_{l=2}^{N}\left(\frac{d_{l1}}{\omega_{lk}}\right)^2 W_1^k. \tag{12}$$

The meaning of the appearance of the zeros as the arguments of the dispersion coefficient will become clear in Sec. 6.

The right-hand side of Eq. (9) contains dispersion and nonlinear terms whose order of smallness relative to the left-hand side of Eq. (9) is $\sim\mu$. Therefore it is natural to use the approximation of unidirectional pulse propagation with velocity near c/N_0. Toward this end, we introduce the local time $\tau = t - N_0 z/c$ and "slow" spatial coordinate $\zeta = \mu z$. Then, neglecting powers of μ higher than the second, we obtain $\partial/\partial t = \partial/\partial\tau$ $\partial^2/\partial z^2 \approx (N_0/c)^2\partial^2/\partial\tau^2 - (2\mu N_0/c)\partial^2/\partial\zeta\partial\tau$. As a result, the given formal approach reduces to a reduction in the order of the derivative in Eq. (9). Transforming back from the slow coordinate ζ to z, we find after integrating Eq. (9) with respect to τ

$$i\frac{\partial\psi}{\partial z} = -\alpha(0,0)\frac{\partial^2\psi}{\partial\tau^2} + i\beta\frac{\partial}{\partial\tau}(|\psi|^2\psi). \tag{13}$$

Equation (13) is known as the "derivative nonlinear Schrödinger equation." (DNSE)[16] In some articles it is called the "modified nonlinear Schrödinger equation."[17,18] The DNSE, as was shown in Ref. 16, can be integrated by the inverse scattering transform method. Its two-soliton solution was obtained in Ref. 19, and its multisoliton solution, in Ref. 20.

Let us find here its single-soliton solution in a form convenient for physical applications. It should describe a pulse with rotating polarization plane. With this goal in mind, we search for the solution in the form

$$\psi = F(\tau - z/b)\exp\{i[\kappa_s z + \varphi(\tau - z/b)]\}, \tag{14}$$

where κ_s and b are parameters. After substituting expression (14) in Eq. (13) and separating the real part from the imaginary, we obtain the system

$$\frac{1}{b}F' = -\alpha(0,0)(2F'\varphi' + F\varphi'') + 3\beta F^2 F', \tag{15}$$

$$\alpha(0,0)(F'' - F\varphi'^2) + \beta F^3 F' - \left(\kappa_s - \frac{1}{b}\varphi'\right)F = 0. \tag{16}$$

Here the prime denotes the derivative with respect to the variable $\tau - z/b$. We now multiply Eq. (15) by F. After integrating, we find, assuming that $F \to 0$ as $\tau - z/b \to \pm\infty$,

$$\varphi' = \frac{1}{2\alpha(0,0)}\left(\frac{1}{b} + \frac{3}{2}\beta F^2\right). \tag{17}$$

Substituting Eq. (17) in Eq. (16), we obtain an equation for F:

$$F'' = \frac{1}{\alpha(0,0)}\left(\kappa_s - \frac{1}{4\beta b^2}\right)F - \frac{\beta}{2\alpha^2(0,0)b}F^3 - \frac{3\beta^2}{16\alpha^2(0,0)}F^5. \tag{18}$$

Integrating Eq. (18) and then invoking Eqs. (17) and (14), we find after converting back to the laboratory frame

$$\psi = |\psi_m|\left[\frac{1 + \sqrt{1 + (\omega\tau_p)^{-2}}}{1 + \sqrt{1 + (\omega\tau_p)^{-2}}\,\mathrm{ch}\left(2\,\dfrac{t - z/v}{\tau_p}\right)}\right]^{1/2} \times \exp\left\{i\left[\kappa_s z + \omega(t - z/v) + 6\arctan\left(\delta\,\mathrm{th}\,\frac{t - z/v}{\tau_p}\right)\right]\right\}, \tag{19}$$

where

$$|\psi_m| = \frac{1}{\omega\tau_p}\left[\frac{\alpha(0,0)\omega/\beta}{1 + \sqrt{1 + (\omega\tau_p)^{-2}}}\right]^{1/2}, \tag{20}$$

$$\kappa_s = \alpha(0,0)(\omega^2 + \tau_p^{-2}), \tag{21}$$

$$\frac{1}{v} = \frac{N}{c} + 2\alpha(0,0)\omega, \tag{22}$$

$$\delta = \frac{\sqrt{1 + (\omega\tau_p)^{-2}} - 1}{\sqrt{1 + (\omega\tau_p)^{-2}} + 1}. \tag{23}$$

A DNSE soliton has two parameters. We choose the frequency ω and the pulse duration τ_p as the free parameters. For $\omega\tau_p \gg 1$ we have an envelope soliton—an isolated pulse, containing within itself many periods of the optical wave. It is obvious that the slowly varying amplitude and phase approximation is valid in this case. Thus, it is easy to obtain the nonlinear Schrödinger equation (NSE) from Eq. (13) for the complex envelope, and the soliton (19)–(23) transforms into the well-known NSE soliton. If $\omega\tau_p \lesssim 1$, solution (19)–(23) describes a video soliton—a pulse containing roughly one period. Since $\beta>0$, it follows from Eq. (19) that $\alpha(0,0)\omega>0$ (the sign of ω determines the direction of rotation of the polarization plane of the pulse for fixed z). Thus, as follows from Eq. (22), $v<c/N_0$. In addition, the pulse velocity v, as in the case of an NSE soliton, does not depend on the amplitude of the pulse (duration τ_p), but is determined only by its frequency. The Faraday rotation angle $\Delta\varphi_s$ of the soliton (19)–(23) after the soliton has traversed a distance Δz, as is evident from Eqs. (19) and (21), is given by

$$\Delta\varphi_s=\kappa_s\Delta z, \tag{24}$$

where the proportionality coefficient between $\Delta\varphi_s$ and Δz is given by relation (21). Invoking Eq. (11), we can write the coefficient $\alpha(0,0)$ in the form

$$\alpha(0,0)=AB_0+CM_0, \tag{25}$$

The first term here describes the diamagnetic contribution, and the second term, the paramagnetic contribution to the Faraday effect for a pulse, $M_0=2\mu_B n(W_1^- - W_1^+)$ is the magnetization of the medium, $\mu_B=e\hbar/2mc$ is the Bohr magneton. In the Paschen–Back case, when the intra-atomic spin–orbit coupling is broken, we have $C=0$. This is explained in the following way. The main atomic states are the s-states of the Zeeman sublevels. In these states paramagnetism of the atom is due to the electron spin. When the spin–orbit coupling is broken, the optical transitions are insensitive to the orientation of the electron spins and, consequently, to the magnetization of the medium. Therefore in strong magnetic fields only the diamagnetic contribution to the Faraday effect is nonzero. For sodium vapor, as for mercury vapor (the case of normal Zeeman splitting), we have $A=8\pi d_{21}^2 e/(\hbar c^2 m\omega_1^3 N_{00})$. For anomalous Zeeman splitting both contributions are nonzero. For sodium vapor we obtain

$$A=\frac{4\pi ne}{\hbar c^2 m\omega_1^3 N_{00}}\left(\frac{4}{3}d_{21}^2+d_{31}^2+\frac{5}{3}d_{41}^2\right), \tag{26}$$

$$C=\frac{4\pi m}{e\hbar\omega_1^2 N_{00}}(d_{21}^2-d_{31}^2+d_{41}^2). \tag{27}$$

Here $N_{00}=[1+(8\pi n/\hbar\omega_1)(d_{21}^2+d_{31}^2+d_{41}^2)]^{1/2}$ is the refractive index for $\mathbf{B}_0=0$. Here it has been taken into account that the frequency separation between the D-lines of a sodium atom is $\omega_2-\omega_1 \ll \omega_1 \sim 10^{15}\ \mathrm{s}^{-1}$, so we can take $\omega_2 \approx \omega_1$. In addition, we have taken into account that $\omega_{2\pm}=\omega_1 \mp (4/3)\omega_B$, $\omega_{3\pm}=\omega_2 \pm \omega_B$, $\omega_{4\pm}=\omega_2 \mp (5/3)\omega_B$ (Ref. 15). The quantities d_{31}^2 and d_{41}^2 can be expressed in terms of d_{21}^2 according to formulas which can be found on p. 201 of Ref. 15. Hence we obtain $d_{31}^2=6d_{21}^2$ and $d_{41}^2=2d_{21}^2$.

If the medium is in a state of thermodynamic equilibrium before the pulse enters it, then

$$M_0=2\mu_B n\ \mathrm{th}\frac{\mu_B B_0}{k_B T}. \tag{28}$$

For gases and vapors $T \gg \mu_B B_0/k_B \sim 1$ K and $M_0 \approx 2\mu_B^2 n B_0/k_B T$. Hence, and from relation (25), we obtain Verde's soliton law

$$\Delta\varphi_s=R_s B_0\Delta z, \tag{29}$$

where the Verde soliton constant R_s is

$$R_s=\left(A+\frac{2\mu_B^2 n}{k_B T}C\right)(\omega^2+\tau_p^{-2}). \tag{30}$$

From Eqs. (26), (27), and (30) it is easy to see that the ratio of the paramagnetic contribution to the Faraday effect to the diamagnetic is equal to $\hbar\omega_1/k_B T \gg 1$.

At present there are a number of ways to achieve resonance optical magnetization of media,[14] including gases.[21] These methods are based on the selective ability of resonance atomic transitions to absorb light of fixed polarization. As a result, after the various optical relaxation processes have run their course, a nonequilibrium distribution of populations of the Zeeman sublevels of the S-state is formed, optical transitions between which are forbidden. Thus it is possible to achieve the result that an overwhelming majority of the atoms at some time will have accumulated in only one of the sublevels of the S-state, leading to saturation of the magnetization of the medium. It is clear that this, in turn, leads to a violation of Verde's soliton law (29) and to a significant enhancement of the Faraday effect.

4. CONNECTION WITH THE LINEAR FARADAY EFFECT

Let us linearize Eq. (13). In this case one of its solutions will be a plane wave of the form

$$\psi \propto \exp[i\kappa_{lin}z+i\omega(t-N_0 z/c)], \tag{31}$$

where

$$\kappa_{lin}=-\alpha(0,0)\omega^2. \tag{32}$$

Relations (31) and (32) tell us that the linear low-frequency ($\omega \ll \omega_{1,2}$) plane wave propagating with phase velocity c/N_0 experiences Faraday rotation. The rotation angle of its polarization plane is given by the expression

$$\Delta\varphi_{lin}=-\alpha(0,0)\omega^2\Delta z.$$

Comparing this formula with expressions (24) and (21) reveals the connection between the Faraday rotation angle of an optical soliton $\Delta\varphi_s$ and of a linear plane wave $\Delta\varphi_{lin}$ of identical frequencies when propagating in the same medium:

$$\Delta\varphi_s=-[1+(\omega\tau_p)^{-2}]\Delta\varphi_{lin}. \tag{33}$$

The minus sign in front of the brackets indicates that the direction of rotation of the polarization plane of the optical soliton is opposite that of the linear plane wave. In addition, the shorter the pulse for a given frequency, the more strongly will the soliton Faraday effect be manifested. In the case of envelope solitons, for which $\omega\tau_p \gg 1$, the difference from the linear effect consists only in the sign. Thus, for a quantitative

description of the nonlinear pulsed Faraday effect given condition (5), it is sufficient to know a quantitative characteristic of the corresponding linear effect and then invoke formula (33). Here it is important that the anti-renormalization factor in front of $\Delta\varphi_{\text{lin}}$ in relation (33) depends only on the pulse parameters and not those of the medium. This allows us to determine $\Delta\varphi_s$ without having to solve the nonlinear constitutive relations. The values of $\Delta\varphi_{\text{lin}}$ can be taken, for example, from experiments in the linear region.

Let us trace out how a wave train of frequency $\omega \ll \omega_{1,2}$ after it has entered the medium in question can be transformed into a soliton of the type (19)–(23). In this case, linearization of Eq. (13) is unavailable. Equation (13), like its linearized variant, admits a solution in the form of a plane wave (31), (32), up to the substitution

$$\kappa_{\text{lin}} \to \kappa = -\alpha(0,0)\omega^2 + \beta\omega a_\omega^2, \tag{34}$$

where a_ω is the amplitude of the plane wave. Here the phase can be represented in the form

$$\varphi = \omega t - [\omega N_0/c + \alpha(0,0)\omega^2 - \beta\omega a_\omega^2]z.$$

Obviously, the expression in brackets plays the role of the wave number k. Hence we can find the dependence $\omega = \omega(k, a_\omega^2)$. Noting that the two last terms inside the brackets are of a higher order of smallness than the first, the given dependence can be found by the method of successive approximations. As a result, we obtain the following nonlinear dispersion relation:

$$\omega(k, a_\omega^2) = \frac{c}{N_0}k - \left(\frac{c}{N_0}\right)^2\left[\alpha(0,0)\frac{c}{N_0}k^2 - \beta k a_\omega^2\right],$$

to which we can apply the Lighthill criterion for the stability of a nonlinear plane wave[22]

$$\left(\frac{\partial\omega}{\partial a_\omega^2}\right) \Big/ \left(\frac{\partial^2\omega}{\partial k^2}\right) > 0. \tag{35}$$

Application of criterion (35) to Eq. (34) leads to the requirement that $\alpha(0,0)\omega/\beta$ be negative. Since in our case $\beta > 0$ [see Eq. (12)], the stability criterion can be written in the form $\alpha(0,0)\omega < 0$. In the opposite case the plane wave experiences a modulational instability, breaking up into solitons. Earlier, we already spoke about the fact that a soliton of the type (19)–(23) under the condition $\beta > 0$ satisfies the inequality $\alpha(0,0)\omega > 0$. It follows from what has been said that this type of soliton can be formed in a nonlinear medium as a result of the modulational instability of the initial wave train.

The physical meaning of inequality $\alpha(0,0)\omega/\beta > 0$ is that the angular momentum of the pulse is conserved. The direction of the angular momentum of the electromagnetic pulse is determined by the direction of the Faraday rotation of the polarization plane. As a result of interaction with the atomic σ-transitions, the optical wave acquires a nonzero spin. It was shown above that given condition (5), the directions of the Faraday rotation for a linear plane wave and an optical soliton are opposite. How then does the breakup of the initial train of plane waves into solitons take place, with the momentum conservation law for the pulse remaining in force? If the inequality $\alpha(0,0)\omega/\beta > 0$ is satisfied, the first and second terms on the right-hand side of Eq. (34) have different signs. In the initial wave packet can be found Fourier components for which $a_\omega^2 > \alpha(0,0)\omega/\beta$. Then the second term prevails over the first, and the sign of κ in Eq. (34) becomes opposite that of κ_{lin}. As a result, the direction of the angular momentum of the given Fourier components is opposite to that for a linear wave pulse. From these components (19–23) is then formed a DNSE soliton.

Let a wave packet of duration τ_0 and spectral width $\Delta\omega \sim \tau_0^{-1}$ be incident upon the boundary of the nonlinear medium under consideration. For this case we can write the threshold condition $|\psi|^2_{\max}(z=0) > |\psi|^2_{\text{th}} \sim \alpha(0,0)/\beta\tau_0$, where $|\psi|_{\text{th}}$ is the minimum value of the amplitude upon entrance to the medium for which a soliton can be formed. Let us estimate the value of $|\psi|_{\text{th}}$ for magnetized sodium vapor. Employing Eqs. (11) and (12), we find that $|\psi|_{\text{th}} \sim \hbar\omega_1/(d_{21}\sqrt{\omega_1\tau_0})$. Setting $\omega_1\tau_0 \sim 10$, $\omega_1 \sim 10^{15}$ s^{-1}, $d_{21} \sim 10^{-18}$ cgs Oe units, we find that $|\psi|_{\text{th}} \sim 5\cdot 10^7$ V/cm. The given field strengths correspond to intensities $I \sim 10^{12}$ W/cm^2, achievable at present with CO_2 lasers. Note that in the case of NSE solitons there also exists a minimum threshold amplitude at the medium boundary:[23] $|\psi|_{\text{th}} \sim \tau_0^{-1}$. It is easy to show that the latter condition can be obtained, like the corresponding condition for the DNSE case, via physical arguments based on the conservation of angular momentum of the pulse.

With the development of the modulational instability a packet of light energy in the form of a soliton begins to lag behind the low-amplitude plane waves moving ahead with velocity c/n_0. These low-amplitude waves are precursors of the soliton. According to Eq. (33), the direction of the Faraday rotation of the precursors should be opposite that of the soliton.

Employing Eqs. (33) and (20), we can write down an expression for $\Delta\varphi_s$ in terms of the intensity, $I = v|\psi_m|^2/4\pi$, of the pulse:

$$\Delta\varphi_s = -\left[1 + \frac{4\pi\beta}{\alpha(0,0)\omega v}I\right]^2\Delta\varphi_{\text{lin}}, \tag{36}$$

where $1/v$ is given by Eq. (22) and, as was noted above, depends only on the frequency of the signal—not on its intensity.

Expression (36), like solution (19)–(23), does not go over to the linear case when $\beta = 0$. This is because the soliton solutions are nonperturbative, i.e., they are fundamentally nonlinear, and depend nonanalytically on the nonlinearity parameter β as $\beta \to 0$.

Let us make some numerical estimates. Let a video soliton, for which $\omega\tau_p \sim 1$, propagate in strongly magnetized sodium vapor (anomalous Zeeman splitting). Then its intensity

$$I \approx \frac{c}{4\pi}|\psi_m|^2 \sim \frac{c}{4\pi}\frac{\alpha(0,0)}{\beta}\omega \sim \left(\frac{\hbar}{d_{21}}\right)^2\frac{c\omega_1}{4\pi}\omega.$$

Setting $\omega_1 \sim 10^{15}$ s^{-1}, $\omega \sim 10^{14}$ s^{-1}, and $d_{21} \sim 10^{-18}$ cgs Oe units, we find that $I \sim 10^{13}$ W/cm^2. In obtaining this estimate, we assumed that the magnetization of the vapor is near saturation: $M_0 \simeq 2\mu_B n$ and the refractive index is essentially equal to unity. We will now estimate the Faraday rotation angle per unit length of the video soliton:

$$\kappa_s \equiv \frac{\Delta\varphi_s}{\Delta z} = CM_0\omega^2 = \frac{24\pi d_{21}^2 n}{\hbar c}\left(\frac{\omega}{\omega_1}\right)^2.$$

Setting $n \sim 10^{17}$ cm^{-3} and $\omega/\omega_1 \sim 0.1$, we find that $\kappa_s \sim 10^{-3}$ rad/cm. Similar estimates show that in the Paschen–Back effect and normal Zeeman splitting, $I \sim 10^{10}$ W/cm^2, and κ_s is $\omega_1/\omega_B \sim 10^3$–$10^4$ times smaller than for anomalous Zeeman splitting.

5. ULTRASHORT PULSES

Let us now consider the case[7–9]

$$\omega_{1,2}\tau_{\max} \ll 1, \tag{37}$$

where $\tau_{\max}$ is the maximum time scale of the pulse. Note that the energy $\hbar\omega_{1,2}$ is of the same order of magnitude as the ionization energy of the atom. Therefore, for condition (37) it is necessary to take ionization processes into account. Strictly speaking, under condition (37), for optical frequencies such processes should play a dominant role, and for a correct solution of the problem it is fundamentally necessary to take them into account. However, in our case the use of approximation (37) has rather a methodological significance. It is the limiting case opposite to Eq. (5), upon which we will base the method of analytic continuation of the dispersion parameters to the complex plane in the following section. Therefore we will not take into account transitions to the continuous spectrum, and will remain within the framework of the model proposed in Sec. 2.

Following Ref. 24 and generalizing the treatment given there, we write

$$\psi = |\psi|\exp(i\varphi), \quad b_1^{\pm} = |b_1^{\pm}|,$$
$$b_j^{\pm} = i|b_j^{\pm}|\exp[i(M_i^{\pm} - M_j^{\pm})\varphi], \quad j = 2,\ldots,N. \tag{38}$$

The idea behind approximation (38) is that the phase of the atomic dipole moment tracks the phase of the electric field of the pulse, but shifted by $\pi/2$. In this case system (1)–(4) can be solved by the method of successive approximations in the parameter $\omega_{1,2}\tau_{\max}$ (Refs. 10 and 11). Substituting Eqs. (38) into Eqs. (1) and (2), taking account of relation (3), we find in the zeroth approximation

$$b_1^{\pm} = |b_{10}^{\pm}|\cos\,\theta/2, \tag{39}$$

$$b_j^{\pm} = id_{j1}\left(\sum_{l=2}^{N} d_{l1}^2\right)^{-1/2} \exp[i(M_1^{\pm} - M_j^{\pm})\varphi]\sin\,\theta/2, \tag{40}$$

where

$$\theta = 2\hbar^{-1}\left(\sum_{l=2}^{N} d_{l1}^2\right)^{1/2} \int_{-\infty}^{t} |\psi(z,t')|dt'.$$

After substituting Eqs. (39) and (40) in the right-hand sides of Eqs. (1) and (2), we obtain expressions in the first approximation for the time derivatives of the atomic amplitudes. Employing these expressions, we find after simple but cumbersome calculations

$$\mathbf{l}\frac{\partial}{\partial t}\sum_{k=+,-}\sum_{j=2}^{N}(\mathbf{d}_{jk}b_j^{k*}b_1^k + \text{c.c.})$$
$$= \exp(i\varphi)\left(\sum_{l=2}^{N} d_{l1}^2\right)^{-1/2}\frac{\hbar c}{\pi n}\left(R\,\sin\,\theta + i\kappa_\infty\cos\,\theta\frac{\partial\theta}{\partial t}\right), \tag{41}$$

where

$$R = \frac{\pi n}{\hbar c}\sum_{k=+,-}\sum_{j=2}^{N} W_1^k d_{jk}^2\omega_{jk}, \tag{42}$$

$$\kappa_\infty = \frac{2\pi n}{\hbar c}\sum_{k=+,-}\sum_{j=2}^{N}(M_1^k - M_j^k)d_{jk}^2 W_1^k. \tag{43}$$

After substituting Eq. (41) in Eq. (4) and using the unidirectional propagation approximation (see Sec. 3), we obtain

$$\frac{\partial^2\theta}{\partial z\partial\tau} = -R\,\sin\,\theta, \tag{44}$$

$$\frac{\partial\varphi}{\partial z} = \kappa_\infty\cos\,\theta. \tag{45}$$

Here $\tau = t - z/c$. Employing the one-soliton solution of the sine–Gordon equation (44)

$$|\psi| = \frac{\hbar}{\tau_p}\left(\sum_{l=2}^{N} d_{l1}^2\right)^{-1/2}\text{sech}\,\frac{t - z/v}{\tau_p}, \tag{46}$$

we find that

$$\varphi = \kappa_\infty z + 2\kappa_\infty v\tau_p\,\text{th}\,\frac{t - z/v}{\tau_p}, \tag{47}$$

where $v^{-1} = c^{-1} + R\tau_p^2$.

From Eq. (47) it follows that κ_∞ is the Faraday rotation angle per unit length of the video soliton (46). From Eq. (43) we find that for the Paschen–Back effect and normal Zeeman splitting $\kappa_\infty = 0$, and for anomalous Zeeman splitting

$$\kappa_\infty = \frac{\pi n}{\hbar c}(d_{21}^2 - d_{31}^2 + d_{41}^2)(W_1^- - W_1^+). \tag{48}$$

Thus, in the presence of spin-orbit interaction, assuming condition (37), there remains only the paramagnetic contribution to the Faraday rotation of a video soliton, which is proportional to the magnetization of the medium.

6. A COMBINATION RULE FOR DETERMINING THE FARADAY ROTATION ANGLE PER UNIT LENGTH OF AN OPTICAL PULSE

The magnitude of the Faraday rotation angle per unit length of an optical pulse for arbitrary ω and τ_p can be found by a generalization to the two-parameter case of the method proposed in Refs. 25 and 26 for one-parameter solitons. This

method allows one to establish relations between the geometric parameters (width, velocity) of the soliton without first solving the nonlinear equation (or system of equations) in explicit form. The method is based on the assumption that the initial nonlinear system has a solution in the form of a traveling, exponentially localized pulse. This solution, generally speaking, must possess the soliton properties of elastic interaction with solitons like itself. Let the initial linearized system generate a dispersion relation of the form

$$F(\omega,k)=0. \tag{49}$$

In the one-parameter case, the frequency and wave number are replaced by the imaginary values[25,26] $\omega\to i/\tau_p$ and $k\to i/v\tau_p$. Substituting these expressions in Eq. (49) gives a relation between the velocity of the soliton v and its duration τ_p.

If the nonlinear system generates two-parameter solitons, it is natural to analytically continue the dispersion parameters ω and k to the complex plane as a generalization of the one-parameter case:

$$\omega\to\Omega\equiv\omega+i/\tau_p,\quad k\to K\equiv k+i/v\tau_p. \tag{50}$$

Substituting expression (50) in Eq. (49), after separating the real and imaginary parts, we obtain two equations:

$$F_1(\omega,\tau_p,k,v)=0,\quad F_2(\omega,\tau_p,k,v)=0. \tag{51}$$

Relations (51) between four parameters leave two free, for which we can choose, for example, ω and τ_p.

The foregoing procedure can be justified in the following way. An exponentially localized two-parameter pulse falls off at its "tails" like $\exp[-|t-z/v|/\tau_p+i(\omega t-kz)]$. Since the intensity of the pulse is low in the tails, we can use an approach here associated with the linear approximation. Recall, however, that the very existence of localized solutions of finite amplitude is fundamentally associated with nonlinearity. The application of this method to the derivative nonlinear Schrödinger equation and the nonlinear Schrödinger equation leads to a relation between the velocity of the soliton and its duration, which coincides with relation (22).

Substitutions (50) admit of a simple quantum-field interpretation. Let $\tau_p\to\infty$. In this case we have a monochromatic wave with frequency ω and wave number k or a photon with energy $\hbar\omega$ and momentum $\hbar k$. The appearance of imaginary additive terms in ω and k means that the photon in a state with energy $\hbar\omega$ and momentum $\hbar k$ has a finite lifetime τ_p (the mean free path of the photon is equal in this case to $v\tau_p$). In quantum field theories such imaginary additive terms in the poles of the Green's functions appear as a result of the interaction of the particles with each other. The interaction of the photons with each other by way of the medium is due to the nonlinearity, which, together with the dispersion, generates the nonperturbative soliton solutions. Thus, the duration τ_p of a two-parameter soliton can be interpreted as the lifetime of a photon in a state with energy $\hbar\omega$. The spatial dimension of the soliton $v\tau_p$ plays the role of the mean free path of the photon forming, among others, the given soliton. In this case, for envelope solitons $\omega\gg\tau_p^{-1}$ and for video solitons $\omega\lesssim\tau_p^{-1}$. In the latter case one can hardly speak any more of individual photons with frequency ω since the spectral width of the soliton τ_p^{-1} exceeds the frequency of the oscillation ω.

Linearization of system (1)–(4) reduces to the equality $b_1^{\pm}=b_{10}^{\pm}$, but $b_1^{\pm}$ and ψ enter the picture in the role of small dynamical parameters. The Faraday effect can be represented in the form of a superposition of right- and left-polarized waves. In this case

$$\psi_{r,l}\propto\exp[\mp i(\omega t-k_{r,l}z)], \tag{52}$$

$$b_{j,r,l}^{\pm}\propto\exp[\mp i(M_1^{\pm}-M_j^{\pm})(\omega t-k_{r,l}z)]. \tag{53}$$

The subscript r corresponds to the right-polarized wave, and the subscript l, to the left-polarized wave. The Faraday rotation angle per unit length of a plane wave is given by

$$\kappa_{lin}\equiv\frac{\Delta\varphi_{lin}}{\Delta z}=\frac{1}{2}(k_l-k_r). \tag{54}$$

Since we have chosen ω and τ_p as the free parameters, they will not differ for the right- and left-polarized waves. The parameters v_r and v_l, and k_r and k_l will differ. The difference between v_r and v_l is a manifestation of soliton circular dichroism, which shows up as a difference in the mean free paths of the left- and right-polarized photons. Assume that system (1)–(4) admits the existence of solutions in the form of exponentially localized, isolated traveling pulses. We represent the phases of the left- and right-polarized waves in the form

$$\varphi_{l,r}=\pm\omega t\mp k_{l,r}z=\pm\omega(t-z/v_{l,r})\mp\tilde{k}_{l,r}z,$$

where we have introduced the effective wave numbers $\tilde{k}_{l,r}=k_{l,r}-\omega/v_{l,r}$. The velocities $v_{l,r}(\omega,\tau_p)$ are found from system (51) after making substitutions (50). Thus, in a frame of reference moving with velocity $v_{l,r}$ we have the effective wave number $\tilde{k}_{l,r}$. Accordingly, for the Faraday rotation angle per unit length of the soliton we have expression (54) with the replacement $k_{l,r}\to\tilde{k}_{l,r}$:

$$\kappa_s\equiv\frac{\Delta\varphi_s}{\Delta z}=\frac{1}{2}[\tilde{k}_l(\omega,\tau_p)-\tilde{k}_r(\omega,\tau_p)]. \tag{55}$$

Substituting relations (52) and (53) in system (1)–(4), making the replacements (50), and invoking relation (55), we find

$$\kappa_s(\omega,\tau_p^{-1})=(\omega^2+\tau_p^{-2})\alpha(\omega,\tau_p^{-1}), \tag{56}$$

where

$$\alpha(\omega,\tau_p^{-1})=\frac{\pi n}{\hbar c}\sum_{k=+,-}\sum_{j=2}^{N}(M_1^k-M_j^k)d_{j1}^2W_1^k\left[\frac{1}{(\omega_{jk}-w)^2+\tau_p^{-2}}+\frac{1}{(\omega_{jk}+\omega)^2+\tau_p^{-2}}\right]. \tag{57}$$

In the derivation of relations (56) and (57) it was assumed that $4\pi d_{j1}^2n/\hbar\omega_{jk}\ll1$. This approximation is equivalent to replacing the refractive index in Eqs. (11) and (22) by unity.

Setting $\omega=\tau_p^{-1}=0$ in Eq. (57), we obtain an expression which coincides with the expression for $\alpha(0,0)$ given by Eq. (11) up to the previously noted replacement $N_0\to 1$. If $\tau_p^{-1}\to\infty$, then from Eqs. (56) and (57) we find that $\kappa_s\to\kappa_\infty$ [see Eq. (43)].

Thus, the expression for κ_s obtained for these two limiting cases coincides with the corresponding expressions obtained previously by other means. This fact can serve as one argument in favor of the method proposed here, which has allowed us to obtain a quantitative characteristic for the pulsed Faraday effect, namely $\kappa_s(\omega,\tau_p^{-1})$ for arbitrary ω and τ_p^{-1}. Here, however, we must mention a restriction related to relaxation: $\tau_p^{-1}\gg T_2^{-1}$, $(T_2^*)^{-1}$, where T_2^* and T_2 are the reversible and irreversible phase relaxation times, respectively. There is another restriction on τ_p^{-1} that is due to the possibility of neglecting ionization processes (see Sec. 5). In other words, the quantity τ_p^{-1} should not exceed $\varepsilon_i/\hbar$, where ε_i is the ionization energy of the atom. The above restrictions on τ_p^{-1} can be written in the form of a double inequality:

$$T_2^{-1},\quad (T_2^*)^{-1}\ll\tau_p^{-1}<\varepsilon_i/\hbar. \tag{58}$$

The restriction on the frequency of the pulse brought about by the possibility of neglecting inhomogeneous broadening has the form

$$|\omega-\omega_{jk}|>(T_2^*)^{-1} \tag{59}$$

for all j and k.

The combination rule (56), (57) allows one, without solving the initial coupled system of material equations and Maxwell's equations, but only making use of the diagram of allowed σ transitions, to determine the value of the Faraday rotation angle per unit length of laser pulses for arbitrary quantitative relations between their frequency and duration. This rule is valid not only for the vapors under discussion, with their specific Zeeman splitting diagrams, but for all rarefied, in particular gaseous, media. Here the criterion of rarefaction has the form

$$4\pi d_{jk}^2 n/\hbar\omega_{jk}\ll 1 \tag{60}$$

for all j and k.

In general the combination rule for finding the Faraday rotation angle per unit length of the pulse, given conditions (58)–(60), can be formulated in the following way. Let the diagrams of the σ transitions corresponding to the Faraday effect be known. Then with each such transition we associate a term of the form

$$\frac{\pi n}{\hbar c}\,d_{jk}^2\left[\frac{1}{(\omega_{jk}-\omega)^2+\tau_p^{-2}}+\frac{1}{(\omega_{jk}+\omega)^2+\tau_p^{-2}}\right]W_k, \tag{61}$$

where W_k is the population of the level from which the transition originates with the arrival of the pulse. We then put a plus sign in front of this term if the transition from the kth to the jth level is accompanied by a decrease in the projection of the total angular momentum onto the direction of the magnetic field, and a minus sign in the opposite case. Allowing in this way for all possible transitions, we obtain an expression for $\alpha(\omega,\tau_p^{-1})$, substitution of which into Eq. (56) gives the desired result. Note that the given rule is also valid in the case in which all of the levels are differently populated before the arrival of the pulse. This assertion can be easily confirmed by calculations for the level diagram described in this section. In the latter case, according to the given rule, each pair of levels in the diagram of the σ transitions is taken into account twice: the transitions "lower to upper" and "upper to lower". We thus conclude that the term (61) corresponds to each pair of levels, up to the replacement $W_k\to W_k-W_j$. In this case, in choosing the sign of the term (61), the more populated level of the given pair of levels is taken as the initial one. Then we must make the replacement $W_1^k\to W_1^k-W_j^k$ in expression (57), which is valid for the level diagrams depicted in Figs. 1–3. Thus, equalization of the populations in a system of optical transitions leads to a decrease in the Faraday rotation angle per unit length of a light pulse.

7. THE MACALUSO–CORBINO SOLITON EFFECT

In the case of linear plane waves, the Macaluso–Corbino effect is manifested as an abrupt increase in the Faraday rotation angle per unit length in the vicinity of the resonance frequencies with multiple changes in the sign of rotation. The frequency of a plane wave in the linear region is its only free parameter. The solitons which have been examined here are two-parameter nonlinear solutions of system (1)–(4). The frequency ω and the duration τ_p have entered as the two independent parameters. Note that the expression for $\kappa_s(\omega,\tau_p^{-1})$ [see Eqs. (56) and (57)] can be written in the form of a real function of the complex variable $\Omega\equiv\omega+i/\tau_p$:

$$\kappa_s(\Omega)=\frac{\pi n}{\hbar c}\,|\Omega|^2\sum_{k=+,-}\sum_{j=2}^{N} d_{j1}^2(M_1^k-M_j^k)(W_1^k-W_j^k)$$
$$\times[|\omega_{jk}-\Omega|^{-2}+|\omega_{jk}+\Omega|^{-2}]. \tag{62}$$

Here we have taken note of the remark made at the end of the previous section regarding the nonzero population of all the levels forming optical σ-transitions. The function $\kappa_s(\Omega)$ has poles on the real axis corresponding to the resonance frequencies of the medium: $\Omega=\pm\omega_{jk}$ for all j and k. At these points $\omega=\pm\omega_{jk}$, $\tau_p^{-1}=0$ [recall, however, restrictions (58)–(60)]. The zeros of $\kappa_s(\Omega)$ lie in the vicinity of the above-mentioned poles and form closed curves around them. The sign of $\kappa_s(\Omega)$ inside the regions formed by these curves and its sign outside them differ. Figures 4 and 5 show such curves for magnetized sodium vapor for anomalous Zeeman splitting and the Paschen–Back effect, respectively. In the latter case we have one such curve. Its closure takes place to the side of ω_{2+}; however, in this case the values of τ_p^{-1} are so large that the left side of condition (58) is violated. For this reason only that part of the curve is depicted in Fig. 5 that has immediate physical meaning. For normal Zeeman splitting, the described curve is identical to the corresponding curve for the Paschen–Back effect. Therefore all conclusions drawn in what follows for the Paschen–Back effect hold equally for normal Zeeman splitting.

Figures 4 and 5 also depict curves along which $\kappa_s(\Omega)=$const. The fact that the zeros and poles of the function $\kappa_s(\Omega)$ are found right next to each other is indicative that in these regions there take place abrupt variations in the be-

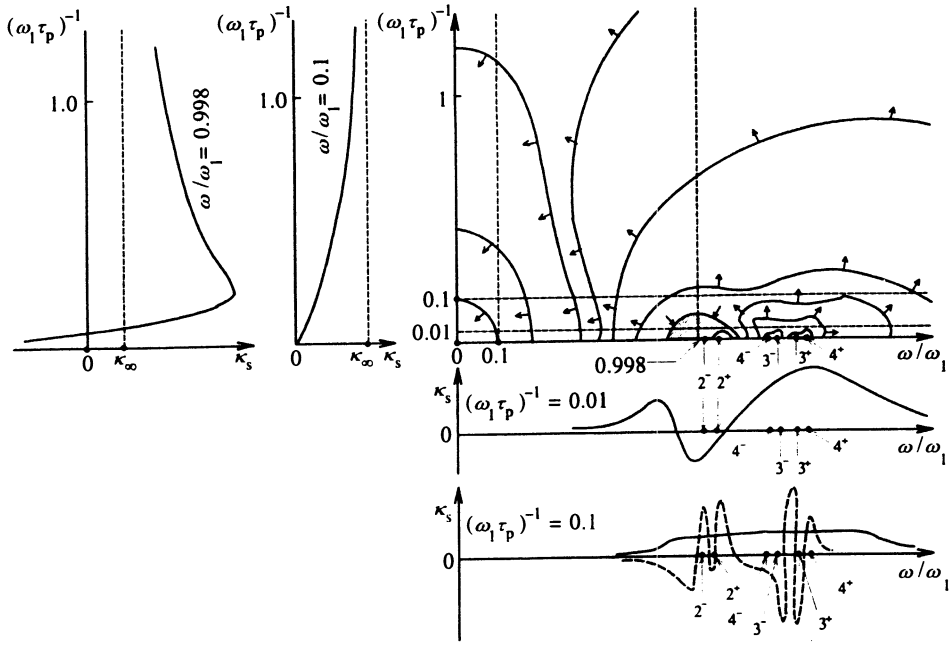

FIG. 4. Isolines κ_s=const in the quadrant $\omega>0$, $\tau_p^{-1}>0$ of the complex Ω plane and schematic graphs of $\kappa_s(\omega)$ and $\kappa_s(\tau_p^{-1})$ for the case of magnetized sodium vapor in the presence of anomalous Zeeman splitting. The straight lines $\omega/\omega_1=0$ and $(\omega_1\tau_p)^{-1}=0$ are the symmetry axes of the isolines. The region $\kappa_s<0$ is hatched in, $\kappa_s>0$ everywhere else. The arrows attached to the isolines indicate the direction of decreasing κ_s. The following notation has been used: $p^{\pm}=\omega_{\pm}/\omega_1$, $p=2, 3, 4$. The dashed curve in the lower graph of $\kappa_s(\omega)$ corresponds to $\kappa_{lin}(\omega)$ with atomic relaxation taken into account.

havior of the Faraday rotation angle per unit length. We will refer to the dependence of the latter on the complex variable Ω, when the values of its modulus $|\Omega|$ lie in the vicinity of the resonance frequencies of the medium, as the Macaluso–Corbino soliton effect. Thus, the Macaluso–Corbino soliton effect consists in the dependence of the Faraday rotation angle per unit length not only on the frequency of the pulse, but also on its duration. With increasing amplitude of the electric field of the pulse, the rate of the induced transitions increases and, as a result, the inverse duration of the pulse grows. Therefore, nonzero values of τ_p^{-1} characterize the degree of optical nonlinearity. In this regard, the functional dependence $\kappa_s(\tau_p^{-1})$ for ω=const should qualitatively coincide with the functional dependence of κ_s on the amplitude of the electric field of the pulse. Note that $\kappa_s(\Omega)$ has an isolated zero at $\Omega=0$. In the vicinity of this point the function $\kappa_s(\Omega)$ has no singularities and behaves like $\kappa_s\sim|\Omega|^2$. This region corresponds to the low-frequency approximation investigated in Sec. 3, which bears no relation to the Macaluso–Corbino effect. Drawing straight lines parallel to the ω axis through the isolines $\kappa_s(\Omega)$=const, we obtain the behavior of $\kappa_s(\omega)$ for various values of τ_p^{-1}=const (Figs. 4 and 5). The straight lines ω=const intersecting the same isolines give us a picture of the dependences $\kappa_s(\tau_p^{-1})$ or $\kappa_s(|\psi_m|)$ for fixed frequency of the pulse (Figs. 4 and 5).

For magnetized vapors in the case of anomalous Zeeman splitting, the behavior of $\kappa_s(\omega)$ depends substantially on the value of τ_p^{-1} (or, equivalently, on the amplitude of the soli-

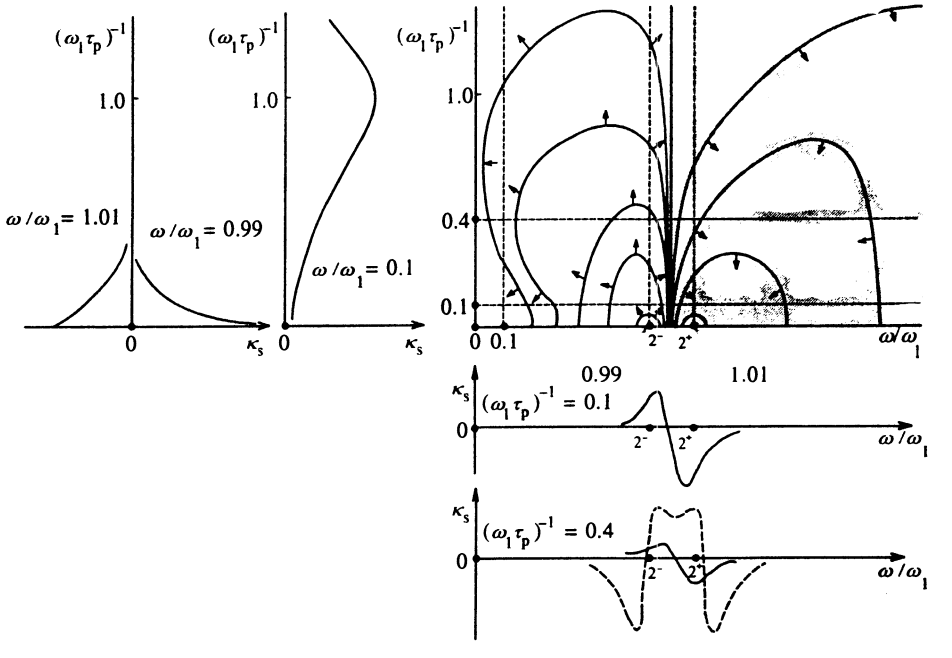

FIG. 5. Isolines κ_s=const in the quadrant $\omega>0$, $\tau_p^{-1}>0$ of the complex Ω plane and schematic graphs of $\kappa_s(\omega)$ and $\kappa_s(\tau_p^{-1})$ in the case of the Paschen–Back effect for sodium vapor. The notation is the same as in Fig. 4. For normal Zeeman splitting, the Faraday spectra coincide with the Faraday spectra in the case of the Paschen–Back effect.

ton). Thus, if $\omega_1\tau_p \sim 10^2$, the function $\kappa_s(\omega)$ changes sign several times in the vicinity of the resonance frequencies. However, for $\omega_1\tau_p \sim 10$, the functional dependence $\kappa_s(\omega)$ "smoothes out" substantially in the vicinity of the resonance frequencies: the function $\kappa_s(\omega)$ has a broad plateau over a region that extends somewhat beyond the resonance frequency band and does not change sign even once. For the Paschen–Back effect and normal Zeeman splitting, and also for the case of nonmagnetized vapors in the presence of anomalous Zeeman splitting, the sign of $\kappa_s(\omega)$ is determined by the offset of the frequency ω from resonance (Fig. 5). Just such a dependence was obtained in Refs. 3 and 27 for Faraday rotation in a high-power continuous laser beam for a system of alkali atoms.

In the case ω=const the behavior of $\kappa_s(\tau_p^{-1})$ is determined to a significant degree by the specific value of ω. If $\omega \ll \omega_{1,2}$, then in nonmagnetized sodium vapors, like for the Paschen–Back effect and normal Zeeman splitting, $\kappa_s(\tau_p^{-1})$ has a broad maximum in the vicinity of $\tau_p^{-1}=\omega_{1,2}$ (Fig. 5). In this region the video soliton acquires spectral components with frequencies near $\omega_{1,2}$, thereby causing resonance amplification of κ_s. However, it is clear that the contribution of these resonance components to the Fourier spectrum of the video soliton is not large in comparison with the frequency components for which $\omega \ll \omega_{1,2}$. This also explains the broadness of this maximum. If the vapor is not magnetized, the dependence $\kappa_s(\tau_p^{-1})$ does not have a local extremum, being everywhere monotonic (Fig. 4). The characteristic absolute value of κ_s for magnetized vapors exceeds the corresponding value in the absence of magnetization, and also in the presence of the Paschen–Back effect and normal Zeeman splitting, by a factor of $\omega_1/\omega_B \sim 10^3$–$10^4$.

The behavior of $\kappa_s(\tau_p^{-1})$ when the soliton frequency is near one of the resonance frequencies is curious. In this case for the Paschen–Back effect and normal Zeeman splitting, κ_s falls off monotonically with increasing τ_p^{-1} (Fig. 5). For anomalous Zeeman splitting, one can observe dependences $\kappa_s(\tau_p^{-1})$ accompanied by a sign change in κ_s (change of direction of the Faraday rotation angle) and the existence of a local extremum (Fig. 4). To observe this effect, the offset of ω from resonance should not exceed a small multiple of ω_B. Then the value of τ_p^{-1} for which κ_s changes sign and has a local extremum varies from several to several tens of ω_B, which corresponds to durations $\tau_p \sim 10^{-12}$ s with a fill frequency of $\omega \sim 10^{15}$ s^{-1}. This is the regime in which self-induced transparency is observed.[12] However, the presence of more than two quantum levels ensuring σ transitions is fundamental here. In a system with one σ transition, there cannot be any changes of sign in the Faraday rotation angle, nor can $\kappa_s(\tau_p^{-1})$ have a local extremum.

The dependences $\kappa_s(\omega)$ do not reduce to similar limiting cases for linear plane waves in the limit $\tau_p^{-1}=0$ (see Figs. 4 and 5). This again emphasizes the fundamentally nonlinear character of solitons.

In conclusion, note that the function $\kappa_s(\Omega)$ defined by Eq. (62) is not analytic, i.e., the Cauchy–Riemann conditions are not fulfilled for it except at the point $\Omega=0$. This is explained by the fact that the propagation velocity of the soliton differs from the phase velocity of the wave. Therefore expression (55) for κ_s is written in terms of the effective wave numbers $\tilde{k}_{r,l}$, which do not reduce to the wave numbers $k_{r,l}$ in the limit $\tau_p^{-1}=0$. Thus, the fundamentally nonlinear nature of solitons (the nonanalytic character of the dependence of the soliton solutions on the nonlinearity parameter as the latter tends to zero) is the reason for the nonanalyticity of the dependence of the Faraday rotation angle per unit length on Ω over the entire complex plane with the exception of the point $\Omega=0$, where the Faraday effect is absent.

8. CONCLUSION

In the present paper we have carried out a theoretical study of the Faraday effect for optical solitons over wide intervals of their frequency and duration, from envelope pulses to video pulses. In this regard, the approach considered here is valid mainly for media which have low atomic density and possess magneto-optical activity. In addition, it is assumed that inhomogeneous broadening of the optical transitions is much less than the distances between the Zeeman sublevels. In the case of sodium and mercury vapors we have examined Faraday rotation of optical pulses from the same viewpoint for anomalous and normal Zeeman splitting and for the Paschen–Back effect. We have obtained the "derivative nonlinear Schrödinger equation" (13) for the electric field intensity in the low-frequency limit $|\Omega| \ll \omega_{jk}$ (for all j and k). In this case the frequency of the pulse is far from resonance. Therefore the assumption of smallness of inhomogeneous broadening is superfluous here. In addition, the low-frequency limit (5) allowed us to obtain equation (13) without resorting to the assumption of low density of the medium. Consequently, Eq. (13) should describe the pulsed Faraday effect for solids as well as gases. Here it is important that the electronic spectra of the atoms responsible for the magneto-optical activity not be subjected to substantial distortions in the crystal. The latter conditions are fulfilled for admixtures of rare-earth elements in solid salts and solutions. The paramagnetism of rare-earth ions is due to the unfilled $4f$ subshell, whose electrons are screened from the influence of neighboring atoms by the outer electrons. As a result, rare-earth ions under such conditions behave like atoms of a rarefied gas.[28] Obviously, relation (33) between the Faraday rotation angles per unit length for a soliton $\Delta\varphi_s$ and for a plane wave $\Delta\varphi_{lin}$ is also valid here. It is remarkable that the ratio $\Delta\varphi_s/\Delta\varphi_{lin}$ does not depend on the properties of the medium, but is determined only by the frequency and duration of the optical soliton. In this case it is sufficient to know the magnitude of $\Delta\varphi_{lin}$, which allows one to predict $\Delta\varphi_s$ without resorting to cumbersome calculations. Here it turns out that the Faraday rotation angle for a pulse and for a linear wave are oppositely directed, which points to the threshold nature of the formation of a DNSE soliton.

At low temperatures ($T \ll \hbar\omega_B/k_B \sim 1$ K) the magnetization of the paramagnetic impurities in solid solutions is near saturation: $M_0 \sim 2\mu_B n$. In this case Verde's soliton law (30) is not fulfilled [see Eqs. (25) and (28)]. The ratio between the paramagnetic contribution to the Faraday rotation angle and the diamagnetic is estimated to be $\omega_1/\omega_B \sim 10^3$–$10^4$, and the first term on the right-hand side of Eq. (24) can be neglected. The diamagnetic contribution to the Faraday rotation angle

in the case of anomalous Zeeman splitting can compete with the paramagnetic term only at very high temperatures $T \gtrsim 10^4$ K, which are encountered, for example, in the vapors of molten metals.

The combination rule for finding the Faraday rotation angle per unit length of an optical soliton formulated in Sec. 6 allows us to state that Eq. (13) remains valid in the low-frequency limit (5) when all possible σ transitions between the discrete levels, not just the levels forming the main spectral series, are taken into account. Thus, for example, the transition from one of the Zeeman sublevels of the 3^2S state to one of the sublevels of the 6^2P state corresponding to the wavelength 2853 Å (Ref. 12) can be taken into account by redefinition of the coefficient $\alpha(0,0)$ in Eq. (13). To be more specific, the expression for $\alpha(0,0)$ that takes account only of the σ transitions within the main spectral series can be augmented by a term of the form $2\pi n d_{j1}^2 W_1^{\pm}/(\hbar c \omega_{j\pm}^2)$. Other σ transitions from the 3^2S sublevels to higher-lying 4^2P and 5^2P sublevels can be taken into account in an analogous way.[12] Transitions from 3^2P sublevels to higher-lying Zeeman sublevels can be taken into account by redefining the nonlinearity coefficient in Eq. (13). Indeed, the contribution of these transitions to the complex polarization is proportional to the product of the population of the 3^2P sublevels to the complex electric field strength $\psi \equiv E_x + iE_y$. However, as is clear from Eq. (8), these populations are of the order of $|\psi|^2$, which proves our assertion. Thus, in the low-frequency limit (5), Eq. (13) is fairly universal. Equation (13) takes account of all possible σ transitions between the discrete levels of a quantum system possessing magneto-optical activity. The more quantum transitions we take account of, the more complicated become the expressions for $\alpha(0,0)$ and β. However, in any case the dynamics of the pulse are described by Eq. (13).

Use of the combination rule (56), (57) [see also Eq. (62)] in the vicinity of one of the resonance frequencies requires great care. First of all, it is necessary to recall inequalities (58)–(60). Secondly, use of this rule is based on the assumption that a stable solution in the form of an exponentially localized pulse exists. In general, clearly, it is not simple to justify such an assumption. The particular solutions analyzed in Secs. 3 and 5 show that in the two opposite limits (5) and (37), this assumption is confirmed. In the case of a resonance pulse with one σ transition we obtain the system of self-induced transparency equations, which describes a circularly polarized envelope soliton (see Ch. 7, Sec. 7 of Ref. 12). The method proposed in Sec. 6 does not yield an answer to the question of whether there exist stable soliton-like solutions in the case of arbitrary pulse durations and frequency, that also take into account all possible resonance and nonresonance transitions. It would be all the more desirable then to have an experimental confirmation of the functional dependences obtained here of the Faraday rotation angle on the frequency and duration (amplitude) of an optical pulse, which are the essence of the Macaluso–Corbino soliton effect.

The studies carried out in the present work could aid the development of pulsed laser polarimetry.[14] The dependence of the Faraday rotation angle per unit length on the two pulse parameters ω and τ_p^{-1} make the Faraday rotation spectra two-dimensional, which allows one to speak of fundamentally new possibilities in laser magnetic spectroscopy.

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[1] S. Giraud-Cotton and U. P. Kaftandjian, Phys. Rev. A **32**, 2211 (1985).
[2] S. B. Borison, I. L. Lyubchanskiĭ, A. S. Zel'tser, and A. D. Petrenko, Opt. Spektrosk. **67**, 228 (1989) [Opt. Spectrosc. **67**, 131 (1989)].
[3] T. Ya. Karagodova, A. A. Zakharov, and A. V. Kolpakov, Izv. RAN: Ser. Fiz. **56**, 209 (1992).
[4] T. Ya. Karagodova, A. A. Zakharov, and A. V. Kolpakov, Opt. Spektrosk. **74**, 1137 (1993) [Opt. Spectrosc. **74**, 676 (1993)].
[5] D. H. Auston, K. P. Cheung, J. A. Valdmanis, and D. A. Kleinman, Phys. Rev. Lett. **53**, 1555 (1984).
[6] J. T. Darrow, B. B. Hu, X. C. Zhang, and D. H. Auston, Opt. Lett. **15**, 323 (1990).
[7] É. M. Belenov, P. G. Kryukov, A. V. Nazarkin *et al.*, JETP Lett. **47**, 523 (1988).
[8] É. M. Belenov and A. V. Nazarkin, JETP Lett. **51**, 288 (1990).
[9] É. M. Belenov, A. V. Nazarkin, and V. A. Ushchapovskiĭ, Zh. Eksp. Teor. Fiz. **100**, 762 (1991) [Sov. Phys. JETP **73**, 422 (1991)].
[10] S. V. Sazonov and E. V. Trifonov, Zh. Eksp. Teor. Fiz. **103**, 1527 (1993) [J. Exp. Teor. Phys. **76**, 744 (1993)].
[11] S. V. Sazonov and E. V. Trifonov, J. Phys. B **27**, L7 (1994).
[12] L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms*, Dover, New York (1987).
[13] P. J. Caudrey, J. C. Eilbeck, J. D. Gibbon, and R. K. Bullough, J. Phys. A **6**, L53 (1973).
[14] E. B. Aleksandrov and V. S. Zapasskiĭ, *Laser Magnetic Spectroscopy* [in Russian], Nauka, Moscow (1986).
[15] I. I. Sobel'man, *Introduction to the Theory of Atomic Spectra* [in Russian], Nauka, Moscow (1977).
[16] D. N. Kaup and A. C. Newell, J. Math. Phys. **19**, 798 (1978).
[17] V. S. Gerdzhikov, M. I. Ivanov, and P. P. Kulish, Teor. Mat. Fiz. **44**, 342 (1980).
[18] A. I. Maĭmistov, Zh. Eksp. Teor. Fiz. **104**, 3620 (1993) [J. Exp. Teor. Phys. **77**, 727 (1993)].
[19] T. Kawata, N. Kobayashi, and H. Inoue, J. Phys. Soc. Jpn. **46**, 1008 (1978).
[20] V. A. Vysloukh and I. V. Cherednik, Teor. Mat. Fiz. **78**, 35 (1989).
[21] A. I. Alekseev, Zh. Eksp. Teor. Fiz. **104**, 2954 (1993) [J. Exp. Teor. Phys. **77**, 371 (1993)].
[22] A. M. Kosevich and A. M. Kovalëv, *Introduction to Nonlinear Physical Mechanics* [in Russian], Naukova Dumka, Kiev (1989), p. 147.
[23] V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskiĭ, *Soliton Theory: The Inverse Transform Method* [in Russian], Nauka, Moscow (1980).
[24] S. V. Sazonov, Kvantovaya Elektron. **20**, 135 (1993) [Quantum Electron. (1993)].
[25] E. Schlömann, Appl. Phys. Lett. **19**, 274 (1971).
[26] V. G. Bar'yakhtar, B. A. Ivanov, and A. L. Sukstanskiĭ, JETP Lett. **27**, 211 (1978).
[27] P. F. Liao and G. C. Bjorklund, Phys. Rev. A **15**, 2009 (1977).
[28] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Pergamon Press, Oxford (1980).

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